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# Completeness and orthogonality of quasinormal modes in leaky optical cavities

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It is shown that for the scalar analog of electrodynamics in one dimension, the quasinormal modes of a leaky cavity form a complete set inside the cavity, provided the cavity is defined by a discontinuity in the refractive index. This condition is sufficiently general to apply to a number of interesting examples. The quasinormal modes are also orthogonal under a modified definition of the inner product. The completeness and orthogonality hold even though the cavity is not a Hermitian system by itself. These properties allow the discrete quasinormal modes to be used as the basis for dynamics of the scalar wave in the cavity.

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## I. INTRODUCTION

An optical cavity is a region of space within which the electromagnetic (em) field is well confined—but not completely confined, if only because of the need for coupling into and out of the cavity. Because of the leakage, the "modes" of the cavity are quasinormal modes (QNM's), characterized by complex frequencies and the outgoing wave boundary condition far from the cavity. The QNM's provide an intuitively appealing description of the allowed em vibrations in the cavity, and are often used heuristically in this manner, e.g., in referring to a particular "mode" of a laser. Moreover, the QNM's are discrete, with wave numbers separated by $\Delta k \sim \pi/a$, where $a$ is the spatial dimension of the cavity. It would be attractive if em processes in the cavity could be described, in a mathematically precise way, in terms of these QNM's.

However, because the cavity by itself is not a Hermitian system, the QNM's are not normal modes, and do not form a complete orthonormal basis, at least not in the usual sense. This problem would seem to preclude their use in the precise formulation of the electrodynamics of the cavity (including second quantization), although approximations valid in the limit of small leakage are possible [1]. Instead, one has to place the cavity in a universe of size $\Lambda$ ($\Lambda \to \infty$), and impose boundary conditions at $\Lambda$ to make the universe as a whole Hermitian. The electrodynamics is then formulated in terms of the modes of the universe [2–4]. The price one pays is threefold. (a) The modes of the universe are continuous, with wave numbers separated by $\Delta k \sim \pi/\Lambda$ ($\Lambda \to \infty$). (b) It is no longer manifest that the electrodynamics of the cavity is independent of the assumptions made about the universe. (c) The intuitive connection to the zero-leakage limit is lost.

The recent surge of interest in so-called cavity QED phenomena has added urgency to this problem. Whether the atomic or molecular transitions occur in microwave cavities [5], in macroscopic [6] or mesoscopic [7] Fabry-Pérot cavities, in monolithic semiconductor heterostructures [8] or dielectric microspheres [9], in all cases the cavities are inevitably leaky and the modes of the universe are difficult to compute explicitly (with the exception of dielectric microspheres on account of symmetry [10]). There is a large class of problems in these systems involving emission *from* the cavity, and thus with wave functions satisfying the outgoing-wave boundary condition. For these problems (though of course not for problems involving radiation going *into* the cavity), it would be of great conceptual and computational advantage to be able to use the QNM's rather than the modes of the universe.

The problems do not relate only to the second quantized level of treatment. Consider a small perturbation of the cavity (for example a change of refractive index in the cavity, induced by temperature change), and the associated change $\Delta\omega$ in the complex frequency $\omega$ of a QNM. To first order in the perturbation, and using the more familiar quantum-mechanical analog, one might expect that

$$\Delta\omega = \frac{\langle \psi | \Delta H | \psi \rangle}{\langle \psi | \psi \rangle} \tag{1.1}$$

in an obvious notation. The perturbation $\Delta H$ is confined to the finite regions of space (e.g., inside the cavity), so the numerator is well defined. However, with the outgoing-wave boundary condition, the denominator would be divergent if understood in the usual sense. It has been found that this problem can be cured by adopting a modified definition of the norm [11,12], which already suggests that the family of QNM's exhibits a modified mathematical structure. The corresponding second-order perturbation formalism [13,14] is somewhat complicated precisely because one has to resort to the states of the universe.

This paper addresses these issues by showing that, subject to certain conditions and under a suitable redefinition of the inner product, the QNM's of an optical cavity are complete and orthonormal. The sufficient mathematical condition are that (a) the refractive index must have a discontinuity (physically an abrupt change over distances

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small compared to a wavelength), which is satisfied whenever there are mirrors or lens surfaces; and (b) the refractive index must approach its constant asymptotic value sufficiently rapidly. These conditions are general enough to be of practical use, and we shall spell out, as examples, the application to a well-known model of a laser cavity [2] and to dielectric microspheres [9,10]. Given this result, it becomes relatively straightforward to implement various standard calculations using the discrete basis set: time-independent and time-dependent perturbation theory, adiabatic perturbation, second quantization, etc.; these will be considerably simpler and more physical than the corresponding formulations using the modes of the universe, and will be discussed in a series of paper to follow.

The present paper will be concerned with the wave equation, i.e., the scalar analog of electromagnetism in one dimension. The extension to three dimensions is trivial when there is spherical symmetry, since each angular momentum sector is a one-dimensional radial problem to which the present result applies, as illustrated by the example in Sec. IV C.

The rest of this paper is organized as follows. Section II gives the derivation, leading to completeness and the definition of the norm and inner product. Section III shows, by a WKB analysis, that the condition necessary for the derivation is satisfied if the refractive index distribution has a discontinuity, and in fact the position of this discontinuity serves to demarcate the inside of the cavity from the outside, with completeness valid only for the inside. Section IV discusses three examples, and verifies the completeness relations numerically and, in one case, also analytically. Attention is paid to zero modes which do not correspond to any mode of the corresponding enclosed cavity. This leads, in Sec. V, to a discussion of the limitations of the usual paradigm for coupling a system to a bath via a term in a Hamiltonian. In the case of optical cavities, the coupling to the outside is through a boundary condition, which, unlike a term in the Hamiltonian, cannot be switched off. Concluding remarks are given in Sec. VI.

## II. FORMALISM

### A. Representation of the Green's function

The scalar analog of electromagnetism in one dimension is described by

$$\left[\rho(x)\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\right]\varphi(x,t)=0\ , \tag{2.1}$$

where $\rho(x)=n(x)^2$, and $n(x)$ is the refractive index distribution, here assumed to be time-independent and to approach unity rapidly for large $x$. Coupling to sources can be introduced if necessary. It is often convenient to think of $\varphi$ as the transverse vibrations of a string with linear mass density $\rho(x)$ and placed under unit tension; such string systems [especially where $\rho(x)$ contains a $\delta$ function as in (2.3) below] are well studied [15], and their analogy to optics well known [16]. We deal with a half-line $(x>0)$ with the boundary condition $\varphi(x=0,t)=0$; this describes the totally reflecting mirror at one end of a cavity, or the origin in cases where $x$ represents the radial variable in three dimensions (see Sec. IV C). The QNM's, labeled by a discrete index $j$, are defined as solutions $\varphi(x,t)=f_j(x)\exp(-i\omega_j t)$, where

$$\frac{d^2}{dx^2}f_j(x)=-\omega_j^2\rho(x)f_j(x)\ , \tag{2.2}$$

with the outgoing-wave condition at $x\to\infty$; the frequencies $\omega_j$ are necessarily complex with a negative imaginary part.

A particular example is

$$\rho(x)=1+M\delta(x-a)\ . \tag{2.3}$$

In the string language, a mass $M$ is attached to the string at $x=a$, with the string being otherwise of unit linear density. For large $M$, the cavity $0<x<a$ is well (but not completely) isolated from the rest of the universe, the point $x=a$ is nearly (but not exactly) a node, the leakage is small (but not zero), and the QNM's are close to (but not the same as) the normal modes of the corresponding enclosed cavity. In optics language, the $\delta$ function in (2.3) represents a thin slab (thickness $\ll$ wavelength) of high dielectric constant, forming a partially transmitting mirror [2]. This and other examples will be used to illustrate the general formalism.

Consider the Green's function for the time-dependent problem

$$\left[\rho(x)\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\right]G(x,y;t)=\delta(t)\delta(x-y) \tag{2.4}$$

representing the effect at $(x,t)$ of an impulse delivered at $(y,t'=0)$. The retarded Green's function is defined by (2.4) together with the initial condition $G(x,y;t)=0$ for $t<0$, so that the Fourier transform $\tilde{G}(x,y;\omega)$ is analytic for $\mathrm{Im}\,\omega>0$, and behaves as $\exp(i\omega x)$ for large $x$. For general $x$ and $y$, $\tilde{G}$ satisfies

$$\Delta(\omega)\tilde{G}(x,y;\omega)\equiv\left[\frac{\partial^2}{\partial x^2}+\omega^2\rho(x)\right]\tilde{G}(x,y;\omega)=-\delta(x-y)\ . \tag{2.5}$$

The strategy is to consider the inverse transform for $G$, and (a) attempt to close the contour in the lower half-plane, and (b) show that the residues correspond to QNM's. To this end, first introduce two auxiliary functions $f$ and $g$, defined as solutions to the homogeneous time-independent equation $\Delta(\omega)f(\omega,x)=\Delta(\omega)g(\omega,x)=0$, with $f(\omega,x=0)=0$ and $g(\omega,x)=\exp(i\omega x)$ for $x\to\infty$. The Wronskian $W(\omega)=g(\omega,x)f'(\omega,x)-f(\omega,x)g'(\omega,x)$ (where $'=d/dx$) is independent of $x$. In terms of these,

$$\tilde{G}(x,y;\omega)=\begin{cases}f(\omega,x)g(\omega,y)/W(\omega), & 0<x<y\\ g(\omega,x)f(\omega,y)/W(\omega), & 0<y<x\ .\end{cases} \tag{2.6}$$

The normalization of $f$ has been left arbitrary, but (2.6) is independent of that normalization.

At a QNM frequency $\omega=\omega_j$, the wave function can satisfy *both* the regularity condition at $x=0$ and the outgoing-wave condition at $x\to\infty$, so that $f(\omega_j,x)\propto g(\omega_j,x)$ and $W(\omega_j)=0$. The residues of $\tilde{G}$ at these poles are given by

$$R_j=f(\omega_j,x)g(\omega_j,y)\Big/\left.\frac{\partial W(\omega)}{\partial\omega}\right|_{\omega_j} . \tag{2.7}$$

We shall show, in Sec. II C, that

$$\left.\frac{\partial W(\omega)}{\partial\omega}\right|_{\omega_j}=-2\omega_j\int_0^\infty dx\,\rho(x)f(\omega_j,x)g(\omega_j,x) \tag{2.8}$$

along a suitably chosen contour [17]. Hence, using the proportionality between $f$ and $g$ at $\omega=\omega_j$, and adopting the notation $f_j(x)=f(\omega_j,x)$,

$$R_j=-\frac{f_j(x)f_j(y)}{2\omega_j\langle\langle f_j|f_j\rangle\rangle} , \tag{2.9}$$

where we have introduced the generalized norm

$$\langle\langle f_j|f_j\rangle\rangle=\int_0^\infty dx\,\rho(x)f_j(x)^2 \tag{2.10}$$

along the same contour, which is chosen so that the integral is well defined. Note that $f_j$ is complex, and that $f_j^2$ rather than $|f_j|^2$ appears, so that the generalized norm is complex. Other properties of this norm are discussed in Sec. II C.

The auxiliary function $f(\omega,x)$ is obtained by integrating the defining equation over a finite distance, from 0 to $x$, and is hence analytic in $\omega$. If $\rho(x)=1$ for $x>b$, then likewise $g(\omega,x)$ is obtained by integrating the defining equation over a finite distance, from $b$ to $x$, and is likewise analytic in $\omega$. The latter condition remains valid if $\rho(x)\to 1$ as $x\to\infty$ sufficiently rapidly [18]. When this condition is satisfied, the zeros of $W(\omega)$, with residue given by (2.9), are the only singularities of $\tilde{G}(x,y;\omega)$.

The condition on $g(\omega,x)$ is sufficient, but may not be necessary. If $g(\omega,x)$ has a factorizable singularity, i.e., $g(\omega,x)=h(\omega)\bar{g}(\omega,x)$, with $\bar{g}$ being analytic in $\omega$, then the singular factor $h(\omega)$ would cancel a similar factor in $W(\omega)$, thus not affecting $\tilde{G}$.

### B. Completeness

Invert the transform for $G$, and close the contour for $t>0$ by a large semicircle in the lower half of the $\omega$ plane. Assume that the contribution from the semicircle is negligible; the conditions necessary for the validity of this assumption will be discussed in Sec. III. Then we obtain

$$G(x,y;t)=\frac{i}{2}\sum_j\frac{f_j(x)f_j(y)}{\omega_j\langle\langle f_j|f_j\rangle\rangle}e^{-i\omega_j t} . \tag{2.11}$$

The defining equation (2.4) for the Green's function and the causal initial condition imply

$$\rho(x)\frac{\partial G}{\partial t}(x,y;t=0^+)=\delta(x-y) . \tag{2.12}$$

Upon combining (2.11) and (2.12), one obtains

$$\frac{1}{2}\sum_j\rho(x)\frac{f_j(x)f_j(y)}{\langle\langle f_j|f_j\rangle\rangle}=\delta(x-y) . \tag{2.13}$$

Because the wave equation is second order in time, the QNM's occur in pairs $[\omega_j,f_j(x)],[-\omega_j^*,f_j^*(x)]$, so (2.13) can also be written as

$$\mathrm{Re}\sum_{\mathrm{Re}\omega_j>0}\rho(x)\frac{f_j(x)f_j(y)}{\langle\langle f_j|f_j\rangle\rangle}=\delta(x-y) , \tag{2.14}$$

where it is assumed that there is no QNM with $\mathrm{Re}\omega_j=0$. (This is not always the case; see Sec. IV C.) However, it will be seen in Sec. III that the condition for neglecting the contribution from the large semicircle in the lower half-plane is valid only for certain $x$ and $y$; roughly speaking, it will be sufficient if both $x$ and $y$ are both inside the cavity. Thus the completeness relations (2.13) or (2.14) will likewise be restricted to such values of $x$ and $y$.

In the limit where the leakage is zero, the eigenfunctions $f_j(x)$ would be real, with a node at the boundary of the cavity, and $\langle\langle f_j|f_j\rangle\rangle$ would reduce to the usual definition of the norm. Then (2.14) reduces to the familiar statement of completeness for a Hermitian system, for which the present work constitutes a generalization. In analogy to the well-known Hermitian case, one expects (2.13) or (2.14) to hold only in a distribution sense.

One may refer to (2.13) or (2.14) as the weak form of completeness, allowing an eigenfunction expansion. In contrast, the strong form of completeness (2.11) allows the eigenfunction expansion to be used for dynamical evolution.

### C. Definition of norm and inner product

To prove (2.8), start with the defining equation for $f(\omega_j,x)$ and $g(\omega,x)$, where $\omega\simeq\omega_j$. The usual manipulations lead to

$$(\omega^2-\omega_j^2)\int_0^R dx\,\rho(x)f(\omega_j,x)g(\omega,x)$$
$$=g(\omega,x)f'(\omega_j,x)-f(\omega_j,x)g'(\omega,x)|_0^R , \tag{2.15}$$

where the integral is taken along any contour from $x=0$ to $R$. We assume that $R$ is large enough so that $\rho(x=R)=1$, and since both $f(\omega_j,x)$ and $g(\omega,x)$ are outgoing waves, the right-hand side of (2.15) becomes

$$i(\omega_j-\omega)f(\omega_j,R)g(\omega,R)$$
$$-[g(\omega,0)f'(\omega_j,0)-f(\omega_j,0)g'(\omega,0)] . \tag{2.16}$$

Differentiating (2.15) with respect to $\omega$ and taking $\omega\to\omega_j$ then gives

$$2\omega_j\int_0^R dx\,\rho(x)f(\omega_j,x)g(\omega_j,x)$$
$$=-if(\omega_j,R)g(\omega_j,R)$$
$$-\left[\frac{\partial}{\partial\omega}(gf'-fg')-\left[g\frac{\partial f'}{\partial\omega}-\frac{\partial f}{\partial\omega}g'\right]\right]_{\substack{x=0\\ \omega=\omega_j}} . \tag{2.17}$$

But $f(\omega,x=0)=0$ for all $\omega$, so $\partial f(\omega,x=0)/\partial\omega=0$; moreover, at a QNM, $g(\omega_j,x=0)\propto f(\omega_j,x=0)=0$. Hence the last two terms in (2.17) vanish, while

$gf'-fg'=W$. So we get

$$\int_0^R dx\,\rho(x)f(\omega_j,x)g(\omega_j,x)+\frac{i}{2\omega_j}f(\omega_j,R)g(\omega_j,R)=-\frac{1}{2\omega_j}\frac{\partial W}{\partial\omega}(\omega_j)\ . \qquad (2.18)$$

Now

$$f(\omega_j,R)g(\omega_j,R)\propto\exp(2i\omega_jR)=\exp[2i|\omega_jR|e^{i(\alpha_j+\beta)}],$$

where $\alpha_j=\arg\omega_j$ and $\beta=\arg R$. Assume $\mathrm{Re}\,\omega_j>0$ (the modifications necessary for the opposite case are straightforward), then $0>\alpha_j>-(\pi/2)$. By choosing $\beta=-\alpha_j+\varepsilon$ ($\varepsilon>0$), $e^{i(\alpha_j+\beta)}$ has a positive imaginary part and $f(\omega_j,R)g(\omega_j,R)$ vanishes as $R\to\infty$. In other words, by choosing the contour shown in Fig. 1 for a suitable $\beta$, the surface term in (2.18) is eliminated and we get (2.8) and consequently the definition of the norm in (2.10).

Alternatively, take the integral along the real axis, then the surface term at $x=R$ cannot be eliminated, but all the results in Sec. II A and II B remain valid provided the norm is now taken to be

$$\langle\langle f_j|f_j\rangle\rangle=\int_0^R dx\,\rho(x)f_j(x)^2+\frac{i}{2\omega_j}f_j(R)^2\ , \qquad (2.19)$$

which is readily seen to be independent of $R$. This latter definition avoids analytic continuation in the $x$ plane, and may be more convenient for numerical integration. The norm (2.19) was introduced by us previously [12–14], and its use in (1.1) gives the correct first-order shift in frequency upon a perturbation.

A third definition [11], again equivalent, is to take

$$\langle\langle f_j|f_j\rangle\rangle=\lim_{\varepsilon\to0+}\int_0^\infty dx\,e^{-\varepsilon x^2}\rho(x)f_j(x)^2\ . \qquad (2.20)$$

The norm is readily generalized to an inner product. Let $\varphi(x)$ and $\psi(x)$ be two functions that can be represented as a sum of QNM's. Then define

$$\langle\langle\varphi|\psi\rangle\rangle=\int_0^\infty dx\,\rho(x)\varphi(x)\psi(x) \qquad (2.21)$$

along the contour in Fig. 2, where $\arg x\to\pi$ as $|x|\to\infty$. Note that this is a special case of the contour in Fig. 1. If we write $\varphi$ and $\psi$ in terms of QNM's, a typical term would go as

$$\varphi\psi\sim e^{i\omega_jx}e^{i\omega_kx}$$

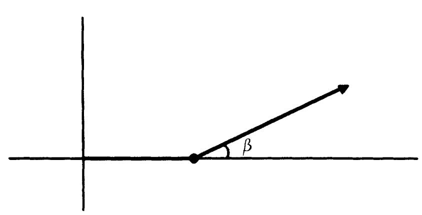


FIG. 1. Contour in the $x$ plane for proving orthogonality. The integral goes along the real axis to $x\gg a$, and then goes along a line making angle $\beta$ with the real axis.

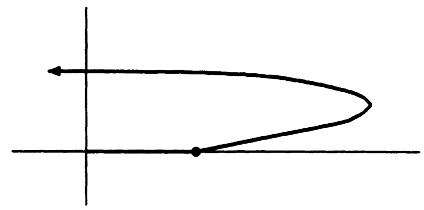

FIG. 2. Contour in the $x$ plane for defining inner products. The line has an asymptotic argument $\pi$, and may be regarded as a special case of Fig. 1. The choice of contour is valid for all pairs of QNM's.

as $|x|\to\infty$. But since $\omega_j+\omega_k$ lies in the lower half-plane, and $\arg x\simeq\pi$, then the exponent has a negative real part as $|x|\to\infty$, and the integrand vanishes at the upper limit. Then by manipulations similar to (2.15), it is readily shown that

$$\langle\langle f_j|f_k\rangle\rangle=0\quad\text{if }\omega_j\neq\omega_k\ , \qquad (2.22)$$

so the basis $\{f_j\}$ is not only complete but also orthogonal. It is likewise possible to define inner products by taking (2.21) along the real axis, but with a regulating factor as in (2.20) [19].

The completeness relation is restricted to the inside of the cavity, whereas the orthogonality relation involves the outside. If one considers the inside of the cavity alone, say $0<x<a$, then

$$\int_0^a dx\,\rho(x)f_j(x)f_k(x)\neq\delta_{jk}\ , \qquad (2.23)$$

and $\{f_j\}$ is overcomplete for the inside of the cavity above, i.e., the representation

$$\varphi(x)=\sum_j b_jf_j(x),\quad 0<x<a \qquad (2.24)$$

is not unique. This makes it impossible to deal with initial value problems for the cavity using QNM's, which is not surprising since a knowledge of the initial condition for $0<x<a$ alone would not lead to a unique solution. This is, however, not a problem for other applications, as will be illustrated elsewhere. Of course the coefficients $b_j$ would be unique if (2.27) were to hold for all $x$.

For many applications (e.g., time-independent perturbation theory; see the following paper), the validity of the QNM representation (2.24) for $x<a$ is already adequate, and the fact that each $f_j(x)$ becomes infinitely strong as $x\to\infty$ presents no conceptual or computational problems. Indeed, if all the fields are emitted by sources in the cavity, and switched on at $t=0$, then $\varphi(x,t)=0$ outside the causal domain $x<t$, simply because of the finite speed of propagation. Therefore one would not attempt to use the QNM representation outside the causal domain. A more thorough discussion of this issue will be given in the context of time-dependent dynamical evolution.

## III. CONDITION FOR COMPLETENESS

The condition for the completeness of the QNM's is that $\tilde{G}(x,y;\omega)$ must vanish as $|\omega|\to\infty$ in the lower half-plane. The behavior for large $|\omega|$ should be given by the WKB approximation. Define

$$I(u,v)=\int_u^v dx\, n(x)\ , \tag{3.1}$$

which is positive for $v>u$. In this paper we consider the case where $\rho(x)=n(x)^2$ has a discontinuity at some $x=a$. Then one should write WKB approximations for $0<x<a$ and $a<x<\infty$ separately, and join the two solutions using the reflection coefficient $R$:

$$R=\frac{n(a^-)-n(a^+)}{n(a^-)+n(a^+)}\ . \tag{3.2}$$

Then, for $0<y\le x<a$,

$$\tilde{G}(x,y;\omega)\simeq\frac{\sin[\omega I(0,y)][e^{-i\omega I(x,a)}+Re^{i\omega I(x,a)}]}{\omega\sqrt{n(x)n(y)}[e^{-i\omega I(0,a)}+Re^{i\omega I(0,a)}]}\ . \tag{3.3}$$

For $\omega=\omega_R+i\omega_I\to\infty$ with $\omega_I<0$, both the numerator and denominator are dominated by the term proportional to $R$, and

$$\tilde{G}(x,y;\omega)\simeq\frac{e^{|\omega_I|I(0,y)}e^{|\omega_I|I(x,a)}}{\omega e^{|\omega_I|I(0,a)}}\ . \tag{3.4}$$

Since $I(0,y)+I(x,a)\le I(0,a)$ for $y\le x$, this vanishes when $|\omega|\to\infty$ in the lower half-plane. As a result, QNM's are complete inside the cavity. In fact, if $n(x)$ is continuous but has a discontinuity in its $p$th derivative at $x=a$, then $R\propto\omega^{-p}$. The term proportional to $R$ still dominates, and the same result is obtained. Thus an extremely "soft" discontinuity is sufficient for the validity of our result. For example, if $n(x)=1$ for $x>a$ and $n(x)-1\propto(1-x/a)^p$ for $x<a$, similar to the potential considered in Ref. [20], the proof will go through.

For $x,y>a$, $G(x,y;\omega)$ does not vanish as $|\omega|\to\infty$ in the lower half-plane, and consequently the QNM's are *not* complete. Since the region outside the cavity has spatial extension $\Lambda(\Lambda\to\infty)$, one does not expect discrete modes with $\Delta k\sim\pi/a$ to be complete.

If $\rho(x)$ does not have a discontinuity (in derivatives of any order) on the real axis, $R$ vanishes for $|\omega|\to\infty$ faster than any power, but it is difficult to estimate how small it becomes, and consequently whether the $R$-dependent terms in (3.3) dominate or not. Note that $R$ can vanish very rapidly, e.g., as $\exp(-\text{const}|\omega|^\sigma)$, $\sigma<1$, and still dominate in (3.3). We leave for future investigation the trickier problem of whether QNM's can still be complete for *some* class of analytic $\rho(x)$, but in a sense the requirement that there has to be a discontinuity in $\rho(x)$ is not surprising. Because the QNM's are discrete, the best that one can hope for is that they are complete "inside the cavity;" however, without a discontinuity, there would be no natural demarcation between the "inside" and the "outside." Fortunately many examples of practical importance, such as those in Sec. IV, do have such a discontinuity.

In the discussion so far, the "inside" is bounded by $x=0$ on one side, and the discontinuity at $x=a$ on the other. More generally, if we consider the wave equation on the full line $-\infty<x<\infty$, and assume that $\rho(x)$ has discontinuities at $a_1,a_2,\ldots,a_N$ (arranged in ascending order), then it can be shown (Appendix B) that completeness holds between the outermost discontinuities, i.e., (2.11) and (2.13) are valid for $a_1<x,y<a_N$.

## IV. EXAMPLES

### A. Model of a laser cavity

A well-studied model of a one-dimensional cavity [2] is given by (2.3). The auxiliary functions are

$$f(\omega,x)=\sin\omega x,\quad x<a \tag{4.1a}$$

$$g(\omega,x)=\begin{cases}e^{i\omega x}, & x>a\\ \alpha\sin\omega x+\beta\cos\omega x, & x<a\ ,\end{cases} \tag{4.1b}$$

where

$$\alpha=i+M\omega\cos\omega a e^{i\omega a}\ , \tag{4.2a}$$

$$\beta=1-M\omega\sin\omega a e^{i\omega a}\ , \tag{4.2b}$$

and $W(\omega)=\omega\beta$. The QNM frequencies are given by the zeros of $\beta$. [The zero of $W(\omega)$ at $\omega=0$ cancels the zero in $f(\omega,x)$ in the numerator in (2.6), and does not contribute to the contour integral.] The QNM's belong to two families which, for small $a/M$, can be written as

$$\omega_j a=j\pi+\frac{1}{j\pi}\left[\frac{a}{M}\right]+\left[-\frac{1}{(j\pi)^3}-\frac{i}{(j\pi)^2}\right]\left[\frac{a}{M}\right]^2+\cdots,\quad j=\pm1,\pm2,\ldots \tag{4.3a}$$

and

$$\omega a=\pm\left[\frac{a}{M}\right]^{1/2}-\frac{i}{2}\left[\frac{a}{M}\right]\mp\frac{7}{24}\left[\frac{a}{M}\right]^{3/2}+\cdots\ . \tag{4.3b}$$

The latter pair will be referred to as the $j=0$ mode, or zero mode for short, and their significance will be discussed in Sec. V. These modes, though not their completeness, have been thoroughly investigated in the literature [21].

The completeness relation then reads, for $x,y<a$,

$$\tfrac{1}{2}\sum_j\frac{\sin\omega_j x\sin\omega_j y}{\langle\langle f_j|f_j\rangle\rangle}=\delta(x-y)\ , \tag{4.4}$$

where the zero mode is understood to be included in the sum, and where the norm is, by (2.19),

$$\langle\langle f_j|f_j\rangle\rangle=\frac{a}{2}\left\{1+\frac{(a/M)}{\omega_j a[\omega_j a+2i(a/M)]}\right\}\ . \tag{4.5}$$

To check (4.4) in a distribution sense, let $S_J(x,y)$ be the partial sum of (4.4) up to $|j|\le J$, and

$$I_J(x,y,\Delta)=\int_{y-\Delta}^{y+\Delta}S_J(x,y')dy' =\sum_{|j|\le J}\frac{\sin\omega_j\Delta}{\omega_j}\frac{1}{\langle\langle f_j|f_j\rangle\rangle}\sin\omega_j x\sin\omega_j y\ . \tag{4.6}$$

Figure 3(a) shows $|I_J|$ vs $J$ for several cases where $x\notin(y-\Delta,y+\Delta)$, and Figure 3(b) shows $|I_J-1|$ vs $J$ for several cases where $x \in (y-\Delta, y+\Delta)$, in all cases evaluated for a relatively large value of $a/M = 0.5$. (These quantities are fluctuating functions of $J$, and Fig. 3 shows the smooth envelope which forms an upper bound.) It is clear that (4.4) holds (in a distribution sense) exactly, and not just to leading order in $a/M$, which is a measure of the amount of leakage. Incidentally the completeness does not hold without the zero mode.

### B. Model of a dielectric rod

A second and even simpler model is a one-dimensional dielectric rod of index $n_0 > 1$, i.e.,

$$\rho(x) = \begin{cases} n_0^2, & x < a \\ 1, & x > a \ . \end{cases} \tag{4.7}$$

Exactly this system has also been discussed as a much simplified model of gravitational radiation from stellar objects [22]. The completeness of the QNM's has been noticed [22], and the generality beyond the specific example has been conjectured, though the connection with a spatial discontinuity has not hitherto been emphasized. We sketch below the verification of completeness, using the language of this paper. The auxiliary functions are

$$f(\omega,x) = \sin\omega n_0 x, \quad x < a \ , \tag{4.8a}$$

$$g(\omega,x) = \begin{cases} e^{i\omega x}, & x > a \\ \alpha \sin\omega n_0 x + \beta \cos\omega n_0 x, & x < a \ , \end{cases} \tag{4.8b}$$

where

$$\alpha = e^{i\omega a}\left[\sin\omega n_0 a + \frac{i}{n_0}\cos\omega n_0 a\right] , \tag{4.9a}$$

$$\beta = e^{i\omega a}\left[\cos\omega n_0 a - \frac{i}{n_0}\sin\omega n_0 a\right] , \tag{4.9b}$$

and the Wronskian is $W(\omega) = \omega n_0 \beta$. Again the zero at $\omega = 0$ does not contribute. The poles are at

$$\omega_j n_0 a = j\pi + (\pi/2 - i\xi), \quad j = 0, \pm 1, \pm 2, \ldots , \tag{4.10}$$

where $\xi = (\frac{1}{2})\ln[(n_0+1)/(n_0-1)]$. Then

$$f_j(x) = \cos\left[\left[\frac{\pi}{2} - i\xi\right]\frac{x}{a}\right]\sin j\pi\frac{x}{a} + \sin\left[\left[\frac{\pi}{2} - i\xi\right]\frac{x}{a}\right]\cos j\pi\frac{x}{a} , \tag{4.11}$$

and $\langle\langle f_j | f_j \rangle\rangle = n_0^2 a/2$. Apart from the $\xi$-dependent factors in (4.11), which are independent of $j$, $\{f_j\}$ is just a standard trigonometric series, and (2.13) can be verified analytically. Again, equality holds only in a distribution sense.

This example is interesting not only because the completeness relation can be verified analytically, but also because the leakage is manifestly not small, thus demonstrating that the present formalism is accurate to all orders in the leakage rate.

We have also checked the completeness relation for the "triangular" function

$$\rho(x) = \begin{cases} x/a, & a > x > 0 \\ 1, & x > a \ , \end{cases}$$

which exhibits a discontinuity only in the derivative.

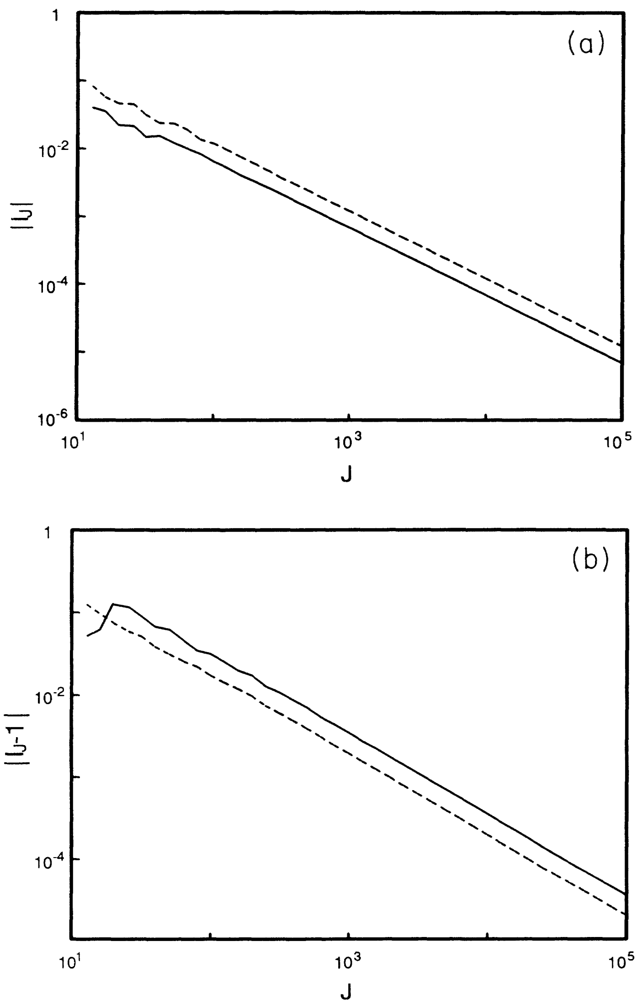


FIG. 3. Numerical data verifying the completeness relation for the model of Sec. IV A with $a/M = 0.5$ (a) Smooth envelope forming upper bound of $|I_J|$ vs $J$ for (i) $x = 0.5a$, $y = 0.2a$, and $\Delta = 0.1a$ (solid line); and (ii) $x = 0.5a$, $y = 0.7a$, and $\Delta = 0.1a$ (broken line). (b) Smooth envelope forming upper bound of $|I_J - 1|$ vs $J$ for (i) $x = 0.3a$, $y = 0.25a$, and $\Delta = 0.1a$ (solid line); and (ii) $x = 0.7a$, $y = 0.8a$, and $\Delta = 0.2a$ (broken line).

### C. Dielectric microsphere

There have been many experimental investigations of the interaction of the em field with dielectric microspheres, starting with Mie scattering [23], fluorescence, and cavity QED effects [9], to nonlinear processes such as stimulated Brillouin scattering [24], stimulated Raman scattering [25], lasing [26], and chemical-energy transfer [27], and the structure of the QNM's (which in this context are often referred to as morphology-dependent resonances) are well studied [10,28,29]. The scalar analog (which is strictly applicable to TE modes) is described by

$$\left[\rho(r)\frac{\partial^2}{\partial t^2}-\nabla^2\right]\Phi(\mathbf{r},t)=0 , \tag{4.12}$$

where $\rho(r)=n(r)^2$ is assumed to be spherically symmetric, and $n(r)=1$ for $r>a$. Spherical symmetry allows each angular momentum sector to be treated separately, so writing [30]

$$\Phi(\mathbf{r},t)=[\varphi(r,t)/r]Y_{lm}(\theta,\phi) \tag{4.13}$$

gives

$$\left[\rho(r)\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial r^2}+\frac{l(l+1)}{r^2}\right]\varphi(r,t)=0 . \tag{4.14}$$

Apart from the replacement $\partial^2/\partial x^2\to\partial^2/\partial r^2-l(l+1)/r^2$, the derivation for the one-dimensional case carriers over. Henceforth we specialize in the case of a uniform microsphere, $n(r)=n_0$ for $r<a$. The auxiliary functions are

$$f(\omega,r)=rj_l(n_0\omega r) , \tag{4.15a}$$

$$g(\omega,r)=\begin{cases} rh_l^{(1)}(\omega r), & r>a \\ r[\alpha j_l(n_0\omega r)+\beta n_l(n_0\omega r)], & r<a , \end{cases} \tag{4.15b}$$

where $j_l$, $n_l$, and $h_l^{(1)}$ are the spherical Bessel, Neumann, and Hankel functions. The coefficients are

$$\alpha=-(n_0\omega a)^2\left[\frac{1}{n_0}n_l h_l^{(1)\prime}-h_l^{(1)}n_l'\right] , \tag{4.16a}$$

$$\beta=(n_0\omega a)^2\left[\frac{1}{n_0}j_l h_l^{(1)\prime}-h_l^{(1)}j_l'\right] , \tag{4.16b}$$

where $j_l$, $n_l$, $j_l'$, and $n_l'$ are evaluated at $n_0\omega a$, and $h_l^{(1)}$ and $h_l^{(1)\prime}$ are evaluated at $\omega a$. The Wronskian is $W(\omega,r)=\beta/n_0\omega$. QNM frequencies $\omega_j$ are found from the roots of $W(\omega)$. The $l=0$ case is exactly the same as the example of the one-dimensional dielectric rod dealt with in Sec. IV B. Figure 4 shows the position of the poles for a microsphere of refractive index $n_0=1.33$, for the angular momentum sectors $l=9$ and 10. The general features are well known [28]. For each $l$, there is one series of poles with $(\text{Re}\omega)a\lesssim l$ and relatively large $|\text{Im}\omega|$. For odd $l$, this series starts at $\text{Re}\omega=0$, while for even $l$ the series starts at $(\text{Re}\omega)a\simeq 1$. These broader QNM's are similar to the zero modes discussed earlier. There is in addition a second series of poles with small imaginary parts, with $(\text{Re}\omega)a\gtrsim l$; these are experimentally more prominent, and their positions are well described by simple asymptotic formulas based on geometric optics [29], and which can be understood physically as follows. Consider a photon of frequency $\omega$, hence momentum in the microsphere $n_0\omega$, striking the microsphere surface at $r=a$ at an angle of incidence $\theta$ (Fig. 5), so that the angular momentum is $l=n_0\omega a\sin\theta$. The low-order QNM's of the narrow series have $n_0\omega a\simeq l$, i.e., $\sin\theta\approx 1$; in other words, the photon strikes the microsphere surface at glancing incidence and suffers total internal reflection. One therefore expects the leakage (i.e., $\text{Im}\omega$) to be small, due only to diffractive effects (i.e., violations of geometric optics) and it is indeed found that theoretical quality factors $Q\propto\text{Re}\omega/|\text{Im}\omega|$ as high as $10^{20}$ or more is possible for $2\pi a/\lambda\gtrsim 500$ [10]. In contrast, the high-order modes ($n_0\omega a\gg l,\sin\theta\ll 1$) correspond to rays propagating in a nearly radial direction, and the situation would be very similar to the example of a dielectric rod in Sec. IV B, and in fact $\text{Im}\omega a\approx(-1/2n_0)\ln[(n_0+1)/(n_0-1)]$.

For this system, norms and inner products are defined by, e.g.,

$$\langle\langle f|g\rangle\rangle=\int_0^\infty dr\, f(\omega,r)g(\omega,r) , \tag{4.17}$$

in which it is understood that the contour is chosen as in Fig. 2 to eliminate the contribution from the upper limit, or a suitable surface term is added in the manner of (2.19). With reference to (4.15), it is sometimes more convenient to write (4.17) as

$$\langle\langle f|g\rangle\rangle=\int_0^\infty dr\, r^2\tilde{f}(\omega,r)\tilde{g}(\omega,r) , \tag{4.18}$$

where $\tilde{f}=f/r,\tilde{g}=g/r$ are given by spherical Bessel, Neumann, or Hankel functions directly. It can be shown that for a QNM given by (4.15a) with $\omega=\omega_j$, the norm is (Appendix A)

$$\langle\langle f_j|f_j\rangle\rangle=(n_0^2-1)(a^3/2)j_l^2(n_0\omega_j a) , \tag{4.19}$$

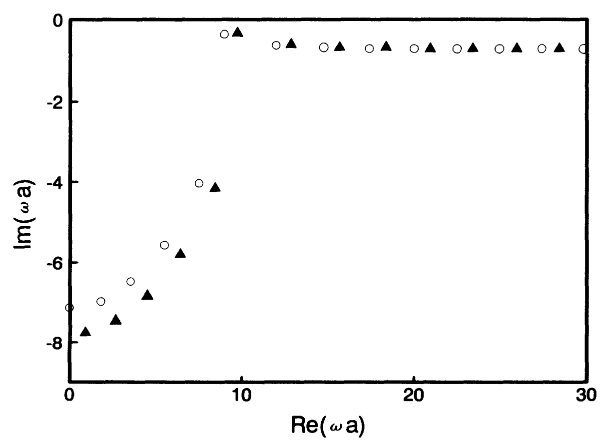


FIG. 4. Schematic of the pole positions in the complex $\omega$ plane for the model of Sec. IV C with $n_0=1.33$, for $l=9$ (circles) and 10 (triangles).

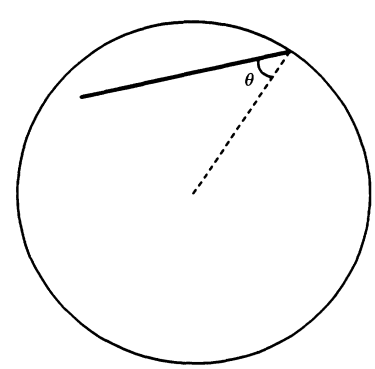


FIG. 5. Geometric optics interpretation of different types of QNM's; low-order QNM's correspond to $\theta\approx\pi/2$, while high-order QNM's correspond to $\theta\approx 0$.

so the completeness relation reads

$$\frac{n_0^2 r^2}{(n_0^2-1)a^3}\sum_j \frac{j_l(n_0\omega_j r) j_l(n_0\omega_j r')}{j_l^2(n_0\omega_j a)} = \delta(r-r') \tag{4.20}$$

for $0<r,r'<a$ and for every integer $l$. This has been verified numerically. This completeness relation is interesting in that the sum includes two series of modes, and the second series contains both very well-confined low-order modes ($Q \gg 1$) and very poorly confined high-order modes [$Q = 0(1)$].

Each partial wave of the corresponding TM modes for the em problem can likewise be described by a scalar field depending on the radial variable only, and completeness is likewise proved.

These results then prove the completeness of QNM's in three dimensions, both for scalar waves and em waves, for systems exhibiting spherical symmetry.

## V. PARADIGM FOR DISSIPATIVE QUANTUM SYSTEMS

Optical cavities are dissipative quantum system—quantum in that one needs to refer to single photons or nonclassically correlated photon states, and dissipative in that energy and probability are not conserved for the cavity alone. In recent years a paradigm has been developed for dealing with dissipative quantum systems [31]. One considers a system $S$ with coordinates and momenta $(Q_i, P_i)$ and a bath with coordinates and momenta $(q_i, p_i)$, described respectively by Hamiltonians $H_S(Q_i,P_i)$ and $H_B(q_i,p_i)$ and coupled by a term $\lambda V(Q_i,q_i)$, i.e., $H = H_S + H_B + \lambda V$. If $\lambda = 0$, the Hilbert space is $\Omega = \Omega_S \times \Omega_B$ in an obvious notation, and it is implicitly assumed that $\Omega$ remains the same even when $\lambda V$ is switched on. In other words, the coupling does not introduce any qualitative change, and a perturbative approach is possible, so each QNM can be regarded as a normal mode of the uncoupled system given a small width.

However, the situation is fundamentally different for open systems such as optical cavities. Here the system is described by $\{\varphi(x,t):0<x<a\}$ and the bath by $\{\varphi(x,t):a<x<\infty\}$; the two are not coupled by a term in the Hamiltonian, but by boundary conditions, e.g., $\varphi(x=a^-,t)=\varphi(x=a^+,t)$. The coupling cannot be switched off, so that the effect of the bath cannot be handled perturbatively, at least not in an obvious way. One way to see this is that $\Omega \neq \Omega_S \times \Omega_B$. This is most evident in the example of Sec. IV A. If $M = \infty$, the system and bath are decoupled, and $\Omega_S = \{\varphi(x):0<x<a, \varphi(a^-)=0\}$, and $\Omega_B = \{\varphi(x):a<x<\infty, \varphi(a^+)=0\}$; on the other hand, the Hilbert space for $M<\infty$ is $\Omega = \{\varphi(x):0<x<\infty\}$ without the nodal condition at $x=a$. An extra dynamical degree of freedom, viz. $\varphi(a,t)$, emerges when the coupling is nonzero. The zero mode in (4.3b), which does not correspond to any modes of the closed cavity (i.e., the $M=\infty$ limit), is a manifestation of the extra degree of freedom. This mode will be missed if the QNM's are modeled by coupling the normal modes of a cavity to the bath via a term in the Hamiltonian [1], as in the usual paradigm [31]. The complete QNM basis precisely provides a means of dealing with this issue.

Nevertheless, in both cases, the totality of system plus bath is conservative. The whole idea of dissipative quantum systems is to attempt to discard the coordinates of the bath, and keep the coordinates of system, which now looks dissipative. In examples of the first type, given by $H = H_S + H_B + \lambda V$, this is best done by integrating out $\{q_i, p_i\}$ in a path integral; in examples of the second type, with coupling by a boundary condition, it appears that the use of QNM's (at least in cases where they are complete) is a convenient way of describing the system without having to mention the coordinates of the bath, viz. $\varphi(x,t)$ for $x>a$.

In referring to the degrees of freedom other than the system itself as the bath, one is primarily concerned with situations in which the bath acts simply as a sink of energy; this is closely tied to the outgoing-wave boundary condition which defines the QNM's. For situations with energy coming into the system, such a description would be inadequate, and one does not in any case expect to be able to eliminate the coordinates outside the system. Nevertheless there are many situations where the ingress of energy (e.g., a short external pulse that pumps an optical system) and the subsequent decay can be thought of as two distinct steps occupying different time domains; in these cases, the QNM's would still provide a useful description for the second stage.

## VI. CONCLUSION

In summary, we have shown that for a leaky cavity (in one dimension, and with the scalar wave analog for em fields) whose spatial extent is defined by a discontinuity (in derivative of any order) of the dielectric constant, the QNM's form a complete basis, which is moreover orthogonal under a suitable definition of the inner product—even though the cavity by itself is not a Hermitian system. While the present paper sets up only the mathematical formalism, this discrete basis will be useful for discussing a variety of physical problems in the electrodynamics of such a cavity, and these applications to concrete phenomena will be taken up in papers to follow.

In addition to the completeness relation, the retarded Green's function in the time domain is expressible in terms of QNM's via (2.11). This provides a powerful tool for the analysis of time-dependent problems.

It will be natural to attempt to generalize the conclusions of this paper to the quantum-mechanical wave equation. For the Schrödinger equation with a potential $V(x)$ which has a discontinuity at some $x=a$, and which vanishes sufficiently rapidly at infinity, completeness in the weak sense analogous to (2.13) can be proved, but not completeness in the strong sense analogous to (2.11). This property is related to the fact that the dispersion relation $k=\sqrt{2\omega}$ implies a cut in the $\omega$ plane. For the Klein-Gordon equation with the mass supplemented by a potential

$$\left[-\frac{\partial^2}{\partial t^2} + \nabla^2 - m^2 - V(x)\right]\varphi(x,t) = 0 \tag{6.1}$$

the situation depends on $m^2$ [where $V(\infty)=0$ by convention]. For $m=0$, the structure is like the wave equation, and completeness holds in both the weak and strong forms provided $V(x)$ has a spatial discontinuity and vanishes sufficiently rapidly at infinity. For $m\neq 0$, the dispersion relation $k=\sqrt{\omega^2-m^2}$ between the frequency $\omega$ and the asymptotic wave number $k$ again leads to a cut, and, similar to the case of the Schrödinger equation, completeness holds only in the weak sense. These results complement some existing works [20,32] which are restricted to potentials of a finite range, and will be reported elsewhere. The results for the Klein-Gordon case has relevance to black holes and relativistic stars coupled to gravitational radiation [33,34].

Finally, the gist of the present work is that under suitable circumstances the resonances are complete, and need not be supplemented by any nonresonant "background." It is of some interest to recall the finite-energy sum rules for high-energy scattering amplitudes [35], which likewise states that resonances are complete, and need not be supplemented by the nonresonant "background" contribution due to the exchange of Regge poles. The equality between resonances and Regge exchange (rather than their additivity) is the basis of duality and string models of elementary particles.

## ACKNOWLEDGMENT

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## APPENDIX A: THE NORM FOR A DIELECTRIC MICROSPHERE

For the model in Sec. IV C and with $f$ defined by (4.15) and the inner product by (4.17),

$$\langle\langle f_j|f_j\rangle\rangle=\int_0^\infty dr\, n^2(r)\varphi_j(r)^2 \; . \tag{A1}$$

The integral is along a contour such as that in Fig. 2; in practice this simply means that the contribution from evaluation at the upper limit can be neglected. Since $\varphi_j(r)=j_l(n_0\omega_j r)$ for $r<a$, and is given by the outgoing Hankel function for $r>a$,

$$\begin{aligned}\langle\langle f_j|f_j\rangle\rangle &= n_0^2\int_0^a dr\, r^2 j_l^2(n_0\omega_j r)\\ &\quad+\left[\frac{j_l(n_0\omega_j a)}{h_l^{(1)}(\omega_j a)}\right]^2\int_a^\infty dr\, r^2 h_l^{(1)}(\omega_j r)^2\\ &=\frac{a^3}{2}\left[n_0^2(j_l^2-j_{l-1}j_{l+1})\right.\\ &\quad\left.-\left[\frac{j_l}{h_l}\right]^2(h_l^2-h_{l-1}h_{l+1})\right] ,\end{aligned} \tag{A2}$$

where henceforth $h_l\equiv h_l^{(1)}$ and the arguments are $j_l\equiv j_l(n_0\omega_j a), h_l\equiv h_l(\omega_j a)$. Rearrangement gives

$$\begin{aligned}\langle\langle f_j|f_j\rangle\rangle &=(n_0^2-1)\frac{a^3}{2}j_l^2\\ &\quad-\frac{a^3}{2}j_l^2\left[n_0^2\frac{j_{l-1}j_{l+1}}{j_l^2}-\frac{h_{l-1}h_{l+1}}{h_l^2}\right] .\end{aligned} \tag{A3}$$

But continuity of the logarithmic derivative and the recursion relation give

$$\begin{aligned} n_0 j_{l-1}/j_l &= h_{l-1}/h_l \; ,\\ n_0 j_{l+1}/j_l &= h_{l+1}/h_l \; .\end{aligned} \tag{A4}$$

Hence the second group of terms in (A3) cancels, and (4.19) is proved.

## APPENDIX B: MULTIPLE DISCONTINUITIES ON A FULL LINE

The proof of completeness can be generalized to cases with multiple discontinuities in the dielectric constant $\rho(x)$, and also to systems defined on the full line $-\infty<x<\infty$. Assume there are discontinuities at $x=a_1,a_2,\ldots,a_N$, with $a_1<a_2<\cdots<a_N$, and that $\rho(x)\rightarrow 1$ as $x\rightarrow\pm\infty$. Introduce the auxiliary functions $g_-(\omega,x)$ and $g_+(\omega,x)$, which are solutions to the time-independent equation (2.2), and defined by the boundary conditions

$$g_\pm(\omega,x)\rightarrow\exp(\pm i\omega x), \quad x\rightarrow\pm\infty \; . \tag{B1}$$

The Green's function in the frequency domain is

$$\tilde{G}(x,y;\omega)=\begin{cases} g_+(\omega,x)g_-(\omega,y)/W(\omega), & y<x\\ g_+(\omega,y)g_-(\omega,x)/W(\omega), & x<y \; ,\end{cases} \tag{B2}$$

where the Wronskian

$$W(\omega)=g_+(\omega,x)g'_-(\omega,x)-g'_+(\omega,x)g_-(\omega,x) \tag{B3}$$

is again independent of $x$. Provided $\rho(x)\rightarrow 1$ sufficiently rapidly, $g_\pm(\omega,x)$ are analytic functions of $\omega$, so $\tilde{G}$ is also analytic except at the zeros of $W(\omega)$. Following the same argument as in Sec. II, if one can show that the integral along a large semicircle in the lower half of the $\omega$ plane is negligible, the QNM's would form a complete set.

It then remains to estimate the $\omega\rightarrow\infty$ behavior using the WKB approximation. The idea is to start in the region $x>a_N$ for $g_+$, and connect to the left by a series of transfer matrices, and likewise to start in the region $x<a_1$ for $g_-$. For $x>a_N$,

$$g_+(\omega,x)\simeq\exp[i\omega I(a_N,x)]g_+(\omega,a_N) \; , \tag{B4}$$

which consists of only an outgoing wave, and $I$ is defined as in (3.1). On the other hand, in the intermediate region $a_j\le x\le a_{j+1}$, $g_+$ consists of two counterpropagating waves:

$$g_+(\omega,x)\simeq A_j\exp[i\omega I(a_j,x)]+B_j\exp[-i\omega I(a_j,x)] \; . \tag{B5}$$

Matching across each discontinuity gives

$$\begin{bmatrix} A_j \\ B_j \end{bmatrix} = \begin{bmatrix} M_{11}(j) & M_{12}(j) \\ M_{21}(j) & M_{22}(j) \end{bmatrix} \begin{bmatrix} e^{i\theta_j} & 0 \\ 0 & e^{-i\theta_j} \end{bmatrix} \begin{bmatrix} A_{j-1} \\ B_{j-1} \end{bmatrix} , \quad \text{(B6)}$$

where

$$\theta_j = \omega I(a_{j-1}, a_j) , \quad \text{(B7)}$$

and the transfer matrix $M$ is given at high frequencies by

$$M_{11}(j) = M_{22}(j) \simeq 1 , \quad \text{(B8)}$$

$$M_{12}(j) = M_{21}(j) \simeq -R(j) , \quad \text{(B9)}$$

in which $R(j)$ is the reflection coefficient at $a_j$, defined in a manner similar to (3.2). Using (B6) and the boundary condition $A_N = g_+(\omega, a_N), B_N = 0$ implied by (B4), it is straightforward to find $g_+$ everywhere.

Considerable simplification is possible when $\text{Im}\omega \to -\infty$, since in this regime one of the counterpropagating waves is exponentially growing; thus

$$g_+(\omega, x) \simeq -R(N)\exp[-i\omega I(a_N, x)]g_+(\omega, a_N) . \quad \text{(B10)}$$

Likewise,

$$g_-(\omega, x) \simeq -R(1)\exp[+i\omega I(a_1, x)]g_-(\omega, a_1) . \quad \text{(B11)}$$

Both of these hold for any $x \in [a_1, a_N]$.

Putting these into (B2), we find

$$\tilde{G}(x, y; \omega) \simeq \frac{1}{2i\omega} \exp\{i\omega[I(a_1, y) + I(x, a_N) - I(a_1, a_N)]\} , \quad \text{(B12)}$$

which vanishes when $\text{Im}\omega \to -\infty$. This then allows the contribution from the semicircle in the $\omega$ plane to be neglected, leading to a proof of completeness in the region $x \in [a_1, a_N]$.

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