Optical pulse propagation at negative group velocities due to a nearby gain line

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For a pulse with carrier frequency detuned less than an atomic plasma frequency but outside a narrow Lorentzian gain line, the group velocity will be negative. Unlike propagation at the center of an absorption line, the energy velocity is approximately equal to the group velocity, and is also negative. We show that a classical Gaussian pulse with such a carrier frequency will propagate at the negative group velocity for many atomic plasma wavelengths, before dispersion deforms the pulse shape. The peak of the transmitted pulse leaves the gain medium before the peak of the incident pulse enters, i.e., the pulse is transmitted superluminally. For a sufficiently long medium the exit pulse is well resolved from a comparison pulse traveling through an equal distance of vacuum. There is no conflict with causality, as numerical simulations with a switched-on Gaussian demonstrate. We propose an experiment to observe this kind of propagation by sending a pulsed probe beam through a Xe gas cell pumped to achieve inversion.

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I. INTRODUCTION

The propagation of a pulse of light through dispersive media has been thoroughly studied, but there still remain some unresolved aspects. It is well known that for transparent media with normal dispersion, the group velocity is less than the vacuum speed of light and correctly describes the motion of a pulse. Anomalous dispersion occurs within an absorption band and results in a group velocity which we term *abnormal* (superluminal or even negative), in apparent violation of relativistic causality. Actually abnormal velocities must always exist in some frequency range in linear media, and will typically occur within absorption lines and outside gain lines [1]. The issue of propagation at abnormal velocities was first raised because of an apparent conflict with the theory of relativity, which forbids any signal to travel faster than light in vacuum. The classic work of Sommerfeld and Brillouin [2] considered the propagation of a step-modulated sine wave with a frequency near an absorption line. They defined five different velocities relevant to this problem, and found that no real signal (defined by the edge of the step) could actually propagate faster than the speed of light in the vacuum. Hence it is frequently claimed in textbooks that the group velocity "is just not a useful concept" when it is abnormal [3]. On the contrary, it has been found that for sufficiently smooth pulses, the group velocity can be meaningful even when abnormal. Recently there has been renewed interest in the question of the time of propagation of Gaussian pulses, since these are more appropriate for experimental work. Garrett and McCumber [4] and Tanaka [5] predicted that the pulse peak would travel with an abnormal group velocity whenever the bandwidth of the pulse lies within the absorption band. Similarly, the tunneling of photons through a 1D photonic band gap is superluminal. These results were confirmed experimentally [6]. The motion of the pulse can be understood as a reshaping effect: most of the pulse is absorbed, leaving only a small pulse in the forward tail which moves superluminally. This type of pulse does not constitute a signal, however, so it still conforms to the idea of relativistic causality.

In this paper we consider an interesting model for which abnormal velocities occur in the nearly transparent regions; in particular, the group velocity becomes negative when the detuning from a gain line is on the order of the plasma frequency. This model is realized in terms of a Lorentz medium with negative oscillator strength, or equivalently by a two-level system in which the level populations are inverted with respect to their thermodynamic values. For this example we calculate the group velocity and its dispersion outside the gain band. Then we examine the propagation of a Gaussian pulse through this material and show that it moves at the group velocity, if the material is not too thick and the spectral width of the pulse not too great. Thus, *the pulse will exit the material before it would have if it had traveled through an equal distance of vacuum.* This is not at variance with relativistic causality, however, because the Gaussian lacks a definite edge. Numerical simulation of the propagation of a modified Gaussian pulse, which has zero intensity until a given time, shows that its edge travels with the front velocity c. The energy velocity is found to be approximately equal to the group velocity, but this does not imply a violation of energy conservation, since the pulse can borrow energy from the excited atoms. We propose an experiment to observe pulses propagating at abnormal group velocities.

II. ABNORMAL GROUP VELOCITIES IN AN INVERTED POPULATION MEDIUM

It is well known that the inversion of level populations in an atomic material leads to gain for light tuned to the resonance frequency of the two levels. Outside this gain

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band, the material appears transparent. Using the Kramers-Kronig relations we have previously shown that abnormal group velocities occur around a strong enough gain line, independent of the line shape [1]. The simplest example, and one which is sufficient for our calculation, is a Lorentz dielectric with negative oscillator strength. Ignoring the effect of inhomogeneous line broadening, the complex index of refraction is

$$n(\omega) = \left[1 - \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\gamma\omega} \right]^{1/2} . \tag{1}$$

Here ω_0 is the frequency difference between the two levels and γ is the linewidth of the resonance. In this classical model, the square of the atomic plasma frequency is defined to be

$$\omega_p^2 = \frac{4\pi N |f| e^2}{m} , \tag{2}$$

where N, e, and m are the atomic density, electron charge, and mass, respectively. The oscillator strength is given by f, which is negative; we have included its sign explicitly in Eq. (1). Note that this expression for the index of refraction is exactly what would be obtained from a semiclassical two-level model in the linear regime. It has been shown from the equations of motion for the density matrix for two levels $|1\rangle$ and $|2\rangle$ (without making the rotating wave approximation) that the average of the dipole moment satisfies [7]

$$\left[\frac{d^2}{dt^2} + \gamma \frac{d}{dt} + \omega_0^2 \right] \langle \mathbf{r}(t) \rangle = \frac{e}{m} f E(t) , \tag{3}$$

where the oscillator strength for polarization ε is defined by

$$f = \frac{2m\omega_0}{\hbar} |\langle 1|\mathbf{r}|2\rangle \cdot \varepsilon|^2 \Delta\rho . \tag{4}$$

Notice that the fractional population inversion $\Delta\rho$ is constant, so that for a given substance f is constant. Thus the index given in Eq. (1) will apply if most of the atoms remain in the excited state while a weak pulse propagates through the medium. Equation (3) is the usual starting point for the Lorentz model, but with the oscillator strength always positive and the dipole moment inter-

preted classically. For our case of an inverted population, f is negative. From the real part of the index we find the phase velocity

$$v_p(\omega) = \frac{c}{\operatorname{Re}(n)} , \tag{5}$$

while the imaginary part results in a gain coefficient

$$g(\omega) = -\frac{2 \operatorname{Im}(\omega n)}{c} . \tag{6}$$

It is usually true that the resonance frequency is much greater than the plasma frequency, which is in turn much larger than the linewidth, so we will assume

$$\gamma \ll \omega_p \ll \omega_0 . \tag{7}$$

For this inequality to hold we require a large gain bandwidth product, but a narrow bandwidth. Typical orders of magnitude for obtainable inversion in noble gases are $\gamma/\omega_p = 10^{-2}$ and $\omega_p/\omega_0 = 10^{-6}$ (see Sec. IV for a numerical example), so this assumption is easily met. [The left inequality of Eq. (7) differs from an example considered by Garrett and McCumber [4], who assumed the plasma frequency was smaller than the gain bandwidth. Here we are detuned sufficiently far from the center of the line so that gain is not very substantial, although for generality we have included it.] To simplify subsequent computation we introduce the dimensionless detuning

$$\xi = \frac{\omega - \omega_0}{\omega_p} , \tag{8}$$

which will be of order unity within the region of interest. More precisely, in order to make power-series expansions in the small ratio ω_p/ω_0, we will require the dimensionless detuning to lie in the range

$$\frac{\omega_p}{\omega_0} \ll |\xi| \ll \frac{\omega_0}{\omega_p} . \tag{9}$$

The power-series expansion of the index of refraction in the small ratios γ/ω_p and ω_p/ω_0 is

$$n(\xi) = 1 + \frac{1}{4\xi} \left[\frac{\omega_p}{\omega_0} \right] - i\frac{1}{8\xi^2} \left[\frac{\gamma}{\omega_p} \right] \left[\frac{\omega_p}{\omega_0} \right] . \tag{10}$$
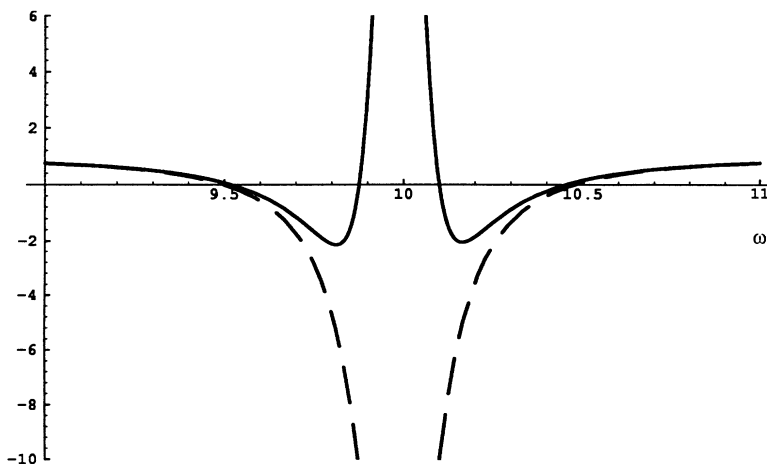
FIG. 1. Reciprocal group velocity (solid curve) and reciprocal energy velocity (dashed) in units of $1/c$, vs angular frequency for a Lorentz-model medium with negative oscillator strength [Eq. (1)]. The parameters used for the plots are $\omega_0 = 10$, $\omega_p = 1$, and $\gamma = 0.2$. Outside the high-gain region, the reciprocal velocities are nearly equal, and are negative in a "window" of width ω_p centered on ω_0.

The real part of the first derivative of the wave number $\omega n(\omega)/c$ with respect to ω is the reciprocal of the group velocity, as calculated by the method of stationary phase. The reciprocal of the group velocity is plotted in Fig. 1, along with the reciprocal energy velocity as found in Appendix A. The energy velocity is defined as the ratio of the time-averaged Poynting vector to the time-averaged energy density of the fields and the atoms. Notice that for the detunings we are considering, in a relatively low-gain region, these two reciprocal velocities are nearly equal, as well as abnormal. The imaginary part of the derivative of the wave number results in preferential gain of frequencies nearer to the gain line. From Eq. (10) we find the power-series expansion

$$\frac{d}{d\omega}[\omega n(\omega)] = 1 - \frac{1}{4\xi^2} + i\frac{1}{4\xi^3}\left[\frac{\gamma}{\omega_p}\right] . \tag{11}$$

It is useful also to consider the group *delay,* defined as the difference in transit times for a pulse traveling through the medium versus one traveling through an equal length of vacuum. This delay will be negative whenever the group velocity is abnormal. For the Lorentz gain medium it is

$$\Delta t(\omega) = \left[\frac{d}{d\omega}\mathrm{Re}\{\omega n(\omega)\} - 1\right]\frac{z}{c} = -\frac{1}{4\xi^2}\frac{z}{c} ; \tag{12}$$

notice that the group delay is always negative. This means that the peak of the transmitted pulse always appears before one which travels through vacuum. A region of particular interest occurs when the group *velocities* are negative, since then certain pulse envelopes actually travel backwards. Setting the group velocity equal to zero and solving for the frequency, we find this region corresponds to

$$|\xi| < \tfrac{1}{2} , \tag{13}$$

which is consistent with Eq. (9). Thus, negative group velocities occur in a frequency "window" centered around (but excluding) the gain line, with a width equal to the plasma frequency.

Differentiating Eq. (11) gives the group-velocity dispersion (real part) and second-order preferential gain (imaginary part),

$$\frac{d^2}{d\omega^2}(\omega n(\omega)) = \frac{1}{\omega_p}\left[\frac{1}{8\xi^3} - i\frac{3}{4\xi^4}\left[\frac{\gamma}{\omega_p}\right]\right] . \tag{14}$$

Note the different functional dependences of Eqs. (12) and (14) on the dimensionless detuning. As the detuning is decreased, the magnitude of the group delay becomes very large, but the dispersion increases more rapidly. We will demonstrate in the next section how the group-

velocity dispersion and preferential gain place a limitation on the distance a pulse can travel without distortion.

III. PROPAGATION OF A GAUSSIAN PULSE AT ABNORMAL GROUP VELOCITY: ANALYTICAL CALCULATION

In this section we consider an initially Gaussian pulse traveling through a finite cell containing the inverted population medium. We make analytical approximations to calculate the shape of the pulse as a function of time and distance traversed. We find that a pulse of spectral width less than the plasma frequency travels a distance of many plasma wavelengths c/ω_p at an abnormal group velocity before becoming very distorted.

The medium extends from $z=0$ to $z=a$. Propagation of a wave at frequency ω is obtained by solving the Helmholtz equation,

$$\left\{\frac{\partial^2}{\partial z^2} + \left[\frac{\omega}{c}n(\omega,z)\right]^2\right\}\widetilde{E}(z,\omega) = 0 , \tag{15}$$

where

$$n(\omega,z) = \begin{cases} n(\omega) & \text{for } 0 < z < a \\ 1 & \text{for } z < 0 \text{ or } z > a . \end{cases} \tag{16}$$

Continuity of the field and its derivative leads to coefficients for the transmitted and reflected waves. However, we will assume that since the index of refraction of a gas is close to 1, all of the wave is transmitted. Including reflections at each surface only introduces a factor with much weaker frequency dependence than the factor due to propagation through the medium. Thus, assuming all waves are traveling towards the right, the field is given by

$$\widetilde{E}(z,\omega) = F(\omega)\begin{cases} \exp\left[i\frac{\omega}{c}z\right] & \text{for } z < 0 \\ \exp\left[i\frac{\omega}{c}n(\omega)z\right] & \text{for } 0 < z < a \\ \exp\left[i\frac{\omega}{c}[n(\omega)a + z - a]\right] & \text{for } z > a , \end{cases} \tag{17}$$

where $F(\omega)$ is some wave packet in the frequency domain.

We choose a Gaussian pulse with carrier frequency ω_c and time width τ, with Fourier transform

$$F(\omega) = \sqrt{2\pi}\tau E_0\exp[-\tfrac{1}{2}\tau^2(\omega - \omega_c)^2] . \tag{18}$$

This corresponds to the peak of the incoming pulse passing $z=0$ at $t=0$; the time-dependent field is then given by

$$E(z,t) = \begin{cases} E_0 \exp\left[-\dfrac{\left[t - \dfrac{z}{c} \right]^2}{2\tau^2} \right] \exp(-i\omega_c t) & \text{for } z < 0 \qquad (19a) \\[3em] \dfrac{\tau}{\sqrt{2\pi}} E_0 \int_{-\infty}^{\infty} d\omega \exp[-\tfrac{1}{2}\tau^2(\omega-\omega_c)^2]\exp\left[i\dfrac{\omega n(\omega)}{c}z - i\omega t \right] & \text{for } 0 < z < a \qquad (19b) \\[3em] \dfrac{\tau}{\sqrt{2\pi}} E_0 \int_{-\infty}^{\infty} d\omega \exp[-\tfrac{1}{2}\tau^2(\omega-\omega_c)^2]\exp\left[i\dfrac{\omega n(\omega)}{c}a + i\dfrac{\omega}{c}(z-a) - i\omega t \right] & \text{for } z > a \; . \qquad (19c) \end{cases}$$

The main point of this section is now to calculate the integrals (19b) and (19c), for propagation within and to the right of the medium. Since they have the same functional form, we only need to evaluate Eq. (19b); then Eq. (19c) can be found by simply replacing z by a and t by $t - (z-a)/c$.

Following the method used in [4], we expand the second exponent in powers of $\omega - \omega_c$, up to second order. This approximation is valid provided that the bandwidth of the pulse does not significantly overlap with the gain line, that is,

$$\frac{1}{\tau} \ll |\omega_c - \omega_0| \; . \qquad (20)$$

Actually we will find below that frequencies very close to resonance are amplified by the gain medium to such an extent that the power-series expansion breaks down; however if the Gaussian spectrum is filtered by first being passes through the same medium with noninverted population, those frequencies can be eliminated. Since the calculation for propagation through the absorbing medium is similar to that for the gain medium, we only need to show that little distortion will occur in the latter case, to justify beginning with the Gaussian spectrum Eq. (18). For actual experiments, and also for numerical simulations, it is important to exclude frequency components which are on resonance; see Secs. IV and V.

Using the power-series approach, the integral Eq. (19b) can be done in closed form to yield the result

$$E(z,t) \cong \frac{1}{\sqrt{A(z)}} E_0 \exp\left[i\left[\frac{\omega_c n(\omega_c)}{c}z - \omega_c t \right] \right]$$

$$\times \exp\left\{ -\frac{\left[t - \dfrac{d}{d\omega_c}[\omega_c n(\omega_c)]\dfrac{z}{c} \right]^2}{2\tau^2 A(z)} \right\}$$

$$\text{for } 0 < z < a \; , \qquad (21)$$

where

$$A(z) = 1 - i\frac{z}{c\tau^2}\frac{d^2}{d\omega_c^2}[\omega_c n(\omega_c)] \; . \qquad (22)$$

By looking at the expression in curly brackets in Eq. (21), we see that for a fixed depth in the medium, the temporal variation of the pulse is nearly that of the original Gauss-

ian, but it now travels at the abnormal group velocity. On the other hand, the spatial variation at a given time is only Gaussian as long as $A(z)$ is close to 1. In order to find the relevant length scales to satisfy this requirement, we substitute Eqs. (11) and (14) for the first and second derivatives of the wave number and simplify, neglecting terms of higher order than the second in γ/ω_p. The envelope of the electric field (excluding the carrier wave) then becomes

$$E(z,T) = \frac{E_0}{\sqrt{1+\Delta(z)}}\exp\left[\frac{gz}{2} \right]$$

$$\times \exp\left\{ -\left[\frac{1 - \dfrac{z}{w_1}}{1+\Delta(z)} \right] \frac{[T-T_1(z)]^2}{2\tau^2} \right\}$$

$$\times \exp\left\{ -i\left[\frac{\dfrac{z}{w_2}}{1+\Delta(z)} \right] \frac{(T-T_2)^2}{2\tau^2} \right\} , \qquad (23)$$

where

$$T = t - \frac{z}{c}\left[1 - \frac{1}{4\xi_c^2} \right] , \qquad (24)$$

$$g = \frac{\gamma}{4c\xi_c^2} , \qquad (25)$$

$$\Delta(z) = -\frac{2z}{w_1} + \left[\frac{z}{w_2} \right]^2 , \qquad (26)$$

$$T_2 = 2\gamma\tau^2 , \qquad (27)$$

and

$$T_1(z) = T_2 \frac{\left[\dfrac{z}{w_2} \right]^2}{1 - \dfrac{z}{w_1}} . \qquad (28)$$

The length at which distortion due to preferential gain occurs is

$$w_1 = \frac{4}{3}\frac{c\omega_p^2\tau^2\xi_c^4}{\gamma} , \qquad (29)$$

and the (signed) length for which group-velocity dispersion becomes evident is

$$w_2 = 8c\omega_p \tau^2 \xi_c^3 \ . \tag{30}$$

The first exponential factor in Eq. (23) is simply the gain at the carrier, which can be made small if the linewidth is sufficiently narrow. The second exponential shows that, aside from gain and chirping effects, the Gaussian propagates at the group velocity for z less than both of the lengths w_1 and $|w_2|$. For very narrow linewidths $|w_2|$ will be the shorter length, so that group-velocity dispersion places an upper limit on the length of the medium:

$$a < |w_2| \ . \tag{31}$$

The last exponential factor in Eq. (23) gives rise to a small frequency chirp which only becomes significant once the intensity profile is already distorted.

The effect of propagation at abnormal velocity will be most striking if the magnitude of the group delay is at least on the order of the time width of the pulse, that is,

$$|\Delta t| > \tau \ . \tag{32}$$

(This condition is similar to Rayleigh's criterion to resolve interference fringes.) Using Eq. (12) for the group delay we find that the length of the medium must be great enough to allow the resolution of the pulse traveling through it and the pulse going through the vacuum,

$$a > 4c\tau \xi_c^2 \ . \tag{33}$$

Assuming that $|w_2| < w_1$, we can combine the inequalities (31) and (33) to obtain

$$\frac{1}{2\tau} \ll |\omega_c - \omega_0| \ , \tag{34}$$

which is consistent with the pulse bandwidth lying outside the gain region [Eq. (20)]. To find the allowable length for the medium, we rewrite the inequalities (31) and (33) in terms of the plasma wave number $k_p = \omega_p/c$,

$$4\omega_p \tau \xi_c^2 < k_p a < 8(\omega_p \tau)^2 |\xi_c^3| \ . \tag{35}$$

For $\omega_p \tau$ sufficiently large (that is, for a pulse with narrow enough bandwidth) we can always find a length to satisfy both inequalities in Eq. (35); in general this will mean the medium should have a length of many plasma wavelengths.

If the inequality Eq. (35) is met, then the comparison pulse which travels through the vacuum and the one which travels through the medium are well resolved from each other, and the pulse in the medium travels undistorted (aside from gain) at the group velocity. If in addition the group velocity is negative we have a situation which seems counterintuitive: most of the pulse has already appeared at the exit face before the main part of the pulse has entered the medium. In the next two sections we will describe this remarkable effect in greater detail, and propose an experiment to confirm it.

IV. PREDICTIONS FOR A PROPOSED EXPERIMENT

Before presenting graphs based on the analytical calculation and numerical simulations, we describe an experiment to demonstrate pulses moving at negative velocities. The most important consideration comes from the inequality Eq. (7), especially the requirement that the linewidth should be narrow compared to the plasma frequency. The plasma frequency can be found most simply from the maximum gain g_0 [setting $\omega = \omega_0$ in Eq. (6)] and the homogeneous linewidth γ,

$$\omega_p = \sqrt{g_0 \gamma c} \ . \tag{36}$$

Substituting into Eq. (7) shows that the gain must be large compared with the linewidth expressed in wave numbers,

$$g_0 \gg \frac{\gamma}{c} \ . \tag{37}$$

Large gains have been reported in noble-gas lasers, although these do not have homogeneously broadened spectra. Instead they are Doppler broadened; the gain spectrum within a Doppler width of the line center is Gaussian, but far from line center the index of refraction is described by Eq. (1) from the Lorentz model. Equation (36) is modified only by a normalization factor,

$$\omega_p = (\pi \ln 2)^{-1/4} \sqrt{2 g_D \delta \omega_D c} = 1.164 \sqrt{g_D \delta \omega_D c} \ , \tag{38}$$

where $\delta \omega_D$ is the full Doppler width at half maximum. For the $5d[\frac{7}{2}]^0_3$-$6p[\frac{3}{2}]_1$ transition (wavelength $= 3.507\mu$) of Xe, gains as high as 60 dB/m have been reported in a Xe:He mixture [8–10]. At a temperature of 300 K, the Doppler width of the gain line is $\delta \omega_D/2\pi = 92$ MHz. Actually, a considerably broader linewidth of 270 MHz, attributed to hyperfine splitting and isotope shifts, is observed [11]. Using this method linewidth we estimated the plasma frequency to be $\omega_p/2\pi = 420$ MHz [where we have used Eq. (38) and ignored an unknown normalization factor of order unity]. To prevent broadening from mass shifts and hyperfine splitting, we propose to use the isotope ^{132}Xe, which has no nuclear spin. Thus the inequality Eq. (7) (where γ is replaced by the Doppler width) is satisfied, so we are in the high-dispersion but low-gain regime. The small parameters used in the propagation calculation of Sec. III are found from the plasma frequency and the homogeneous linewidth of $\gamma/2\pi = 4$ MHz,

$$\frac{\gamma}{\omega_p} = 10^{-2} \ , \tag{39}$$

$$\frac{\omega_p}{\omega_0} = 5 \times 10^{-6} \ . \tag{40}$$

Of course, the power-series Eqs. (10), (11), and (14) are only valid for detuning outside the Doppler width, so for small detunings it may be necessary to narrow the linewidth by cooling.

In this narrow-linewidth regime superfluorescence has been observed, but we do not expect that to occur for a system which is continuously and incoherently pumped

(by electron-impact excitation). Simulations of a continuously pumped, pointlike system of three-level atoms show that after a transient superradiant pulse, the populations approach a steady state with inversion of the upper two levels, as we have assumed. Since superfluorescence in an extended gain medium requires more stringent conditions to occur, these simulations support our expectations.

As an illustration, suppose we choose a dimensionless detuning of $\xi = \frac{1}{3}$. We must take the product of the plasma frequency and the pulse duration to be much greater than $\frac{3}{2}$ to satisfy Eq. (35), which now reads

$$\frac{4}{9}\omega_p\tau < k_p L \ll \frac{8}{27}(\omega_p\tau)^2 \ . \tag{41}$$

Let us take $\omega_p\tau = 30$; this corresponds to a pulse duration of $\tau = 10$ ns. Thus the Xe cell can have a length of 50 reduced plasma wavelengths, or about 5 m, to avoid distortion and yet still provide sufficient separation of the pulses.

As mentioned earlier, because the pulse frequency is so close to the gain line, filtering of the on-resonance frequencies is necessary to avoid large amplification of these components. A convenient way to achieve this is to first pass the laser pulse through an absorbing, i.e., uninverted population sample of the same medium. The form of the index of refraction for the absorbing section is then identical to Eq. (1) but with the opposite sign in front of the square of the plasma frequency. For simplicity, assume there is total inversion in the gain section and all atoms in the absorbing section are in the lower state. Then a gain cell of length L preceded by an absorbing cell of the same length will compensate exactly for the attenuation. Of course, it will also cancel out the dispersion so that the final pulse is identical to the original and arrives as though it had traveled through a vacuum of length $2L$. Thus we must measure the time between when the pulse exits the absorbing cell and when it exits the gain cell. In practice, complete inversion is difficult to obtain, but exact compensation of the gain can still be made by increasing the atomic density or length of the gain section, as we show explicitly in Appendix B.

The complete schematic for the proposed experiment is shown in Fig. 2. An InAs diode laser is tuned to the carrier frequency and the current is modulated to produce a pulse, which then passes through the absorbing cell filled with Xe, pumped to the lower level by an electrical discharge. This cell is followed by a beam splitter; note that at this point both the reflected and the transmitted pulse are identical, although slightly distorted due to the absorber. One pulse travels through vacuum to a detector $D2$; the path length to $D2$ is varied by a delay line. The second pulse travels through the Xe gain cell (pumped to the upper level), and is then detected at $D1$. This pulse will be restored by the gain cell to the original laser-pulse shape. To find the relative velocities of the two pulses, the path length to $D2$ is adjusted until the peaks of detector signals coincide. The anticipated result is that the path length from the beam splitter to $D2$ must be made *shorter* than the distance from the beam splitter to $D1$. Correlation techniques may also be used to compare the pulse shapes.
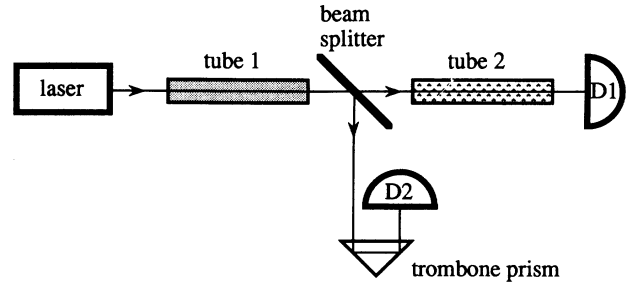

FIG. 2. Schematic of proposed experiment. The laser diode produces a pulse with carrier detuned about a plasma frequency from resonance. This propagates through cell with uninverted Xe population, which absorbs on-resonance frequencies. After the beam splitter, the reflected pulse goes along a path with a delay line (trombone prism), and to detector $D2$. The transmitted pulse travels through the Xe cell with inverted population, and is detected at $D1$. The delay line is adjusted until the signals from $D1$ and $D2$ coincide.

V. COMPARISON OF ANALYTICAL RESULTS WITH NUMERICAL COMPUTATION

In Fig. 3 we show a time sequence of the expected propagation for the values given above for Xe, based on the analytical calculation. The pulse duration is 26.4 inverse plasma frequencies, and the dimensionless detuning is $\frac{1}{3}$. Propagation through the absorber, which begins at $z = 0$, a vacuum gap, and the gain section are shown; this is equivalent to the path from the laser to $D1$. (The calculation for the absorber is similar to that of the gain cell done in Sec. III.) The sequence of events can be described as follows. First, the incoming pulse begins propagating through the absorber, and its leading edge creates a cusped peak at the exit face of the gain section. This peak then splits into a forward- and a backward-moving pulse. The forward-moving pulse continues through the vacuum (and is detected at $D1$), while the backwards-moving pulse collides with the main part of the incoming pulse at the entrance of the gain cell, and both are annihilated by destructive interference. Although at $t = 200$ we see there are three pulses (two moving forward and one moving backward), energy is still conserved since the "extra" pulses have borrowed the energy from the inverted atoms. In a future paper we will examine the microscopic process underlying this "virtual gain" effect. For the macroscopic model considered here, we note that after a long enough time, there is only one final pulse with the same energy as the initial pulse.

Although the results of the calculation of Sec. III may appear preposterous, we have done numerical calculations which agree with them. Using a fast Fourier transform in time for each z in steps of two plasma wavelengths gave results virtually identical to those plotted in Fig. 3. Since the appearance of the backwards-moving pulse depends crucially on the small amplitude of the leading edge of the Gaussian, the question naturally

arises: how can the backwards-propagating pulse emerge if one switches the Gaussian on at a definite time (e.g., by suddenly opening a shutter just before the peak arrives)? To answer this question, and to more thoroughly understand what is occurring in the gain cell, we have also done numerical simulations starting with this step-modulated Gaussian pulse, where the shutter is opened at time t_0,

$$E(z=0,t)= \begin{cases} 0 \quad \text{for } t<t_0 \, , \\ E_0\exp\left[-\dfrac{t^2}{2\tau^2} \right]\exp(-i\omega_c t) \quad \text{for } t>t_0 \, . \end{cases}$$

(42)

Beginning with the analytical Fourier transform,

$$F(\omega)= \left[\frac{\pi}{2}\right]^{1/2} \tau E_0\exp[-\tfrac{1}{2}\tau^2(\omega-\omega_c)^2]\left\{1+\text{erf}\left[\frac{1}{\sqrt{2}\tau}[t_0+i\tau^2(\omega-\omega_c)]\right]\right\}$$

(43)

decreases the amount of computation needed. Propagation of this pulse with $t_0=\tau$ is shown in Fig. 4. Notice that the front defined by the sharp cutoff moves at exactly the vacuum speed of light regardless of the medium. This delays the formation of the small cusp at the exit face of the gain cell until the front has passed through [compare Figs. 3 and 4 at $t=100$]. Once that has occurred, the motion is qualitatively the same as that shown in Fig. 3. Thus we can expect that the particular pulse shape is not

crucial to producing a well resolved backwards-propagating pulse. As long as it has an unambiguous maximum and its duration satisfies Eq. (35), the pulse must travel at the group velocity. The numerical simulations illustrate three important facts: (a) causality is always maintained; (b) the group velocity is useful for describing motion of the peak of a pulse; and (c) the gain medium is unstable in the sense that it can easily be per-
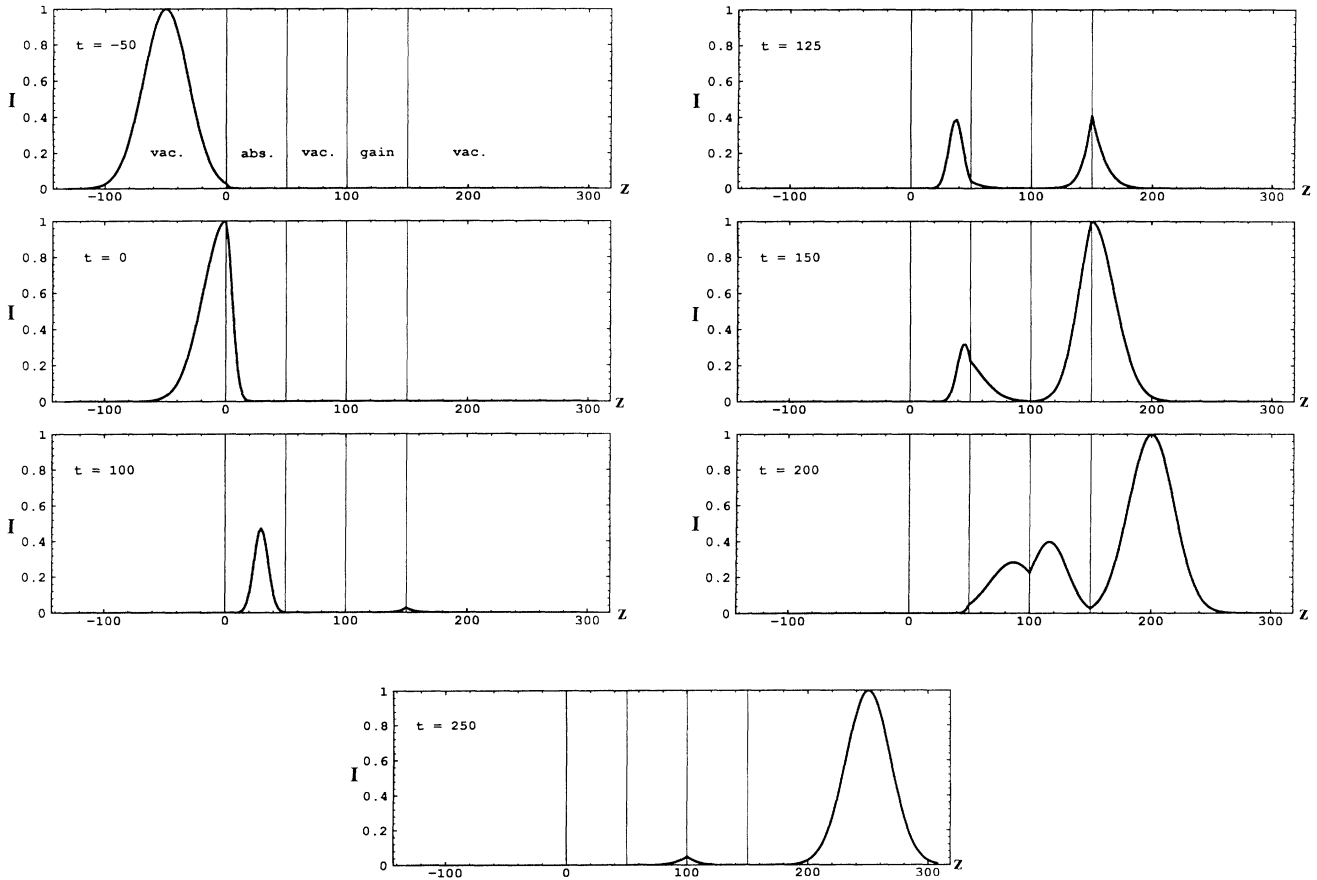

FIG. 3. Time sequence of Gaussian pulse propagation through cells filled with uninverted and inverted atomic population of the same medium, based on analytical calculation. Uninverted medium extends from $z=0$ to 50, inverted medium from $z=100$ to 150, and all other positions are vacuum. Carrier frequency is detuned $\omega_p/3$ above resonance. Distances are in reduced plasma wavelengths, times in inverse plasma frequencies. The peak of pulse is at $z=0$ at $t=0$. The pulse velocity is subluminal in the uninverted section, but negative in the inverted section.
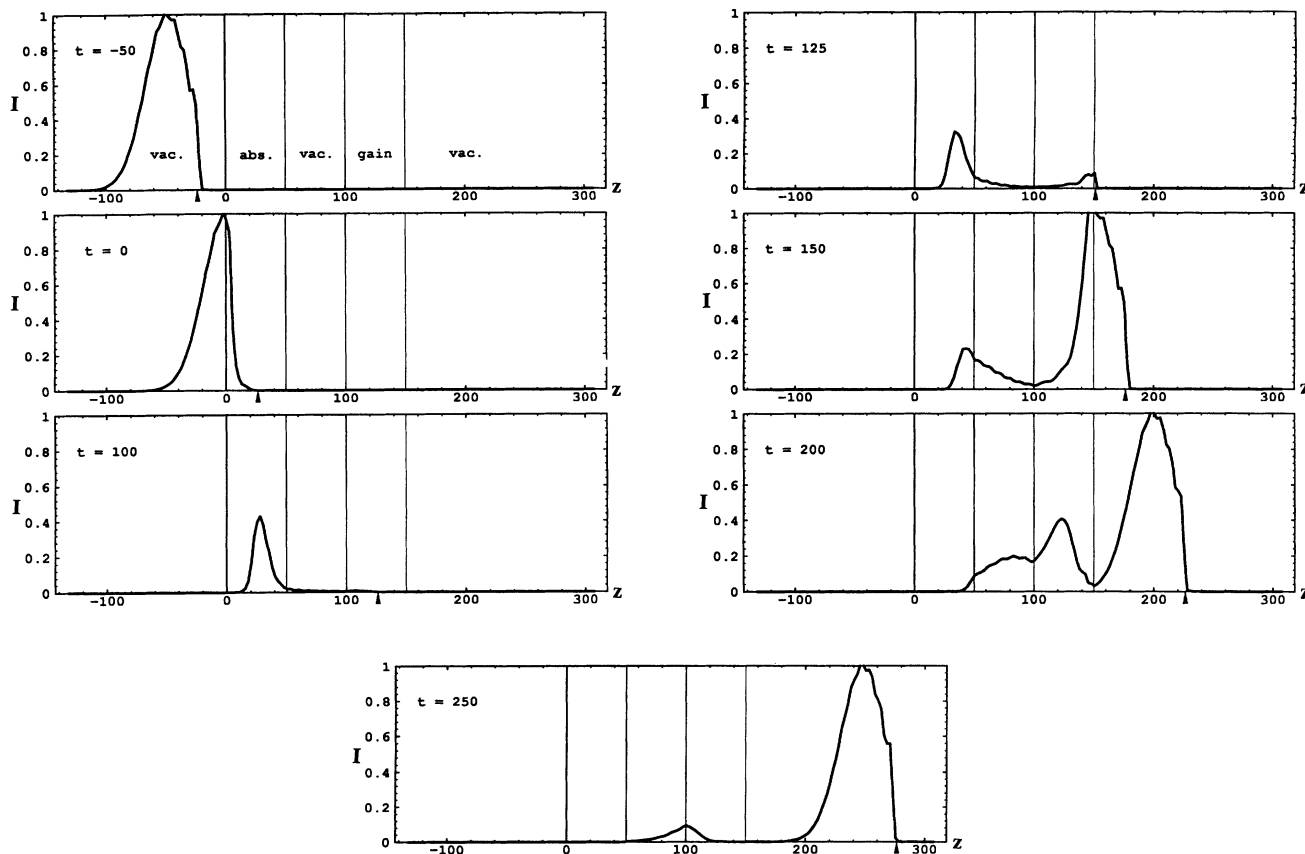
FIG. 4. Same configuration as for Fig. 3, but the Gaussian is step modulated; the step occurs at $1/e$ point of intensity. Graphs are based on numerical simulations. The location of the front $(z = ct)$ is indicated by the pointer at each time. (Ripples near the step are due to numerical error.)

turbed into producing a backwards-moving pulse from a small leading edge.

VI. CONCLUSION

Of the five velocities defined in [2], we have considered the phase, group, energy, and front velocities. For detuning near a narrow gain line, the group velocity becomes negative, but still remains meaningful for near-Gaussian pulses, as we have demonstrated analytically and numerically. The energy velocity is similar to the group velocity, so within the gain medium energy travels in the opposite direction as the phase velocity. In spite of these peculiarities, the front velocity is exactly the vacuum speed of light, so causality is not violated. While it is tempting to define a signal velocity using the time it takes for the intensity to reach a fraction of its maximum, this is unsuitable for our problem [12].

The motion of the peak(s) can be summarized in a space-time diagram, Fig. 5. One should not be misled by its resemblance to a Feynman diagram used in discussing virtual electron-positron pair creation and annihilation. The figure should not be interpreted in this way, because although the atoms are making virtual transitions, the backwards-moving pulse is composed entirely of photons with k vectors in the *positive* z direction. (To see this

more physically, notice that a beam splitter placed at normal incidence inside the gain medium would reflect light only to the left.)

Throughout this paper, we have dealt with the inverted-population medium in a purely macroscopic way, by prescribing a uniform, time-invariant index of refraction. We also assumed classical (as opposed to quantum) fields. It remains to be shown that the medium actually does approach a steady state as long as the pump remains on, i.e., there are no oscillations or superfluorescent pulses. For accurate comparison with experiment, we must also consider possible saturation effects
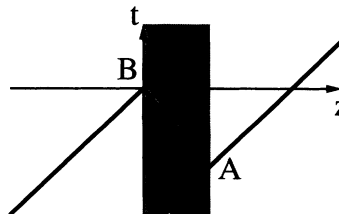

FIG. 5. Motion of the peak(s) of the pulse. The shaded region indicates the inverted medium. Point A is the exit face at which a pair of pulses appears, point B is the entrance face at which two pulses disappear due to destructive interference.

due to on-resonance amplified spontaneous emission, which may cause reduction of the inversion towards the ends of the gain cell. Note that although the spontaneous-emission field undergoes power broadening, the (weak) probe pulse sees the same gain spectrum as in our linear model. These problems require a detailed analysis of fluctuations and stability of the medium, and will be investigated in a future article. Carrying out such a fully quantum-mechanical calculation would also answer two questions related to the backwards-moving pulse. First, the energy transfer between the atoms and fields could be seen to produce the temporary "extra" pulse. Second, the results of this paper could be extended to single-photon pulses. Since the carrier frequency is outside the gain line, spontaneous emission should not destroy the backwards-pulse effect, even as one approaches the single-photon limit. Using quantum propagation theory, we can interpret the electric field as the probability amplitude for finding a photon, as long as the quasimonochromatic approximation is valid [13,14]. Therefore we expect the result of a fully quantum-mechanical calculation will agree with the classical one given above, but its interpretation will differ.

ACKNOWLEDGMENTS

APPENDIX A: CALCULATION OF THE ENERGY VELOCITY

This calculation is based on a similar discussion of the propagation of electromagnetic energy in the classical Lorentz model [15]. Here we will use the semiclassical equation of motion for the dipole moment (allowing population inversion) and show that in the transparent region, the energy velocity is approximately equal to the group velocity, and is abnormal.

The energy velocity at frequency ω is defined by

$$\mathbf{v}_E(\omega) = \frac{\overline{\mathbf{S}}(\omega)}{\overline{W}(\omega)} , \qquad (\text{A1})$$

where

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} \qquad (\text{A2})$$

and W is the energy density of the fields and mechanical energy density of the medium; bars indicate time averages over one cycle. From the relation between electric and magnetic fields, the Poynting vector is easily found to be

$$\overline{\mathbf{S}}(\omega) = \frac{c n_r(\omega)}{8\pi} |\mathbf{E}|^2 , \qquad (\text{A3})$$

so the remaining problem is to determine the energy den-

sity. [In this Appendix we use the notation for real and imaginary parts of the index of refraction, $n(\omega) = n_r(\omega) + i n_i(\omega)$.] The equation of energy conservation for a volume $\mathcal{V}$ enclosed by a surface $\partial\mathcal{V}$ is

$$\oint_{\partial\mathcal{V}} \mathbf{S} \cdot d\mathbf{a} = -\int_{\mathcal{V}} \left[\frac{\partial}{\partial t} \left[\frac{E^2 + B^2}{8\pi} \right] + \mathbf{E} \cdot \frac{\partial \mathbf{P}}{\partial t} \right] dV . \qquad (\text{A4})$$

To identify the energy density of the atoms, we will express the last term in the integrand as a sum of a perfect time derivative and an energy gain or loss term:

$$\mathbf{E} \cdot \frac{\partial \mathbf{P}}{\partial t} = \frac{\partial W_a}{\partial t} + \Gamma . \qquad (\text{A5})$$

This can be done by using the definition of the polarization and the equation of motion [Eq. (3)],

$$\mathbf{E} \cdot \frac{\partial \mathbf{P}}{\partial t} = \frac{m}{ef} \left[\frac{d^2}{dt^2} + \gamma \frac{d}{dt} + \omega_0^2 \right] \langle \mathbf{r} \rangle \cdot N e \frac{d}{dt} \langle \mathbf{r} \rangle , \qquad (\text{A6})$$

$$= \frac{mN}{2f} \frac{d}{dt} (\langle \dot{\mathbf{r}} \rangle^2 + \omega_0^2 \langle \mathbf{r} \rangle^2) + \frac{mN\gamma}{f} \langle \dot{\mathbf{r}} \rangle^2 . \qquad (\text{A7})$$

Here the first and second terms of the last expression can be interpreted classically as the time derivative of kinetic and potential energy of the dipole oscillators, respectively, and the third term is the power loss (for positive f) or gain (negative f). Thus the total energy density must be

$$W = \frac{E^2 + B^2}{8\pi} + \frac{mN}{2f} (\langle \dot{\mathbf{r}} \rangle^2 + \omega_0^2 \langle \mathbf{r} \rangle^2) . \qquad (\text{A8})$$

For a sinusoidally varying electric field of frequency ω, the time average over one cycle is

$$\overline{W} = \frac{1}{16\pi} (1 + |n_r + i n_i|^2) E^2 + \frac{mN}{4f} (\omega^2 + \omega_0^2) \langle \mathbf{r} \rangle^2 . \qquad (\text{A9})$$

To express this in terms of the fields, we use the solution

$$\langle \mathbf{r} \rangle = \frac{ef}{m} \mathbf{E} \frac{1}{\omega_0^2 - \omega^2 - i \gamma \omega} \qquad (\text{A10})$$

of Eq. (3) to eliminate $\langle \mathbf{r} \rangle$:

$$\overline{W} = \left[\frac{1}{16\pi} (1 + n_r^2 + n_i^2) + \frac{N e^2 f}{4m} \frac{\omega^2 + \omega_0^2}{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2} \right] |\mathbf{E}|^2 . \qquad (\text{A11})$$

By using the explicit expressions for real and imaginary parts of the square of the index of refraction, we obtain after some algebra

$$\overline{W} = \frac{1}{8\pi} \left[\frac{2\omega n_r n_i}{\gamma} + n_r^2 \right] |\mathbf{E}|^2 . \qquad (\text{A12})$$

Substituting this expression and Eq. (A2) for the Poynting vector leads to the exact result for the energy velocity,

$$v_E(\omega) = c \left[\frac{2\omega n_i(\omega)}{\gamma} + n_r(\omega) \right]^{-1} . \qquad (\text{A13})$$

By finding the minimum and maximum values of $v_E(\omega)$, it can be shown that for $f < 0$ $v_E(\omega)$ is always

greater than c or is negative; this is because energy is being added to the wave. It is generally true [2,16] that as the linewidth goes to zero, the reciprocal of the energy velocity reduces to the reciprocal group velocity in the transparent region. This can also be verified in this case by explicitly substituting narrow-linewidth approximations for $n_r(\omega)$ and $n_i(\omega)$ into Eqs. (13) and (11).

APPENDIX B: CANCELLATION OF GAIN AND ABNORMAL DISPERSION BY AN ABSORBER

In this appendix we will show that by sending any pulse through an absorbing medium followed by the same medium but with inverted population, the original pulse can be recovered. Subscripts g and a refer to properties of the gain and absorbing medium, respectively. Expanding the square root in Eq. (1), and substituting the plasma frequency Eq. (2), the index for the gain medium becomes

$$n_g(\omega) \cong 1 - \frac{2\pi|f|e^2}{m} \frac{N_g|\Delta\rho_g|}{\omega_0^2 - \omega^2 - i\gamma\omega} , \tag{B1}$$

where for generality we have included a fractional population inversion $\Delta\rho_g$. Similarly, the index for an absorber

with the same resonance and linewidth is

$$n_a(\omega) \cong 1 + \frac{2\pi|f|e^2}{m} \frac{N_a|\Delta\rho_a|}{\omega_0^2 - \omega^2 - i\gamma\omega} . \tag{B2}$$

On propagating through a length of absorber and length of gain medium, each frequency component acquires a phase

$$\phi(\omega) = \frac{1}{c}(\omega n_a L_a + \omega n_g L_g) \tag{B3}$$

$$= \frac{\omega}{c}(L_a + L_g) + \frac{2\pi|f|e^2}{mc} \frac{\omega}{\omega_0^2 - \omega^2 - i\gamma\omega}$$

$$\times (N_a|\Delta\rho_a|L_a - N_g|\Delta\rho_g|L_g) . \tag{B4}$$

The second term will vanish if we take

$$N_a|\Delta\rho_a|L_a = N_g|\Delta\rho_g|L_g , \tag{B5}$$

so that we are left with the first term, which is what we would obtain for free-space propagation through $L_a + L_g$. Therefore, both the gain and the dispersion can be completely canceled out.

[1] R. Y. Chiao, Phys. Rev. A **48**, R34 (1993); E. L. Bolda, R. Y. Chiao, and J. C. Garrison, *ibid.* **48**, 3890 (1993); A. M. Steinberg and R. Y. Chiao, *ibid.* **49**, 2071 (1994).

[2] L. Brillouin, *Wave Propagation and Group Velocity* (Academic, New York, 1952).

[3] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975), p. 302.

[4] C. G. B. Garrett and D. E. McCumber, Phys. Rev. A **1**, 305 (1969).

[5] M. Tanaka, Plasma Phys. Control. Fusion **31**, 1049 (1989).

[6] S. Chu and S. Wong, Phys. Rev. Lett. **48**, 738 (1982); A. M. Steinberg, P. G. Kwiat, and R. Y. Chiao, Phys. Rev. Lett. **71**, 708 (1993).

[7] R. W. Boyd, *Nonlinear Optics* (Academic, San Diego, 1992), p. 209.

[8] D. R. Armstrong, IEEE J. Quantum Electron. QE-4, 968 (1968).

[9] P. O. Clark, IEEE J. Quantum Electron. QE-1, 109 (1965).

[10] R. A. Paananen and D. L. Bobroff, Appl. Phys. Lett. **2**, 99 (1963).

[11] Lee Casperson and Amnon Yariv, Phys. Rev. Lett. **26**, 293 (1971).

[12] In [2] we find a more precise definition of the signal velocity, using the method of steepest descents: the signal arrives when the path of integration reaches the pole from the signal spectrum $F(\omega)$. Since a Gaussian spectrum does not have a pole, this definition cannot be made.

[13] Ivan H. Deutsch and John C. Garrison, Phys. Rev. A **43**, 2498 (1991).

[14] L. Mandel, Phys. Rev. **144**, 1071 (1966).

[15] R. Loudon, J. Phys. A **3**, 233 (1970).

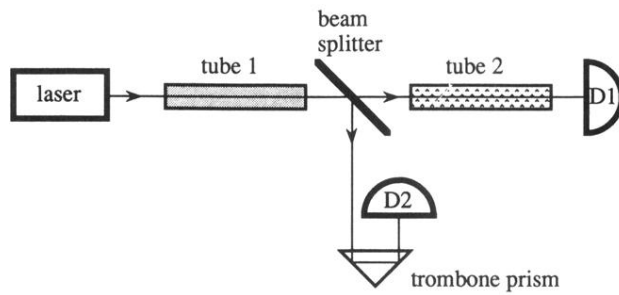[16] L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media* (Pergamon, Oxford, 1960).

FIG. 2. Schematic of proposed experiment. The laser diode produces a pulse with carrier detuned about a plasma frequency from resonance. This propagates through cell with uninverted Xe population, which absorbs on-resonance frequencies. After the beam splitter, the reflected pulse goes along a path with a delay line (trombone prism), and to detector $D2$. The transmitted pulse travels through the Xe cell with inverted population, and is detected at $D1$. The delay line is adjusted until the signals from $D1$ and $D2$ coincide.
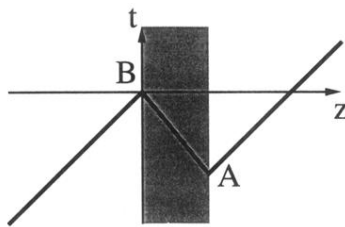
FIG. 5. Motion of the peak(s) of the pulse. The shaded region indicates the inverted medium. Point A is the exit face at which a pair of pulses appears, point B is the entrance face at which two pulses disappear due to destructive interference.