Time-independent perturbation theory for quasinormal modes in leaky optical cavities

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If a cavity is leaky, its "modes" are quasinormal modes (QNM's) with complex frequencies, and they do not constitute a Hermitian system. Nevertheless, the time-independent perturbation arising from a small change of the dielectric-constant distribution can be formulated in terms of these discrete QNM's, resulting in a generalization of the usual perturbation scheme to a non-Hermitian situation. In particular, shifts for the imaginary parts of the eigenvalues are obtained as well. This paper presents this theory for scalar waves in one dimension.

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I. INTRODUCTION

The scalar analog of electromagnetic waves in a one-dimensional model optical cavity is described by

$$\left[\rho(x)\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\right]\varphi(x,t)=0 , \tag{1.1}$$

where $\rho(x)$ is the dielectric constant, here assumed to be nondispersive and nonabsorptive, but spatially inhomogeneous. (The notation ρ emphasizes the analogy to the transverse vibrations of a string with linear density ρ, which has been extensively studied [1] as models of dissipative systems.) We shall first specialize, without much loss of generality, to situations where there is a totally reflecting mirror placed at $x=0$, represented by the boundary condition $\varphi(x=0,t)=0$. We further assume that $\rho(x)$ is such as to create an optical cavity. A simple example is $\rho(x)=1+M\delta(x-a)$, where the δ function is a large mass attached to the string, making the point $x=a$ nearly a node; in optics language, a thin slab of high refractive index forms a partially transmitting dielectric mirror. The case of a cavity defined by partial transmission on both sides, with outgoing waves as $x\rightarrow\pm\infty$, can be handled as well. The "modes" of the cavity are quasinormal modes (QNM's) because energy is lost from the cavity due to leakage. The QNM's are solutions to (1.1)

$$\varphi(x,t)=f_j(x)e^{-i\omega_j t} , \tag{1.2}$$

with outgoing-wave boundary condition at infinity. The frequencies ω_j are complex, with negative imaginary parts. For a cavity that has a small leakage, $\mathrm{Im}\omega_j\simeq 0$, f_j is nearly real, and the QNM's are in some sense close to the normal modes of the corresponding enclosed cavity. The latter is a Hermitian system, whose normal modes form a complete orthogonal basis, to which all the usual mathematical formalisms can be applied. It is therefore natural to ask whether the QNM's $\{f_j\}$ of a leaky cavity can play the same role; the answer is by no means obvious since a leaky cavity by itself is not a Hermitian system.

In a previous paper [2], hereafter called I, we have shown that provided $\rho(x)$ has a discontinuity at some $x=a$ (either in the function itself, or in any of its derivatives), and provided that $\rho(x)$ approaches its asymptotic constant value sufficiently rapidly, then $\{f_j\}$ forms a complete set in the domain $0<x<a$ for functions satisfying the outgoing wave conditions at infinity. In many circumstances, a set of "zero modes," which have no counterparts in the corresponding enclosed cavity, are necessary to ensure completeness. Moreover, the retarded Green's function $G(x,y;t)$ for $t>0$ can be expressed as a sum over $f_j(x)f_j(y)e^{-i\omega_j t}$. In the case of a system defined on the full line, with $\rho(x)$ having discontinuities at $x=a_1,\ldots,a_N$, the completeness holds for all x,y lying between the leftmost discontinuity a_1 and the rightmost discontinuity a_N [2].

Hitherto, all rigorous treatments of optical phenomena in leaky cavities have relied on the modes of the universe, which involve $\varphi(x,t)$ not just inside the cavity but also outside. These are continuous in frequency and mathematically more complicated. Moreover, they tend to obscure the often simple physics, which at a heuristic level is readily described in terms of discrete "modes," e.g., the "modes" of a laser. The results in I open the way to a systematic program of recasting all these works in terms of the discrete QNM's, in a way that is both mathematically precise and physically intuitive. The works that could be so reformulated include discussions on the density of states [3], stimulated emission [4], cavity QED effects [5], quantization [6], lasing [7], etc. In this paper, we take the first step in this program by dealing with time-independent perturbation of the dielectric constant

$$\rho(x)\rightarrow\rho(x)+\mu V(x) ,$$

where $|\mu|\ll 1$, and the "potential" $V(x)$ is confined to the cavity. The object, in parallel with the standard perturbation series, is to express the corrections to the frequencies ω_j and QNM eigenfunctions $f_j(x)$ as a series in

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μ, in terms of the matrix elements of V.

One might at first attempt to approach this problem by making use of the completeness and orthogonality of $\{f_j\}$ proved in I, in much the usual way. This turns out to be impossible, because the functions f_j are complete within the cavity, but are orthogonal only under a suitably defined inner product that also involves the outside [2]. Consequently, the concepts of completeness and orthogonality are incongruent and cannot be used together, and we have to resort to a derivation that relies only on the completeness and the corresponding representation of the Green's function.

The time-independent perturbation of QNM's has immediate application to any situation in which the dielectric constant inside a cavity is changed slightly, e.g., by a temperature fluctuation. Our work on this general formalism has in particular been motivated by experiments involving laser interaction with microdroplets [8], the physics of which is dominated by the sharp resonances, i.e., QNM's, which in this context are generally referred to as morphology-dependent resonances. Specifically one is often interested in the shifts in the frequencies and widths of the resonances when the droplet departs slightly from sphericity, or undergoes other small changes. These shifts, the splitting of the originally degenerate angular momentum multiplet, and their mutual interference have all been observed [9], and these have also been used to infer the degree of departure from sphericity. Largely in this context, the perturbative formalism for the frequencies has been developed, without using the completeness relation of the quasimodes [10], which makes the formalism (especially the second- and higher-order formalism) rather complicated. The present paper will instead generate a perturbative series using the discrete QNM basis, which is therefore no more complicated than the usual Rayleigh-Schrödinger series. More importantly, it manifestly goes over to the corresponding series in the case of a totally enclosed cavity defining a Hermitian system.

II. FORMALISM

A. Series for the exact Green's function

The exact system in the frequency domain is governed by the Green's function $\tilde{G}(x,y;\omega)$ satisfying

$$-\left\{\frac{\partial^2}{\partial x^2}+\omega^2[\rho(x)+\mu V(x)]\right\}\tilde{G}(x,y;\omega)=\delta(x-y)\ . \tag{2.1}$$

The causal boundary condition on $G(x,y;t)$ implies that $\tilde{G}$ is analytic in the upper half of the ω plane. The zero-order Green's function $\tilde{G}^{(0)}$, henceforth denoted as $\tilde{D}$, satisfies

$$-\left\{\frac{\partial^2}{\partial x^2}+\omega^2\rho(x)\right\}\tilde{D}(x,y;\omega)=\delta(x-y)\ . \tag{2.2}$$

In I, we have shown that for $0<x,y<a$, $D(x,y;t)$ can be represented in terms of the QNM's of the system, labeled by an index j. Recast into the frequency domain, this statement reads

$$\tilde{D}(x,y;\omega)=-\tfrac{1}{2}\sum_j \frac{f_j^{(0)}(x)f_j^{(0)}(y)}{\omega_j^{(0)}(\omega-\omega_j^{(0)})}\ , \tag{2.3}$$

where the unperturbed eigenvalues and eigenfunctions, denoted by a superscript (0), satisfy

$$-\left\{\frac{\partial^2}{\partial x^2}+\omega_j^{(0)^2}\rho(x)\right\}f_j^{(0)}(x)=0\ , \tag{2.4}$$

with $f_j^{(0)}$ $(x=0)=0$ and the outgoing-wave boundary condition as $x\to\infty$. In writing (2.3), we have assumed that $f_j^{(0)}$ is normalized to

$$\langle\langle f_j^{(0)}|f_j^{(0)}\rangle\rangle=1\ , \tag{2.5}$$

in which the generalized norm has been defined in I.

Put

$$\tilde{G}=\tilde{D}+\tilde{G}'\ , \tag{2.6}$$

so that (2.1) becomes

$$-\left\{\frac{\partial^2}{\partial x^2}+\omega^2\rho(x)\right\}\tilde{G}'=\mu\omega^2V(x)\tilde{G}\ . \tag{2.7}$$

By virtue of (2.2), $\tilde{D}$ is the inverse of the operator on the left of (2.7), so that one has exactly

$$\tilde{G}'(x,y;\omega)=\mu\int dx_1\tilde{D}(x,x_1;\omega)W(x_1,\omega)\tilde{G}(x_1,y;\omega)\ , \tag{2.8}$$

where

$$W(x,\omega)=\omega^2V(x)\ . \tag{2.9}$$

In an obvious notation, and henceforth dispensing with $\sim$ since there is no danger of confusion with the t domain, (2.8) is

$$G'=\mu DWG\ , \tag{2.10}$$

from which one obtains the formal iterative solution

$$G=D+\mu DWD+\mu^2DWDWD+\cdots\ . \tag{2.11}$$

Now assume that $V(x)=0$ for $x>a$; then all the coordinates involved in (2.11) are inside the cavity, for which (2.3) applies. Then define the matrices G_{jk}, D_{jk}, and W_{jk} by

$$\tilde{G}(x,y;\omega)=\sum_{jk} f_j^{(0)}(x)G_{jk}(\omega)f_k^{(0)}(y)\ , \tag{2.12}$$

$$\tilde{D}(x,y;\omega)=\sum_{jk} f_j^{(0)}(x)D_{jk}(\omega)f_k^{(0)}(y)\ , \tag{2.13}$$

$$\begin{aligned}W_{jk}(\omega)&=\omega^2V_{jk}(\omega)\\&=\int dx\, f_j^{(0)}(x)W(x,\omega)f_k^{(0)}(x)\ .\end{aligned} \tag{2.14}$$

The matrix D is diagonal and given explicitly by

$$D_{jk}(\omega)=-\frac{1}{2}\frac{1}{\omega_j^{(0)}(\omega-\omega_j^{(0)})}\delta_{jk}\ . \tag{2.15}$$

The matrix W_{jk} is symmetric, but not Hermitian since the eigenfunctions are not real. Then (2.11) is recovered in exactly the same form as an equation for the corresponding matrices. In other words we have obtained a perturbative series for the exact Green's function in the basis set $\{f_j^{(0)}\}$, without having to invoke the orthogonality of the set.

The exact eigenvalues ω_p are the poles of the exact Green's function G as calculated from (2.11). Let G behave for $\omega \to \omega_p$ as

$$G_{jk} \simeq -\frac{1}{2\omega_p} A_p^2 \frac{R_p^{jk}}{\omega-\omega_p} \; . \tag{2.16}$$

The normalization constant A_p^2 will be chosen for convenience later; this then defines the residue matrix R_p^{jk}. We shall later see that this residue factorizes as

$$R_p^{jk} = \alpha_p^j \alpha_p^k \; , \tag{2.17}$$

thus we have, for $\omega \to \omega_p$,

$$\tilde{G}(x,y;\omega) \simeq -\frac{1}{2\omega_p}\left\{\sum_j \alpha_p^j f_j^{(0)}(x)\right\} \times \frac{A_p^2}{\omega-\omega_p}\left\{\sum_k \alpha_p^k f_k^{(0)}(y)\right\} . \tag{2.18}$$

But the exact Green's function must have an expansion similar to (2.3), in terms of the normalized exact eigenfunctions $f_p(x)$, i.e.

$$\tilde{G}(x,y;\omega) \simeq -\frac{1}{2\omega_p}\frac{f_p(x)f_p(y)}{\omega-\omega_p} \; . \tag{2.19}$$

By comparing (2.18) and (2.19), we recognize

$$f_p(x) = A_p\left\{\sum_j \alpha_p^j f_j^{(0)}(x)\right\} . \tag{2.20}$$

The strategy is therefore to use the perturbation series (2.11) for G, and from it find the poles ω_p. The residues R_p^{jk} should then factorize in the manner of (2.17), from which one then obtains the coefficients α_p^j for the exact eigenfunctions. In the rest of this paper we shall assume that the eigenvalues are nondegenerate.

B. Eigenvalues

In order to obtain the eigenvalues ω_p, one has to seek the poles of the matrix G defined by (2.11). It must be noted in particular that D is strongly dependent on ω, and since we are interested in $\omega = \omega_p^{(0)} + O(\mu)$, the element

$$D_{pp} \propto \frac{1}{\omega-\omega_p^{(0)}} = O\left[\frac{1}{\mu}\right] \tag{2.21}$$

is large. Thus, even though $|\mu| \ll 1$, the series for G in the form given by (2.11) is not rapidly convergent, and must be resummed by picking out all the terms with the large factor D_{pp}. Such resummation of Feynman diagrams involves standard techniques, and will be sketched only briefly.

First consider the pp element, and notice that D is diagonal; then, from (2.11),

$$G_{pp} = D_{pp} + \mu D_{pp} W_{pp} D_{pp} + \mu^2 \sum_i D_{pp} W_{pi} D_{ii} W_{ip} D_{pp} + \mu^3 \sum_{ij} D_{pp} W_{pi} D_{ii} W_{ij} D_{jj} W_{jp} D_{pp} + \cdots \; . \tag{2.22}$$

If we display explicitly the large factor D_{pp}, a typical term looks like

$$D_{pp}(\;)D_{pp}(\;)D_{pp}\ldots(\;)D_{pp} \; . \tag{2.23}$$

Each bracket in (2.23) has the following properties. (a) It has the structure $WDW\ldots WDW$, where the number of factors of W ranges from 1 to ∞. (b) However, in the factor D, only terms D_{ii} with $i \neq p$ are present, since by definition all the D_{pp} factors have been separately displayed in (2.23). One is then led to define a matrix W^p by

$$W_{jk}^p = W_{jk} + \mu \sum_{m\neq p} W_{jm} D_{mm} W_{mk} + \mu^2 \sum_{m\neq p}\sum_{i\neq p} W_{jm} D_{mm} W_{mi} D_{ii} W_{ik} + \cdots \; . \tag{2.24}$$

The superscript p indicates the exclusion of p for the dummy indices; in diagrammatic language, all terms with an internal leg p have been removed. There are no large factors going like $O(1/\mu)$ in (2.24), so for $|\mu| \ll 1$ it is legitimate to truncate the series. In terms of this matrix,

$$G_{pp} = D_{pp} + \mu D_{pp} W_{pp}^p D_{pp} + \mu^2 D_{pp} W_{pp}^p D_{pp} W_{pp}^p D_{pp} + \cdots = [D_{pp}^{-1} - \mu W_{pp}^p]^{-1} \; . \tag{2.25}$$

Note that the infinite series in (2.25) involves numbers, not matrices. The eigenvalue condition is then

$$D_{pp}^{-1} - \mu W_{pp}^p \Big|_{\omega=\omega_p} = 0 \; , \tag{2.26}$$

or, explicitly,

$$-2\omega_p^{(0)}(\omega-\omega_p^{(0)}) - \mu\omega^2 V_{pp} - \mu^2\omega^4 \sum_{m\neq p} V_{pm} D_{mm}(\omega) V_{mp} - \mu^3\omega^6 \sum_{m\neq p}\sum_{k\neq p} V_{pm} D_{mm}(\omega) V_{mk} D_{kk}(\omega) V_{kp} + \cdots = 0 \; . \tag{2.27}$$

Denoting the expansion of the exact eigenvalue by

$$\omega_p = \omega_p^{(0)} + \mu\omega_p^{(1)} + \mu^2\omega_p^{(2)} + \cdots \; , \tag{2.28}$$

we find from (2.27) that

$$\omega_p^{(1)} = -\frac{\omega_p^{(0)}}{2} V_{pp} , \tag{2.29}$$

$$\omega_p^{(2)} = \frac{\omega_p^{(0)}}{4} \left\{ 2(V_{pp})^2 + \sum_{m \neq p} V_{pm} \frac{\omega_p^{(0)^2}}{\omega_m^{(0)}(\omega_p^{(0)} - \omega_m^{(0)})} V_{mp} \right\} , \tag{2.30}$$

$$\omega_p^{(3)} = \frac{\omega_p^{(0)}}{8} \left\{ -5(V_{pp})^3 + V_{pp} \sum_{m \neq p} V_{pm} \frac{\omega_p^{(0)^2}(6\omega_m^{(0)} - 5\omega_p^{(0)})}{\omega_m^{(0)}(\omega_m^{(0)} - \omega_p^{(0)})^2} V_{mp} - \sum_{\substack{m \neq p \\ i \neq p}} V_{pm} \frac{\omega_p^{(0)^2}}{\omega_m^{(0)}(\omega_m^{(0)} - \omega_p^{(0)})} V_{mi} \frac{\omega_p^{(0)^2}}{\omega_i^{(0)}(\omega_i^{(0)} - \omega_p^{(0)})} V_{ip} \right\} . \tag{2.31}$$

These are similar to the usual Rayleigh-Schrödinger series, and, as in that case, there are no small denominators. The factor of 2 and the $(V_{pp})^2$ term in (2.30) arise because the natural eigenvalue is ω^2 rather than ω. These formulas give corrections to both the real parts and imaginary parts of the QNM frequencies, i.e., both the shifts in the resonance positions and the changes in the widths. We have previously derived the first-order result (2.29) [10], which does not involve other modes and therefore could be dealt with without the knowledge of completeness. We have also given a limited version of the second-order result [10] involving the contributions to $\mathrm{Im}\omega_p^{(2)}$ due to transitions to other channels. The third-order correction is new, and higher orders can be written down as well. Somewhat simpler expressions can be obtained if we leave the exact ω_p implicitly on the right-hand side, and in evaluation substitute a lower-order numerical value.

From (2.16), only the combination $A_p^2 R_p^{jk}$ is defined. We adopt the convention $\alpha_p^p = 1$, so that

$$G_{pp} = [D_{pp}^{-1} - \mu W_{pp}^p]^{-1} \simeq -\frac{1}{2\omega_p} \frac{A_p^2}{\omega - \omega_p} \tag{2.32}$$

or

$$A_p^{-2} = -\frac{1}{2\omega_p} \frac{\partial}{\partial \omega} [D_{pp}^{-1} - \mu W_{pp}^p]|_{\omega_p} . \tag{2.33}$$

In the above derivation, we have assumed that as $\omega \to \omega_p$, only one element D_{pp} is large. If there are degeneracies, then an entire submatrix D_{pq} in the degenerate subspace is large, and the large elements must all be singled out and resummed, leading to slightly more complicated results than (2.29)–(2.31). The modifications are relatively straightforward, and will not be detailed in this paper.

C. Eigenfunctions

Next consider the series (2.11) for G_{pj} $(j \neq p)$ near $\omega = \omega_p$. Again using (2.24) and, isolating the large element D_{pp}, we have

$$\begin{aligned} G_{pj} &= \mu D_{pp} W_{pj}^p D_{jj} + \mu^2 D_{pp} W_{pp}^p D_{pp} W_{pj}^p D_{jj} + \mu^3 D_{pp} W_{pp}^p D_{pp} W_{pp}^p D_{pp} W_{pj}^p D_{jj} + \cdots \\ &= D_{pp} \sum_{n=0}^{\infty} (\mu W_{pp}^p D_{pp})^n \mu W_{pj}^p D_{jj} \\ &= \frac{1}{D_{pp}^{-1} - \mu W_{pp}^p} \mu W_{pj}^p D_{jj} \\ &\approx -\frac{1}{2\omega_p} \frac{A_p^2}{\omega - \omega_p} \mu W_{pj}^p D_{jj} . \end{aligned} \tag{2.34}$$

The structure of (2.34) is readily understood as follows. Of all the dummy indices that must be summed over, all indices equal to p have been explicitly displayed in D_{pp}, whereas all indices not equal to p have been incorporated into the matrix W^p. Then, by comparison with (2.16) and (2.17),

$$\begin{aligned} \alpha_p^p \alpha_p^j &= R_p^{pj} = \mu W_{pj}^p D_{jj} , \\ \alpha_p^j &= \mu W_{pj}^p D_{jj} . \end{aligned} \tag{2.35}$$

Finally, by considering the series (2.11) for G_{jk} $(j,k \neq p)$ near $\omega = \omega_p$, in a similar manner one finds

$$\alpha_p^j \alpha_p^k = (\mu W_{pj}^p D_{jj})(\mu W_{pk}^p D_{kk}) , \tag{2.36}$$

which is consistent with (2.35). In other words, the expected factorization of the residue is verified.

The results (2.33) and (2.35) give a formally exact representation of $f_p(x)$ in terms of $f_j^{(0)}(x)$:

$$f_p(x)=A_p\left\{f_p^{(0)}(x)+\mu\sum_{j\neq p}W_{pj}^p(\omega_p)D_{jj}(\omega_p)f_j^{(0)}(x)\right\}. \quad (2.37)$$

We have indicated explicitly that the frequency argument in W_{pj}^p and D_{jj} is to be set to $\omega=\omega_p$; this is obvious since all these results come from comparing the residue at $\omega=\omega_p$. The quantity W_{pj}^p can be evaluated to any order by truncating the series (2.24), which does not contain any large term. If an explicit representation in terms of unperturbed quantities is desired, then ω_p must be further expressed via (2.28). We show explicitly only the first nontrivial correction:

$$f_p(x)=A_p\left\{f_p^{(0)}(x)-\frac{\mu}{2}\omega_p^{(0)^2}\sum_{j\neq p}\frac{V_{pj}}{\omega_j^{(0)}(\omega_p^{(0)}-\omega_j^{(0)})}\times f_j^{(0)}(x)\right\}. \quad (2.38)$$

III. NUMERICAL EXAMPLES

As an example, let the unperturbed system be a dielectric rod:

$$\rho(x)=\begin{cases}n_0^2, & 0<x<a\\ 1, & x>a.\end{cases} \quad (3.1)$$

The QNM's of this system have been discussed in detail in I. Subject this to a perturbation $\rho(x)\to\rho(x)+\mu V(x)$, with

$$V(x)=\begin{cases}1, & 0<x<a\\ 0, & x>a.\end{cases} \quad (3.2)$$

In other words, the dielectric constant in the cavity is changed uniformly from n_0^2 to $n_0^2+\mu$. In the numerical results below, $n_0^2=4, \mu=0.9$.

The perturbed system can be solved directly. The result for $\omega_p^{(1)}$ can be obtained by expanding the explicit solution in μ; this agrees with (2.29). We have then compared the second- and third-order coefficients obtained directly from the explicit solution with the perturbative result in (2.30) and (2.31), in which the sums are truncated at $|m|\le M$. The fractional error in the real and imaginary parts of these quantities are shown against M in Fig. 1, for the mode $p=4$. It is clear that the error converges to zero as $M\to\infty$ and the perturbative series is indeed correct. The verification has been pursued to much higher accuracy than the leakage, which is of order $n_0^2-1=0$ (1) in this case. Although the QNM basis is motivated by the resemblance to the normal mode basis in the nonleaking limit, the results are emphatically *not* limited in validity to small leakage. Indeed, the zero modes [2], which have no counterparts in the nonleaking limit, are essential.

To compare the eigenfunctions, let $f_p(x)$ be the normalized exact QNM function, and $f_p^{(k)}(x)$ be the corresponding perturbative solution up to order k, as given by (2.37). In evaluating the perturbative solution, all ω_p are expanded about $\omega_p^{(0)}$, and only terms up to the indicated order are kept. The normalization of the perturbative solution is fixed by evaluating A_p^{-2} in (2.33) to the same order. Figure 2 shows the difference $\Delta f_p^{(k)}(x)=f_p(x)-f_p^{(k)}(x)$ for $p=4$ and other parameters as described above. Since the function themselves are 0(1), the remaining difference is minuscule, and constitutes a verification of the present algorithm for eigenfunctions.

We have considered several other examples numerically, including the unperturbed systems

$$\rho(x)=\begin{cases}n_0^2+M\delta(x-a), & 0<x<a\\ 1, & x>a,\end{cases} \quad (3.3)$$

where the dielectric mirror represented by M renders this

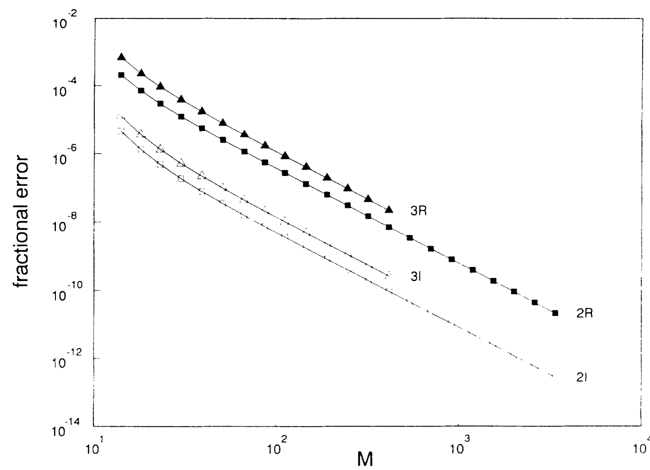

FIG. 1. The fractional error in the real and imaginary parts of the second- and third-order coefficients (denoted as $2R$, $2I$, $3R$, and $3I$) vs the maximum value $|m|=M$ at which the sums (2.27) and (2.28) are truncated. All curves for the imaginary part have been shifted down by a factor of 100 for clarity. For large M, the $2R$ and $2I$ lines overlap, as do $3R$ and $3I$.

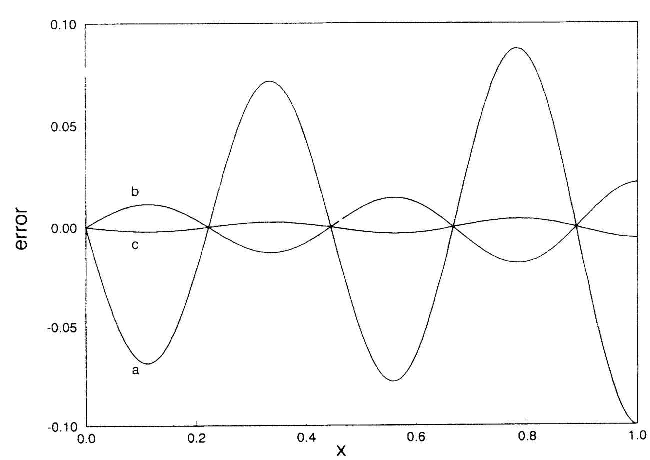

FIG. 2. The difference $\Delta f_p^{(k)}(x)=f_p(x)-f_p^{(k)}(x)$, where $f_p^{(k)}(x)$ is the perturbative solution up to kth order, and $f_p(x)$ is the exact solution, for the mode $p=4$, and (a) $k=0$, (b) $k=1$, and (c) $k=2$.

a more realistic model of a laser cavity. This is subject to the same perturbation as (3.2). Again the first-order perturbative correction to the eigenvalue has been checked analytically, while the second- and third-order coefficients have been verified to high accuracy numerically.

Another example of interest is that of a string $\rho(x)=1+M\delta(x-a)$, subject to a perturbation $\mu V(x)=\mu\delta(x-b)$, $0<b\le a$, i.e., an additional small mass is attached. The same analytic and numerical checks have been performed. Of special interest is the case $b=a$, i.e., a perturbation applied right at the edge of the cavity, in effect changing $M\rightarrow M+\mu$. The first-order perturbative correction is

$$\omega_p^{(1)}=-\frac{\omega_p^{(0)}\sin^2\omega_p^{(0)}a}{a+M\sin^2\omega_p^{(0)}a}\ , \tag{3.4}$$

which agrees with expanding the known solution in μ. The second-order perturbative correction is given by (2.30), where, on the right-hand side,

$$V_{pp}=\frac{\sin^2\omega_p^{(0)}a}{N_p}\ , \tag{3.5}$$

$$V_{pm}=V_{mp}=\frac{\sin\omega_p^{(0)}a\ \sin\omega_m^{(0)}a}{N_pN_m}\ , \tag{3.6}$$

where

$$N_p=[\tfrac{1}{2}(a+M\sin^2\omega_p^{(0)}a)]^{1/2}\ . \tag{3.7}$$

We have checked that the sum in (2.30) agrees numerically with the result of expanding the known solution to $O(\mu^2)$. In fact the convergence of the sum is very rapid. This example is of particular interest because the completeness and hence the perturbative solution applies to the inside of the cavity; this example shows that it applies all the way to the edge of the cavity.

IV. CONCLUSION

We have formulated the time-independent perturbation theory for scalar waves in a one-dimensional leaky cavity in terms of the discrete QNM's of the system. This work represents the generalization of the familiar tools of perturbation theory to such a non-Hermitian system, in principle to all orders. In particular, results are obtained for the shifts of both the real (resonance positions) and imaginary parts (resonance widths) of the frequencies of the QNM's. The formalism has been checked explicitly through examples. The validity is not limited to small leakage, and the "zero modes," which have no counterparts in the corresponding Hermitian systems, are not negligible. In these regards, the generalization to the non-Hermitian case is nontrivial.

The generalization to scalar waves in three dimensions is straightforward if the system is spherically symmetric. In the presence of such symmetry, the generalization to the Maxwell equation is also straightforward, since the electric field can be decomposed into transverse electric and transverse magnetic parts, each of which can be related to a scalar potential which satisfies an equation similar to (1.1) in the radial variable. This then already allows various applications to realistic optical systems, which will be reported elsewhere.

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