PHYSICAL REVIEW A VOLUME 49, NUMBER 4 APRIL 1994

Similarity reduction and perturbation solution of the stimulated-Raman-scattering equations in the presence of dissipation

D. Levi,[1] C. R. Menyuk,[2] and P. Winternitz[3]
[1]*Dipartimento di Fisica, Università di Roma III e Istituto Nazionale di Fisica Nucleare, Sezione di Roma, piazzale Aldo Moro 2, 00185 Roma, Italy*
[2]*Department of Electrical Engineering, University of Maryland, Baltimore, Maryland 21228*
[3]*Centre de Recherches Mathématiques, Université de Montréal, Case Postale 6128-A, Montréal, Quebec, Canada H3C 3J7*
(Received 15 July 1993)

Similarity and other group-invariant solutions of stimulated Raman scattering (SRS) in the presence of dissipation are studied. The group-theoretical analysis reduces the SRS equations to ordinary differential equations which in the most interesting cases, i.e., in the case of the self-similar and soliton solutions, are studied perturbatively to derive the effect of dissipation.

PACS number(s): 42.65.Dr, 02.20.−a, 02.30.Mv, 02.30.Jr

I. INTRODUCTION

The original observation of the Raman effect by Raman and Krishnan [1] and Landsberg and Mandelstam [2] goes back to 1928. However, the observation of *stimulated,* as opposed to *spontaneous* Raman scattering, was not possible before the development of the laser, as powerful coherent sources of light did not exist previously. The first observation of stimulated Raman scattering (SRS) by Woodbury and Ng [3] quickly followed the invention of the laser.

Basic studies of the Raman effect are almost always carried out in molecular gases, because gases are far less susceptible than liquids or solids to self-focusing and have far less dispersion. Among the molecular gases H_2 and D_2 have both high gain and simple behavior, which makes them excellent candidates for basic studies.

Transient Raman interactions in H_2 and other gases were first observed by Hagenlocker, Minck and Rado [4]. In transient interactions, the light pulse has a short duration compared to T_2, the molecular deexcitation time, due to molecular collisions. Theoretical studies by Wang [5], and by Carman *et al.* [6], and further experimental studies by Carman and co-workers rapidly followed [7–9].

While good qualitative agreement was found between theory and experiment, detailed quantitative comparisons were not made. Moreover, all these experiments were carried out in the linear gain regime. Hence, pump depletion and nonlinear pump evolution were not observed. In later experiments by Duncan *et al.* [10] these defects in the early work were rectified and a careful comparison between theory and experiments showed good agreement. Unfortunately, second Stokes generation limited how far into the depleted regime the experiment could go, but more recent work by MacPherson, Swanson, and Carlsten [11] has shown that this limitation can be overcome.

Shortly after the experimental work by Duncan *et al.* [10], Hilfer and Menyuk [12,13] carried out simulations which indicated that in the highly depleted regime the system always tends toward a self-similar solution, an accordion. This result might seem surprising at first, since the system possesses a Lax pair in the transient limit [14] and one might anticipate that solitons should emerge. This does not happen in either the experiments or simulations carried out in the transient limit. Indeed, Menyuk [15] has shown that the system cannot have any permanent solitons. A key step in resolving this difficulty was a careful study of the similarity solutions by the present authors [16], in which the self-similar, or accordion solution, was identified. Shortly thereafter, Menyuk and coworkers [17,18], using an inverse-scattering approach originally developed by Kaup [19–21], were able to show that the system will always tend toward an accordion solution when there is no frequency mismatch. It was possible to relate the accordion's parameters to the initial data [17,18]. An experiment to observe accordions was suggested.

All the work described in the last three paragraphs is concerned with the limit in which pulse durations are short compared to T_2. Transient phenomena can also be observed when pulse durations are long compared to T_2. The most important of these phenomena are solitons which have only been observed when a phase flip that is short compared to T_2 is imposed upon a pulse that is long compared to T_2 [22,23]. Theory indicates that the dissipation provided by a finite T_2 plays a crucial role in soliton formation [20]. Because of this fact and because attenuation plays an important role in realistic scenarios for observing accordions, it is important to have a better understanding of its effects on the similarity solutions.

In this paper we study similarity solutions and other group-invariant solutions of the SRS equation in the presence of dissipation (ignored in our previous work [16,17]). In the physical setting under consideration, these equations, after suitable normalization, can be written as

$$\frac{\partial A_1}{\partial x} = -XA_2, \quad \frac{\partial A_2}{\partial x} = -X^* A_1 ,$$
$$\frac{\partial X}{\partial t} + gX = A_1 A_2^*, \quad g \geq 0 , \tag{1.1}$$

where A_1 and A_2 correspond to the complex pump and Stokes wave envelopes and X corresponds to the material excitation. The real positive constant g represents the attenuation due to finite T_2. The variable x denotes distance along the Raman cell and t represents retarded time.

Equation (1.1) for $g\neq 0$ is not integrable—i.e., no Lax pair exists—so that inverse-scattering techniques do not apply. The group-theoretical methods that were applied in the $g=0$ case [16] do not rely on integrability and we apply them systematically in the present article. The symmetry groups which for $g=0$ gave accordion solutions and travelling waves such as solitons and cnoidal waves still exist for $g\neq 0$. They provide reductions to ordinary differential equations (ODE's) that for $g\neq 0$ do not have the Painlevé property [24]. We are not able to solve them in terms of any known functions. Instead we analyze the reduced ODE's qualitatively and perturbatively. We also obtain phase-wave and stationary solutions.

In Sec. II we obtain the symmetry group of the SRS equations with dissipation. The group and its Lie algebra are much smaller than in the $g=0$ case. We classify the one-dimensional subgroups of the symmetry group into conjugacy classes. In Sec. III we use the individual subgroups to perform different reductions to ODE's and we solve some of the reduced equations. The effect of dissipation on travelling waves and self-similar solutions is analyzed in Sec. IV.

II. SRS EQUATIONS AND THEIR SYMMETRY GROUP

As in our previous article [16], in which we treated SRS with no dissipation, we simplify notation by putting

$$v_1=iA_1^*,\quad v_2=A_2,\quad v_3=X \tag{2.1}$$

to obtain

$$v_{1,x}=-iv_2^*v_3^*,\quad v_{2,x}=iv_3^*v_1^*,\quad v_{3,t}+gv_3=iv_1^*v_2^* . \tag{2.2}$$

The symmetry group of the SRS equations (2.2), i.e., the group of local Lie point transformations taking solutions of Eq. (2.2) into solutions can be found using standard methods [25,26]. In particular, we applied a MACSYMA program [27] that provided the determining equations for the symmetries.

From the determining equations we obtain the Lie algebra of the symmetry group (the "symmetry algebra," for short). As a basis for the symmetry algebra L we choose the following differential operators:

$$\begin{aligned}&P_1=\partial_x,\quad D=x\partial_x-\tfrac{1}{2}(\rho_1\partial_{\rho_1}+\rho_2\partial_{\rho_2}+2\rho_3\partial_{\rho_3}) ,\\ &P_0=\partial_t,\quad V=-\partial_{\phi_2}+\partial_{\phi_3},\quad U(h)=h(t)(-\partial_{\phi_1}+\partial_{\phi_2}) ,\end{aligned} \tag{2.3}$$

where $h(t)$ is an arbitrary smooth function of t, and we have introduced the moduli and phases of waves v_k:

$$v_k=\rho_k e^{i\phi_k},\quad 0\le\rho_k<\infty,\quad 0\le\phi_k<2\pi,\quad k=1,2,3 . \tag{2.4}$$

The nonzero commutation relations of the Lie algebra L are

$$[P_1,D]=P_1,\quad [P_0,U(h)]=U(\dot h) , \tag{2.5}$$

where the dot denotes a time derivative. Thus the Lie algebra is a direct sum of three Lie algebras, namely,

$$\begin{aligned}&L=L_1\oplus L_2\oplus L_3,\quad L_1=\{P_1,D\},\\ &L_2=\{V\},\quad L_3=\{P_0,U(h)\} .\end{aligned} \tag{2.6}$$

The subalgebra L_3 is infinite dimensional since $h(t)$ can be expanded into a Taylor series (or a Laurent series) involving infinitely many arbitrary coefficients. Both V and $U(h)$ generate gauge transformations, constant and time-dependent changes of phase, respectively. The Lie group transformations corresponding to the Lie algebra (2.3) are

$$\begin{aligned}&\tilde t=t-t_0,\quad \tilde x=\exp(d)(x-x_0) ,\\ &\tilde v_1(\tilde x,\tilde t)=\exp(-d/2)\exp[-i\lambda h(t)]v_1(x,t) ,\\ &\tilde v_2(\tilde x,\tilde t)=\exp(-d/2)\exp\{i[\lambda h(t)-\mu]\}v_2(x,t) ,\\ &\tilde v_3(\tilde x,\tilde t)=\exp(-d)\exp(i\mu)v_3(x,t) ,\end{aligned} \tag{2.7}$$

where $d,\lambda,\mu,x_0,t_0\in\mathbb{R}$ are the group parameters.

Comparing the Lie algebra L with the one obtained in the integrable case (i.e., $g=0$) [16], we see that the symmetry is greatly reduced. Indeed, for $g=0$ Eq. (2.2) is invariant under arbitrary reparametrizations of time, i.e., the symmetry algebra contains an element of the form

$$\tilde V(f)=f(t)\partial_t-\tfrac{1}{2}\dot f(t)(\rho_1\partial_{\rho_1}+\rho_2\partial_{\rho_2}) . \tag{2.8}$$

The presence of $g\neq 0$ restricts the arbitrary function $f(t)$ to $f(t)=1$, i.e., time translations only. From (2.3) we see that the symmetries of the SRS Eq. (2.2) are simply space and time translations, dilations, and the gauge transformations corresponding to V and $U(h)$.

We shall use the symmetry group to obtain invariant solutions of the SRS system (2.2) with $g\neq 0$. We shall construct solutions invariant under each of the one-dimensional subgroups of the symmetry group. To do this we need a list of all one-dimensional subalgebras of the symmetry algebra. This can be obtained using standard methods [26,28,29]. The result is that any one-dimensional subalgebra of the symmetry algebra L is conjugate under the symmetry group G to precisely one of the following ones:

$$\begin{aligned}&A_1(\alpha,\beta)=\{D+\alpha P_0+\beta V\},\quad \alpha\neq 0 ,\\ &A_2(\alpha,h)=\{D+\alpha V+U(h)\} ,\\ &A_3(\epsilon,\beta)=\{P_0-\epsilon P_1+\beta V\},\quad \epsilon=\pm 1 ,\\ &A_4(k,h)=\{P_1+kV+U(h)\},\quad k=0,\pm 1 ,\\ &A_5(\alpha)=\{P_0+\alpha V\} ,\\ &A_6(h)=\{U(h)\},\quad h\neq 0 ,\end{aligned} \tag{2.9}$$

where α and β are arbitrary real constants and $h(t)$ is an arbitrary function of t.

III. INVARIANT SOLUTIONS

Let us now run through the list of subalgebras (2.9) and use each of them to reduce the SRS Eq. (2.2) to a system of ODE's. These we decouple and, whenever possible, solve. The difference between the present case and the integrable case with $g=0$ treated in Ref. [16] is that the reduced equations for $g\neq 0$ do not necessarily have the Painlevé property [24]. The solutions may hence have movable critical points, e.g., logarithmic branch points, the positions of which depend on the initial conditions. Nonlinear ODEs that do not have the Painlevé property are much more difficult to integrate than those that do.

The groups corresponding to $A_1,\ldots,A_5$ have a nontrivial action on space-time and hence provide reductions to ODE's. A_6, on the other hand, is a purely gauge transformation and will not lead to a reduction.

The SRS system (2.2) allows a first integral providing the x-independent quantity

$$\tilde{I}_1=|v_1|^2+|v_2|^2,\quad \frac{\partial\tilde{I}_1}{\partial x}=0\ , \tag{3.1}$$

which we shall use throughout. We shall only be interested in "nontrivial" solutions, satisfying

$$v_1v_2v_3\neq 0\ . \tag{3.2}$$

A. Algebra $A_1(\alpha,\beta)$ and self-similar solutions

A solution invariant under the group generated by $D+\alpha P_0+\beta V$ will have the form

$$\begin{aligned} v_1&=\exp\left[-\frac{t}{2\alpha}\right]R_1(\xi)\exp[i\psi_1(\xi)],\quad \xi=x\exp\left[-\frac{1}{\alpha}t\right],\\ v_2&=\exp\left[-\frac{t}{2\alpha}\right]R_2(\xi)\exp\left[i\left[\psi_2(\xi)-\frac{\beta}{\alpha}t\right]\right],\quad \alpha\neq 0\ ,\\ v_3&=\frac{1}{x}\xi^{\alpha g}R_3(\xi)e^{i[\psi_3(\xi)+\beta\ln x]}\ , \end{aligned}\tag{3.3}$$

where R_i and ψ_i are real functions of ξ and the factor $\xi^{\alpha g}$ in v_3 was separated out from $R_3(\xi)$ for future convenience. We substitute (3.3) into the SRS Equations (2.2), separate the real and imaginary parts, and obtain the six real ODE's:

$$\begin{aligned} &\dot{R}_1=-\xi^{\alpha g-1}R_2R_3\sin\psi,\quad R_1\dot{\psi}_1=-\xi^{\alpha g-1}R_2R_3\cos\psi\ ,\\ &\dot{R}_2=\xi^{\alpha g-1}R_3R_1\sin\psi,\quad R_2\dot{\psi}_2=\xi^{\alpha g-1}R_3R_1\cos\psi\ ,\\ &\dot{R}_3=-\alpha\xi^{-\alpha g}R_1R_2\sin\psi,\quad R_3\dot{\psi}_3=-\alpha\xi^{-\alpha g}R_1R_2\cos\psi\ ,\\ &\psi=\psi_1+\psi_2+\psi_3+\beta\ln\xi\ . \end{aligned}\tag{3.4}$$

The reduced system has a first integral inherited from formula (3.1), namely,

$$I_1=R_1^2+R_2^2\ . \tag{3.5}$$

For $g=0$ another first integral exists, namely,

$$\tilde{I}_2=R_1R_2R_3\cos\psi-\frac{\beta}{2}R_1^2\ . \tag{3.6}$$

For $\beta\neq 0$ this integral does not survive if $g\neq 0$. Now, in order to get another first integral and thus be able to decouple Eqs. (3.4), we shall restrict ourselves to the case

$$\beta=0\ . \tag{3.7}$$

Then we have

$$\psi=\psi_1+\psi_2+\psi_3\ , \tag{3.8}$$

and

$$I_2=R_1R_2R_3\cos\psi \tag{3.9}$$

is a first integral (i.e., $I_{1,\xi}=I_{2,\xi}=0$).

We express R_2, R_3, and ψ in terms of R_1 as

$$\begin{aligned} R_2&=\sqrt{I_1-R_1^2}\ ,\\ R_3\cos\psi&=\frac{I_2}{R_1\sqrt{I_1-R_1^2}}\ ,\\ R_3\sin\psi&=-\frac{\xi^{-\alpha g+1}\dot{R}_1}{\sqrt{I_1-R_1^2}}\ . \end{aligned}\tag{3.10}$$

We differentiate the first equation in the system (3.4) and obtain an ODE for R_1:

$$\begin{aligned} \ddot{R}_1=&\frac{R_1\dot{R}_1^2}{I_1-R_1^2}+\frac{\alpha g-1}{\xi}\dot{R}_1-\frac{I_2^2\xi^{2(\alpha g-1)}}{R_1(I_1-R_1^2)}\\ &+\frac{\alpha}{\xi}R_1(I_1-R_1^2)+\frac{I_2^2\xi^{2(\alpha g-1)}}{R_1^3}\ . \end{aligned}\tag{3.11}$$

To simplify Eq. (3.11) we perform the transformation [16] that in the case when we have $g=0$ would take Eq. (3.11) into the equation for the fifth Painlevé transcendent [30] P_V, namely,

$$R_1=\left[I_1\frac{W}{W-1}\right]^{1/2}\ . \tag{3.12}$$

The function $W(\xi)$ satisfies

$$\begin{aligned} \ddot{W}=&\left[\frac{1}{2W}+\frac{1}{W-1}\right]\dot{W}^2+\frac{\alpha g-1}{\xi}\dot{W}+\frac{2\alpha}{\xi}I_1W\\ &+\frac{2I_2^2}{I_1^2}\xi^{2(\alpha g-1)}(W-1)^2\left[-W+\frac{1}{W}\right]\ . \end{aligned}\tag{3.13}$$

For $g=0$ Eq. (3.13) does indeed reduce to the equation for P_V $(\xi,\alpha,\beta,\gamma,\delta=0)$ [16]. For $g\neq 0$ Eq. (3.9) does not pass the Painlevé test [24,31]. The "offensive" ξ dependence due to the fact that $g\neq 0$ cannot be transformed away. Still less can the solution be expressed in terms of any known functions.

The asymptotic behavior of the solutions of Eq. (3.13) will be discussed in Sec. IV. It is worth mentioning that a different variable was used in Ref. [16], namely,

$$\tilde{\xi}=xt^{\alpha},\quad \alpha=-\epsilon=\mp 1\ . \tag{3.14}$$

This is due to the fact that for $g=0$ the subalgebra

$D+\alpha P_0$ is equivalent to $D+\tilde{V}(f)$ for any $f(t)\neq 0$, with $\tilde{V}(f)$ as in Eq. (2.8). In Ref. [16] we chose $f(t)=\epsilon t$ so as to get the similarity variable (3.14). For $g\neq 0$ this choice is not available, since we have $f=\text{const}$.

The corresponding equations of Ref. [16] can be transformed into those of this section (with $g=0$) by the group transformation (allowed for $g=0$)

$$\tilde{t}=\ln t,\quad \tilde{x}=x\ ,$$
$$\tilde{R}_{1,2}(\tilde{x},\tilde{t})=\sqrt{t}R_{1,2}(x,t),\quad \tilde{R}_3(\tilde{x},\tilde{t})=R_3(x,t)\ . \tag{3.15}$$

A difficulty with interpreting the similarity solution physically for $g\neq 0$ is that a reparametrization of time no longer leaves the SRS equations (2.2) invariant. Hence it is not possible to transform any initial pulse shape into a constant, as was the case for $g=0$. In view of Eq. (3.1) only very special initial pulses will develop into the self-similar solution for $g\neq 0$. Moreover, the similarity solution corresponds to the situation in which the waves have "lost memory" of their initial condition. For a pulse that is zero when $t<0$ we expect the self-similar solution to be observed for $gt\gg 1$. More work, both analytical and computational, is needed, before definitive conclusion about the physical role of self similar solutions for $g\neq 0$ can be drawn.

B. Algebra $A_3(\alpha,\beta)$ and traveling-wave solutions

We take $A_3(\alpha,\beta)=\{P_0-\alpha P_1+\beta V\}$, where $\alpha\neq 0$; we could actually scale α to $\alpha=\epsilon=\pm 1$ as in Eq. (2.9), but we find it convenient to keep track of $\alpha\neq 0$ as a velocity. The reduction formulas are

$$\begin{aligned}
&v_1=R_1(\xi)\exp[i\psi_1(\xi)],\quad \xi=x+\alpha t\ ,\\
&v_2=R_2(\xi)\exp\{i[\psi_2(\xi)-\beta t]\}\ ,\\
&v_3=\exp[-(g/\alpha)\xi]R_3(\xi)\exp\left[i\left[\psi_3(\xi)-\frac{\beta}{\alpha}t\right]\right]\ ,
\end{aligned}\tag{3.16}$$

where $\alpha\neq 0$, and the exponential in v_3 was introduced for convenience. The reduced equations are

$$\begin{aligned}
&\dot{R}_1=-\exp\left[-\frac{g}{\alpha}\xi\right]R_2R_3\sin\psi\ ,\\
&R_1\dot{\psi}_1=-\exp\left[-\frac{g}{\alpha}\xi\right]R_2R_3\cos\psi\ ,\\
&\dot{R}_2=\exp\left[-\frac{g}{\alpha}\xi\right]R_3R_1\sin\psi\ ,\\
&R_2\dot{\psi}_2=\exp\left[-\frac{g}{\alpha}\xi\right]R_3R_1\cos\psi\ ,\\
&\dot{R}_3=\frac{1}{\alpha}\exp\left[\frac{g}{\alpha}\xi\right]R_1R_2\sin\psi\ ,\\
&R_3\dot{\psi}_3=\frac{1}{\alpha}\exp\left[\frac{g}{\alpha}\xi\right]R_1R_2\cos\psi\ ,\\
&\psi=\psi_1+\psi_2+\psi_3-\frac{\beta}{\alpha}\xi\ .
\end{aligned}\tag{3.17}$$

As in the case of the algebra $A_1(\alpha,\beta)$ corresponding to the self-similar solutions, only one simple first integral of the system (3.17) exists for $g\neq 0$, $\beta\neq 0$. In order to be able to decouple the system in a simple manner for $g\neq 0$, we again make the restriction $\beta=0$, so that we have Eq. (3.8) again. In this case we again have the two first integrals I_1 and I_2 of Eqs. (3.5) and (3.9).

We again express R_2, R_3, and ψ in terms of R_1:

$$R_2=(I_1-R_1^2)^{1/2} \tag{3.18}$$

$$\begin{aligned}
&R_3\cos\psi=\frac{I_2}{R_1(I_1-R_1^2)^{1/2}}\\
&R_3\sin\psi=-\frac{\dot{R}_1\exp\left[\frac{g}{\alpha}\xi\right]}{(I_1-R_1^2)^{1/2}}\ .
\end{aligned}\tag{3.19}$$

For R_1 we obtain a second-order equation

$$\begin{aligned}
\ddot{R}_1=&-\frac{1}{R_1(I_1-R_1^2)}\left[R_1^2\dot{R}_1^2+I_2^2\exp\left[-\frac{2g}{\alpha}\xi\right]\right]\\
&-\frac{1}{\alpha}R_1(I_1-R_1^2)-\frac{g}{\alpha}\dot{R}_1+\frac{I_2^2}{R_1^3}\exp\left[-\frac{2g\xi}{\alpha}\right]\ .
\end{aligned}\tag{3.20}$$

Equation (3.20) does not have the Painlevé property, even for $g=0$. However, if we put

$$R_1=\sqrt{H}\ , \tag{3.21}$$

we obtain

$$\begin{aligned}
\ddot{H}=&\left[\frac{1}{H}-\frac{1}{I_1-H}\right]\left[\frac{1}{2}\dot{H}^2+2I_2^2\exp\left[-\frac{2g\xi}{\alpha}\right]\right]\\
&-\frac{g}{\alpha}\dot{H}-\frac{2}{\alpha}H(I_1-H)\ .
\end{aligned}\tag{3.22}$$

For $g=0$ Eq. (3.22) does have the Painlevé property. Moreover, in this case, it has a first integral that is quadratic in $\dot{H}$ and provides a first-order equation for H. Denoting

$$H_0=H|_{g=0}\ , \tag{3.23}$$

we rewrite the equation for H_0 as

$$\dot{H}_0^2=\frac{4}{\alpha}(H_0-Z_1)(H_0-Z_2)(H_0-Z_3)\ , \tag{3.24}$$

where the constants Z_i satisfy

$$\begin{aligned}
&Z_1+Z_2+Z_3=2I_1+\frac{2\alpha I_2^2}{I_1^2}-\frac{K\alpha}{4}\ ,\\
&Z_1Z_2+Z_2Z_3+Z_3Z_1=I_1^2+\frac{2\alpha I_2^2}{I_1}-\frac{KI_1\alpha}{4}\ ,\\
&Z_1Z_2Z_3=\alpha I_2^2\ .
\end{aligned}\tag{3.25}$$

The solutions for H_0 were given earlier [16]. Since we shall need them below, let us reproduce them here in the notations that we are now using. We have the following.

Solitary waves:
(1) For $\alpha>0$, $Z_1\le H_0\le Z_2=Z_3$,

$$H_0=Z_2-\frac{Z_2-Z_1}{\cosh^2\left[\frac{Z_2-Z_1}{\alpha}\right]^{1/2}(\xi-\xi_0)} . \qquad (3.26)$$

(2) For $\alpha<0$, $0=Z_1=Z_2\le H_0\le Z_3=I_1=K\alpha/4$, $K<0$,

$$H_0=\frac{I_1}{\cosh^2\left[\frac{I_1}{-\alpha}\right]^{1/2}(\xi-\xi_0)} . \qquad (3.27)$$

Periodic (cnoidal) waves:
(3) For $\alpha>0$, $0\le Z_1\le H_0\le Z_2<Z_3$,

$$H_0=Z_1+(Z_2-Z_1)\,\mathrm{sn}^2[p(\xi-\xi_0),k] ,$$
$$p=\left[\frac{Z_3-Z_1}{\alpha}\right]^{1/2},\quad k^2=\frac{Z_2-Z_1}{Z_3-Z_1} . \qquad (3.28)$$

(4) For $\alpha<0$, $Z_1<0<Z_2<H_0<Z_3$,

$$H_0=Z_3-(Z_3-Z_2)\,\mathrm{sn}^2[p(\xi-\xi_0),k] ,$$
$$p=\left[\frac{Z_3-Z_1}{-\alpha}\right]^{1/2},\quad k^2=\frac{Z_3-Z_2}{Z_3-Z_1} . \qquad (3.29)$$

When dissipation is present—$g\neq 0$—Eq. (3.22) no longer allows the first integral K, so the solitary and periodic waves are no longer exact solutions.

Moreover, it can easily be shown that Eq. (3.22) does not have the Painlevé property for $g\neq 0$ and that the ξ dependence cannot be transformed away. This equation cannot be solved in terms of any known functions, nor can it be linearized. Some properties of solutions will be discussed in Sec. IV.

C. Algebra $A_4(k,h)$ and phase-wave solutions

The reduction formulas in this case are

$$\begin{aligned} v_1&=R_1(t)\exp\{i[\psi_1(t)-h(t)x]\} ,\\ v_2&=R_2(t)\exp(i\{\psi_2(t)+[h(t)-k]x\}) ,\\ v_3&=R_3(t)\exp\{i[\psi_3(t)+kx]\} . \end{aligned} \qquad (3.30)$$

Substituting into the SRS equation (2.2) we find that nontrivial solutions exist only for a specific function $h(t)$, namely,

$$h(t)=\tfrac{1}{2}\{k+\sqrt{k^2+4\exp[-2g(t-t_0)]}\} . \qquad (3.31)$$

We obtain the phase-wave solutions,

$$\begin{aligned} &R_1=R_1(t),\quad R_2=\left[1+\frac{k+\sqrt{k^2+4\exp[-2g(t-t_0)]}}{2}\right]^{1/2}R_1(t),\quad R_3=\exp[-g(t-t_0)] ,\\ &\psi_1=\psi_1(t),\quad \psi_2=-\psi_1-\psi_3 ,\\ &\psi_3(t)=\int\left[1+\frac{k+\sqrt{k^2+4\exp[-2g(t-t_0)]}}{2}\right]^{1/2}R_1^2(t)\exp[g(t-t_0)]dt , \end{aligned} \qquad (3.32)$$

where $R_1(t)$ and $\psi_1(t)$ are arbitrary functions; $k=0,\pm1$, and t_0 is an arbitrary constant. The entire x dependence is in the phases in Eq. (3.30). For $g=0$ we have $h(t)=k=0,\pm1$, in agreement with Ref. [16].

D. Algebra $A_5(\alpha)$

The reduction formulas are

$$\begin{aligned} v_1&=R_1(x)\exp(i\psi_1(x)) ,\\ v_2&=R_2(x)\exp(i[\psi_2(x)-\alpha t]) ,\\ v_3&=R_3(x)\exp(i[\psi_3(x)+\alpha t]) . \end{aligned} \qquad (3.33)$$

The reduced ODEs are easy to solve and we obtain

$$\begin{aligned} R_1&=\sqrt{I_1}\exp(-\mu(x-x_0))(1+\exp[-2\mu(x-x_0)])^{-1/2},\\ R_2&=\sqrt{I_1}(1+\exp[-2\mu(x-x_0)])^{-1/2} ,\\ R_3&=\frac{I_1\exp(-\mu(x-x_0))}{\sqrt{\alpha^2+g^2}}(1+\exp[-2\mu(x-x_0)])^{-1} ,\\ \mu&=\frac{gI_1}{\alpha^2+g^2},\quad \alpha^2+g^2\neq 0 ,\\ \psi_1&=-\frac{\alpha x}{\alpha^2+g^2}-\frac{\alpha}{2g}\ln(1+\exp[-2\mu(x-x_0)])+\psi_1^0 ,\\ \psi_2&=-\frac{\sqrt{\alpha^2+g^2}}{2g}\ln(1+\exp[-2\mu(x-x_0)])+\psi_2^0 ,\\ \psi_3&=-\psi_1-\psi_2+\arctan\frac{g}{\alpha} . \end{aligned} \qquad (3.34)$$

For $g=0$ the result simplifies and we obtain plane waves [16]. For $\alpha=0$ we obtain well-known stationary solutions. The algebra $A_2(\alpha,h)$ was left out in this section since it leads to trivial solutions with $v_1=v_2=0$.

IV. DISCUSSION OF SOLUTIONS

The two cases to discuss are the traveling-wave and self-similar solutions. As was pointed out in Sec. III, in these cases the solutions of the reduced ODE cannot be expressed in terms of known special or elementary functions. We shall apply a perturbative approach in these cases.

A. Traveling-wave solution

At first let us consider the dissipation coefficient g/α to be small, positive, but nonzero. We can then expand the exponentials in Eq. (3.22) and keep only the first terms. We also expand the solution putting

$$H=H_0(\xi)+gH_1(\xi)+\cdots \quad (4.1)$$

The leading term H_0 satisfies Eq. (3.24). More specifically, H_0 is a real positive and finite solution and hence has one of the forms (3.26), . . . , (3.29). In the linear approximation in g we obtain a linear equation for $H_1(\xi)$, namely,

$$\ddot{H}_1\left[-\frac{1}{H_0}+\frac{1}{I_1-H_0}\right]\dot{H}_0\dot{H}_1+\left\{\left[\frac{1}{H_0^2}+\frac{1}{(I_1-H_0)^2}\right]\left[\frac{\dot{H}_0^2}{2}+2I_2^2\right]+\frac{2}{\alpha}(I_1-2H_0)\right\}H_1 +\frac{1}{\alpha}\dot{H}_0-\frac{4}{\alpha}I_2^2\xi\left[-\frac{1}{H_0}+\frac{1}{I_1-H_0}\right]=0\,. \quad (4.2)$$

For soliton solutions of the $g=0$ equation, i.e., H_0 as in Eq. (3.26), Eq. (4.2) is quite simple. Indeed the correction term in this case satisfies the equation

$$\ddot{H}_1-2A\frac{2-\cosh^2A\xi}{\sinh A\xi\cosh A\xi}\dot{H}_1+2A^2\frac{1}{\sinh^2A\xi}H_1+\frac{2I_1A}{\alpha}\frac{\sinh A\xi}{\cosh^3A\xi}=0,\quad A=\left[\frac{I_1}{|\alpha|}\right]^{1/2}. \quad (4.3)$$

The general solution of Eq. (4.3) is

$$H_1=c_1\frac{\sinh A\xi}{\cosh^3A\xi}+c_2\left[A\xi\frac{\sinh A\xi}{\cosh^3A\xi}+\frac{\sinh^2A\xi}{\cosh^2A\xi}\right]-\frac{I_1}{A}\frac{\sinh A\xi}{\cosh A\xi}, \quad (4.4)$$

where c_1 and c_2 are arbitrary constants. This correction term can be viewed as modifying the constant background of the soliton, equal to Z_1 for H_0 as in (3.26). It must however be remembered that the expansion (4.1) is only meaningful for $0<(g\xi/\alpha)<1$, i.e., for values of ξ such that $0<\xi<\alpha/g$ as $\alpha>0$, i.e., for small ξ. Similar results can be obtained for the "cnoidal wave" solutions (3.28).

More meaningful results can be obtained when looking for asymptotic solutions, i.e., for $\xi\to\infty$. We are interested in the asymptotic behavior of R_1, solution of Eq. (3.20), and as before we consider $\alpha>0$. Applying the Boutroux transformation [30] as was done in Ref. [16] for $g=0$, we get that

$$R_1(\xi)=\sqrt{v(\xi)}\exp(-g\xi/2\alpha)\,, \quad (4.5)$$

and $v(\xi)$ will satisfy the following nonlinear ODE:

$$v_{\xi\xi}-\frac{1}{2}\frac{(v_\xi)^2}{v}+\left[\frac{2}{\alpha}I_1-\frac{1}{2}\frac{g^2}{\alpha^2}\right]v-2\frac{I_2^2}{v}+\frac{E}{I_1}\left[-v_{\xi\xi}v+\left[v_\xi\right]^2-\frac{g}{\alpha}vv_\xi-\left[\frac{4}{\alpha}I_1-\frac{g^2}{\alpha^2}\right]v^2+4I_2^2\right]+\frac{2}{\alpha I_1}v^3E^2=0,$$

$$E=\exp(-g\xi/\alpha)\,, \quad (4.6)$$

where for $\xi \to +\infty$ and g/α positive E is a small quantity. We can thus expand v in powers of E,

$$v = v_0(\xi) + E v_1(\xi) + \cdots ,$$

and at lowest order we get

$$v_0 = \frac{I_2}{C}\{\cosh\gamma + \sinh\gamma \sin 2C(\xi - \xi_0)\} ,$$

$$C = \left[\frac{I_1}{\alpha} - \frac{g^2}{4\alpha^2}\right]^{1/2} , \quad (4.7)$$

where γ and ξ_0 are two integration constants.

This is a bounded solution for $g < 2\sqrt{\alpha I_1}$ while for $g > 2\sqrt{\alpha I_1}$ it is exponentially diverging. However, as one can easily see from the numerical integration of Eq. (3.20) plotted in Fig. 1 for $g = 0$, Fig. 2 for $g = 0.1$, and Fig. 3 for $g = 2.2$, and deduce from Eq. (4.5), the asymptotic solution for $R_1(\xi)$ is always bounded and for $g < 2\sqrt{\alpha I_1}$ goes to zero oscillating, while for $g > 2\sqrt{\alpha I_1}$ goes exponentially to zero.

B. Self-similar solutions

Our starting point in this case is Eq. (3.11) for the real function $R_1(\xi)$. We are interested in the asymptotic behavior of $R_1(\xi)$ for $\xi \to +\infty$. More specifically, since $R_1(\xi)$ is related to the pump amplitude v_1, we require $R_1(\xi) \to 0$ for $\xi \to +\infty$.

By analogy with the case $g = 0$ of Ref. [16] we look for a decreasing behavior of the type

$$R_1 = A\xi^p, \quad A, p, \text{const}, p < 0 , \quad (4.8)$$

substitute into Eq. (3.11) and compare leading terms for $\xi \to +\infty$. Thus we find

$$p = -\frac{1}{4} + \frac{\alpha g}{2}, \quad \alpha g < \frac{1}{2} . \quad (4.9)$$

In order to obtain an asymptotic expansion we perform the transformation

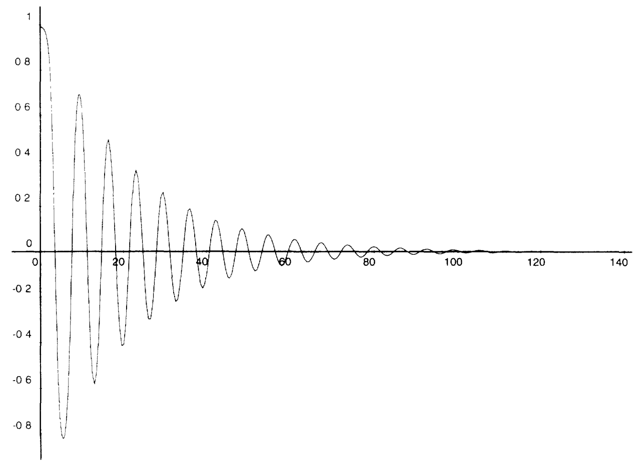

FIG. 2. Numerical solution of Eq. (3.20) for $g = 0.1$ with the same initial conditions and values of parameters as in Fig. 1.

$$R_1(\xi) = \xi^p \sqrt{F(\eta)}, \quad \eta = 2\sqrt{\xi} , \quad (4.10)$$

where $F(\xi)$ will be expanded into a series

$$F(\eta) = \sum_{\{n\}} F_n(\eta) \frac{1}{\eta^n} , \quad (4.11)$$

with $F_n(\eta)$ finite for $\eta \to +\infty$ and n a sequence of increasing positive numbers starting with zero. The redefinition of the independent variable ($\xi \to \eta$) is performed to obtain an equation with constant coefficients for the leading term $F_0(\eta)$. The overall leading asymptotic behavior is extracted in the factor ξ^p, with p the same as in (4.9). Finally the square root in the transformation $R_1 \to F$ was needed to simplify the equation for the leading term. Moreover, for $g = 0$, as we saw before, this square root is needed to obtain an equation having the Painlevé property.

Substituting (4.10) into Eq. (3.11) we obtain an equation for $F(\eta)$:

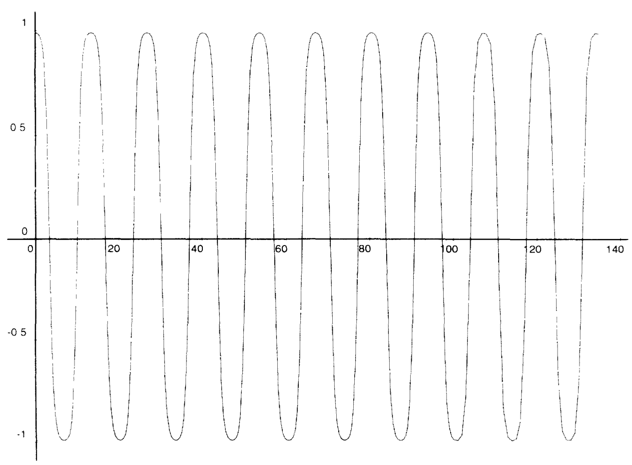

FIG. 1. Numerical solution of Eq. (3.20) for $g = 0$ with $I_2 = 0$, $I_1 = 1$, and $\alpha = 1$. As initial conditions we put $R_1(0) = 0.99$ and $\dot{R}_1(0) = 0$.

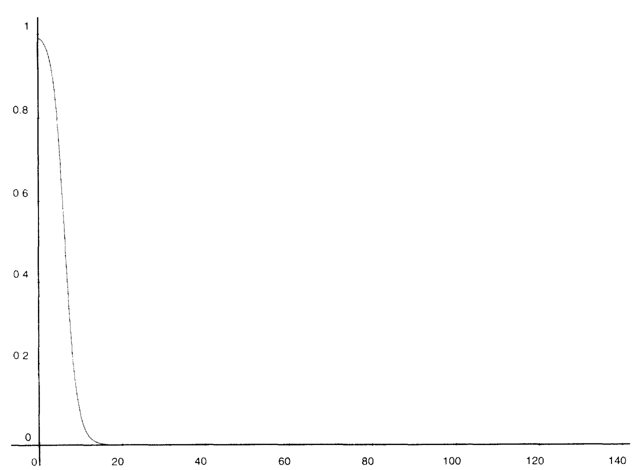

FIG. 3. Numerical solution of Eq. (3.20) for $g = 2.2$ with the same initial conditions and values of parameters as in Fig. 1.

$$F\ddot{F}-\frac{\dot{F}^2}{2}-2\alpha I_1F^2-2I_2^2+\frac{1}{\eta}\left\{\frac{1}{I_1}2^{1-2\alpha g}\eta^{2\alpha g}[-F^2\ddot{F}+F\dot{F}^2+4I_2^2F+4I_1\alpha F^3]\right\}$$
$$+\frac{1}{\eta^2}\left\{\frac{1}{2}(1-4\alpha^2g^2)F^2-\frac{1}{I_1}2^{1-2\alpha g}\eta^{2\alpha g}(1-2\alpha g)F^2\dot{F}-\frac{\alpha}{I_1}2^{3-4\alpha g}\eta^{4\alpha g}F^4\right\}$$
$$+\frac{1}{\eta^3}\left\{\frac{-1}{I_1}2^{2-2\alpha g}(1-2\alpha g)\alpha g\,\eta^{2\alpha g}F^3\right\}=0 . \qquad (4.12)$$

For $g=0$ in (4.12) the first three terms are of the order η^0, η^{-1}, and η^{-2}, respectively, and the η^{-3} term is absent. For $g\neq 0$ we do not wish to interchange the order of terms and so we impose the further restriction

$$\alpha g > -\tfrac{1}{2} . \qquad (4.13)$$

We now substitute the expansion (4.11) into Eq. (4.12). The equation for the leading term does not involve the dissipative constant g and as in Ref. [16] we obtain

$$F_0(\eta)=\frac{-2I_2}{\sqrt{-\alpha I_1}}[\cosh\delta+\sinh\delta\sin 2\sqrt{-\alpha I_1}(\eta-\eta_0)] , \qquad (4.14)$$

where δ and η_0 are integration constants. This solution is real and bounded for

$$\alpha<0 . \qquad (4.15)$$

In order to go beyond the leading term (4.14) for $g\neq 0$ we shall look for a solution of the form

$$F(\eta)=F_0(\theta)+\eta^{2\alpha g-1}F_1(\theta)+o(\eta^{2\alpha g-1}) , \qquad (4.16)$$

where $\theta(\eta)$ is given by

$$\theta=2\sqrt{-\alpha I_1}\,\eta+2^{-2\alpha g}\frac{I_2}{\alpha g I_1}[\eta^{2\alpha g}-1] . \qquad (4.17)$$

The term $\eta^{2\alpha g}$ in θ is introduced to avoid secular terms in the first-order term of the expansion while the constant term is added to provide a reasonable limit as $g\to 0$.

Introducing this ansatz for $F(\eta)$ in Eq. (4.12) we get

$$F_1(\theta)=C_1[\sin\theta+\tanh\delta]+C_2\cos\theta$$
$$+\frac{I_2^2\sinh(2\delta)}{2^{2\alpha g+1}\alpha I_1^2}\sin\theta+\frac{I_2^2\sinh^2\delta}{2^{2\alpha g}\alpha I_1^2}\sin^2\theta . \qquad (4.18)$$

We see that this first-order correction to $F_0(\theta)$ is bounded for any choice of the parameters C_1, C_2, and δ. We also note that in the limit of no dissipation ($g\to 0$) we have

$$\theta=2\sqrt{-\alpha I_1}\,\eta+2\frac{I_2}{I_1}\ln(\eta) . \qquad (4.19)$$

To illustrate this result and to show the influence of the dissipative parameter g, we have plotted a numerical solution of Eq. (3.11) for $g=0$ and $g=0.49$ on Figs. 4 and 5, respectively. As one can see by comparing Fig. 4 with Fig. 5, the presence of the dissipation decreases the amplitude of $R_1(\xi)$ and slightly decreases the frequency of the oscillations.

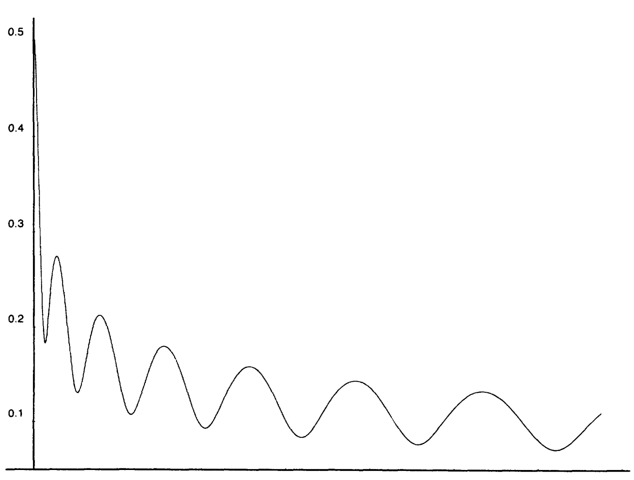

FIG. 4. Numerical solution of Eq. (3.11) for $g=0$ with $I_2=0.1$, $I_1=1$, and $\alpha=-1$. As initial conditions we put $R_1(0)=0.5$ and $\dot{R}_1(01)=0$.

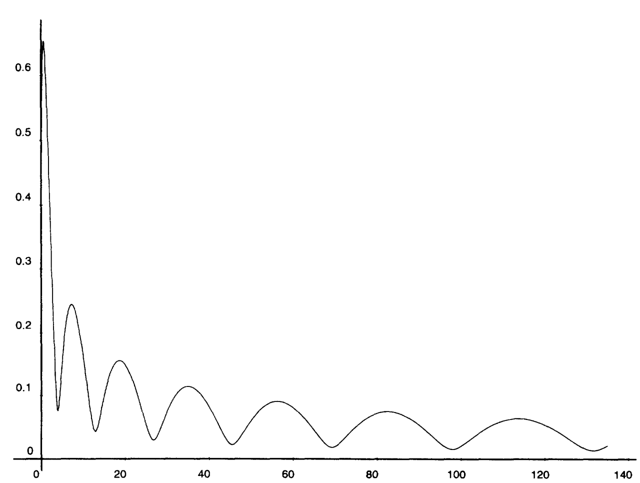

FIG. 5. Numerical solution of Eq. (3.11) for $g=0.49$ with the same initial conditions and values of parameters as in Fig. 4.

ACKNOWLEDGMENTS

We thank A. R. Its for helpful discussions. The research of D.L. was partially supported by the Italian MURST, that of P.W. by grants from NSERC of Canada and FCAR du Québec. The work of C.R.M. was supported by the National Science Foundation, Grant No. ECS-9114191.

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