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Simultaneous fourth-order squeezing of both quadrature components

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It is shown that a particular combination of so-called *even coherent states,* dubbed the *orthogonal-even coherent state,* while exhibiting no "ordinary" or second-order squeezing, can furnish near-optimal simultaneous-quadrature fourth-order squeezing [where the latter term is defined in the sense of Hong and Mandel, Phys. Rev. Lett. **54**, 323 (1985)]. The Wigner function of the particular quantum-mechanical superposition of states is calculated and various plots compared to the Wigner function of a simple statistical mixture of the same states. Rather uninteresting behavior is found in the latter case, which illustrates that the higher-order squeezing effect is due to nonclassical, quantum-mechanical "interference in phase space."

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I. INTRODUCTION

Some time ago Hong and Mandel [1] generalized the standard concept of *squeezing* [2], which involves second-order moments, to arbitrary (even) Nth-order moments of the electromagnetic field. Their criterion for higher-order squeezing of the quadrature field components X_1, X_2 (to be defined below) is

$$\langle (\Delta X_i)^N \rangle < \langle (\Delta X_i)^N \rangle_{\text{coh}} \quad (i = 1 \text{ or } 2) , \tag{1}$$

i.e., the Nth-order fluctuation in the given state is less than that which obtains in the coherent state (which, of course, includes the important special case of the vacuum).

In a Comment [3] following publication of their papers, I was able to display a state infinitesimally different from the coherent state, in which, by Hong and Mandel's criterion, *both* quadrature components are infinitesimally squeezed in fourth order ($N=4$). Subsequently [4] I produced a state with *finite* simultaneous-quadrature fourth-order squeezing.

For $N=2$ ("ordinary" squeezing) one is unable to construct such simultaneous-quadrature component-squeezed states, since, as is well known, in the coherent state the uncertainty product $U_2 \equiv \langle (\Delta X_1)^2 \rangle \langle (\Delta X_2)^2 \rangle$ takes on its minimal value. However, such simultaneous-quadrature component-squeezed states exist in Hong and Mandel's higher-order squeezing scheme for the simple reason that the coherent state is *not* the minimum-uncertainty state for the Nth-order uncertainty product $U_N \equiv \langle (\Delta X_1)^N \rangle \langle (\Delta X_2)^N \rangle$, when $N \neq 2$. It is of interest to inquire what the minimum values are for $U_N, N \neq 2$, and this has been explored numerically in a series of papers by myself and Mavromatis [5]. In the interests of successfully pursuing this work the problem was abstracted and moved onto a more or less purely mathematical plane.

In the course of the research much has been learned about such higher-order uncertainty products. However, during these endeavors the original quantum-optics origins of the problem have become almost completely attenuated. For example, these somewhat abstract numerical calculations involved a general superposition of number states, each of whose weighting in the overall trial wave function was varied to achieve a numerical minimum for simultaneous-quadrature squeezing. Hence although this work originated with Hong and Mandel's [1] generalization of "ordinary" *optical* squeezing, those interested in optical problems might well question the relevance to their field of the results [4,5] ultimately found. Thus the motivation of the current research was to discover whether a connection could be established between the abstract, variational states and states of common currency in quantum optics, viz., the *coherent states* [6]. The attempt was in fact successful, for it is found that a certain superposition of coherent states also leads to simultaneous-quadrature-component higher-order squeezing. In the rest of this paper these states will be described and some of their properties explored.

II. ORTHOGONAL-EVEN COHERENT STATES

The *even coherent states* [7] are defined by

$$|\alpha\rangle\rangle \equiv A^{1/2}[|\alpha\rangle + |-\alpha\rangle] . \tag{2}$$

Here the $|\alpha\rangle$ kets are eigenstates of the single-mode electromagnetic field operator a, the well-known coherent states [6], whence the normalization constant A is easily found to be

$$A = \frac{1}{2[1+e^{-2|\alpha|^2}]} . \tag{3}$$

These states have been utilized by Hillery [8] in the study of a form of higher-order squeezing that differs completely from the Hong-Mandel version, termed by Hillery as *amplitude-squared squeezing.* For completeness one might mention that a third flavor of higher-order squeezing has been proposed by Braunstein and McLachlan [9]. However, aside from mentioning these two latter schemes, in this paper only the Hong-Mandel form of higher-order squeezing will be considered.

Bužek, Knight, and Barranco [10] have examined some properties of the even coherent states in an excellent semitutorial paper. For completeness we mention that Gerry [11] (see also Zhu, Wang, and Li [12]) has studied the nonclassical properties of these states, and their consideration in the context of "Schrödinger cat states" is found in a recent paper by Bužek, Gantsog, and Kim [13]. Finally, Schleich, Pernigo, and Le Kien [14] consider a generalized form of the superposition in Eq. (2), which reveals some highly interesting features of these states.

In particular it is found [10,14] that the even coherent states exhibit "ordinary" ($N=2$) squeezing. Furthermore it has been shown by Bužek, Jex, and Quang [15] that these states also display fourth-order squeezing. In this paper we will examine the second- and fourth-order squeezing properties of a particular superposition of even coherent states

$$|\phi\rangle\rangle \equiv B^{1/2}[|\alpha\rangle\rangle + |i\alpha\rangle\rangle] , \tag{4}$$

where it can be shown that the normalization constant B satisfies

$$B = \frac{\cosh|\alpha|^2}{2[\cosh|\alpha|^2 + \cos|\alpha|^2]} . \tag{5}$$

In this paper the $|\phi\rangle\rangle$ states will be termed *orthogonal-even coherent states.* [The origin of this name is revealed by the study of the defining Eq. (4). In the complex α plane the vector representing $i\alpha$ is rotated 90 degrees from α, hence the even coherent state $|i\alpha\rangle\rangle$ is "orthogonal" to the state $|\alpha\rangle\rangle$. Alternatively one might follow the terminology of Jex and Bužek [16], who in the context of so-called *multiphoton coherent states* would term these *four-photon states.*]

III. HIGHER-ORDER SQUEEZING CALCULATIONS

Given the field operator a and its Hermitian conjugate $a^\dagger$, satisfying $[a,a^\dagger]=1$, the quadrature field components are defined as [17]

$$X_1 \equiv \frac{a+a^\dagger}{\sqrt{2}} , \tag{6a}$$

$$X_2 \equiv \frac{a-a^\dagger}{\sqrt{2}i} , \tag{6b}$$

with the resulting commutation relation $[X_1,X_2]=i$. Calculations of the degree of higher-order squeezing are facilitated by a formula due to Hong and Mandel [1],

$$\begin{aligned}\langle(\Delta X_1)^N\rangle = \langle :(\Delta X_1)^N:\rangle + \frac{N^{(2)}}{1!}(\tfrac{1}{2})^2\langle :(\Delta X_1)^{N-2}:\rangle \\ + \cdots + \frac{N^{(4)}}{2!}(\tfrac{1}{2})^4\langle :(\Delta X_1)^{N-4}:\rangle \\ + \cdots + (N-1)!!(\tfrac{1}{2})^{N/2} ,\end{aligned} \tag{7}$$

where $N^{(r)} \equiv N(N-1)\cdots(N-r+1)$, $:f(a,a^\dagger):$ denotes normal ordering of the $a,a^\dagger$ operators in the function f, and $\Delta X_1 \equiv X_1 - \langle X_1\rangle$.

IV. SECOND-ORDER SQUEEZING

For the case of second-order squeezing ($N=2$), Eq. (7) becomes

$$\langle(\Delta X_1)^2\rangle = \tfrac{1}{2} + \langle :(\Delta X_1)^2:\rangle . \tag{8}$$

In the coherent state the second term on the right-hand side (rhs) of Eq. (8) vanishes. Second-order squeezing of X_1 thus obtains whenever one has $\langle(\Delta X_1)^2\rangle < \frac{1}{2}$.

For the orthogonal-even coherent state one finds after a short calculation that

$$\langle\langle\phi|(\Delta X_1)^2|\phi\rangle\rangle = \tfrac{1}{2} + |\alpha|^2 \frac{\sinh|\alpha|^2 - \sin|\alpha|^2}{\cosh|\alpha|^2 + \cos|\alpha|^2} . \tag{9}$$

The second term on the rhs of Eq. (9) is always positive, hence X_1 is never squeezed in second order. One can obtain the corresponding equation for X_2 by rotating the coordinate system by 90°, i.e., $\alpha \to i\alpha$. Since Eq. (9) depends only on $|\alpha|$, the identical result is obtained for X_2. Thus one concludes that the orthogonal-even coherent state does not exhibit any second-order squeezing. It will be seen later when the Wigner function plots of these states are exhibited that this result can be traced to the symmetry of the states in X_1 and X_2.

V. FOURTH-ORDER SQUEEZING

After a long and tedious calculation one obtains the following result for the fourth-order moment of X_1 in the orthogonal-even coherent state:

$$\langle\langle\phi|(\Delta X_1)^4|\phi\rangle\rangle = \frac{3}{4}\left[1 + 4\frac{|\alpha|^2(\sinh|\alpha|^2 - \sin|\alpha|^2) + |\alpha|^4(C\cosh|\alpha|^2 - D\cos|\alpha|^2)}{\cosh|\alpha|^2 + \cos|\alpha|^2}\right] . \tag{10a}$$

Here

$$C \equiv \frac{1}{2}\left[1 + \frac{\cos 4\phi}{3}\right] , \tag{10b}$$

$$D \equiv \frac{1}{2}\left[1 - \frac{\cos 4\phi}{3}\right] , \tag{10c}$$

where $\phi = \arg(\alpha)$. From Eq. (7) we conclude that X_1 will exhibit fourth-order squeezing whenever $S \equiv \langle\langle\phi|(\Delta X_1)^4|\phi\rangle\rangle < \frac{3}{4}$.

If one differentiates Eq. (10a) with respect to ϕ and sets the resulting expression equal to zero, it is found that a necessary condition for S to be an extremum is that $\sin 4\phi = 0$, *independent* of $|\alpha|$. For $0 \le \phi \le \pi/2$, numerical calculations of S reveal that of the possible solutions of the extremum condition, $\phi = 0, \pi/4, \pi/2$, only $\phi = \pi/4$ yields any fourth-order squeezing in X_1. This fourth-order squeezing exists over a range of $|\alpha|$, $0 < |\alpha| < 0.79$. Minimal squeezing is found at $|\alpha| \approx 0.67$, with value $S(\min) \approx 0.6999$, or about 7% below the value obtaining

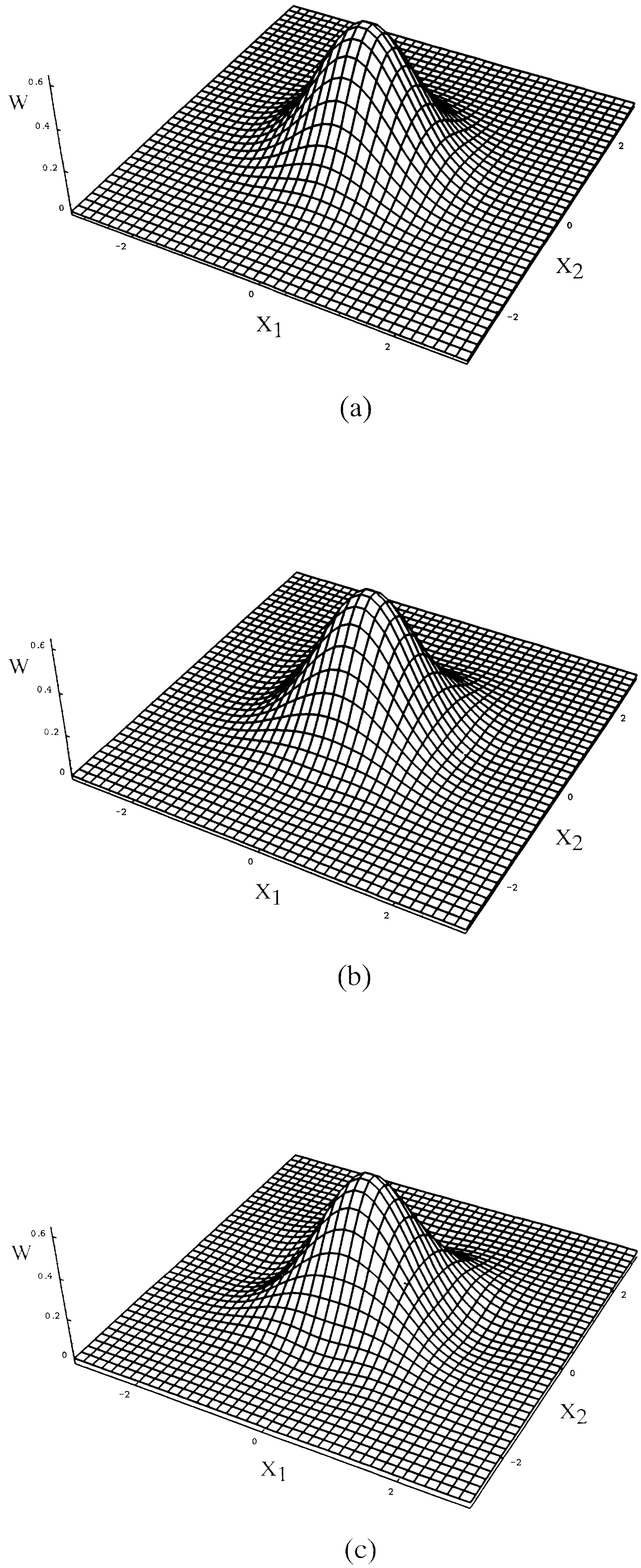
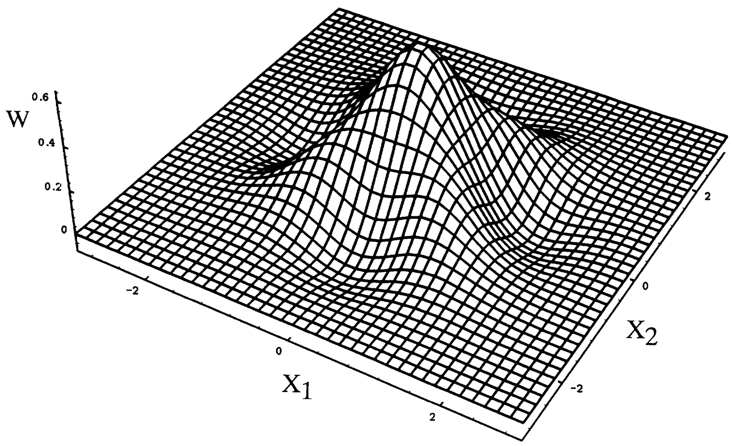
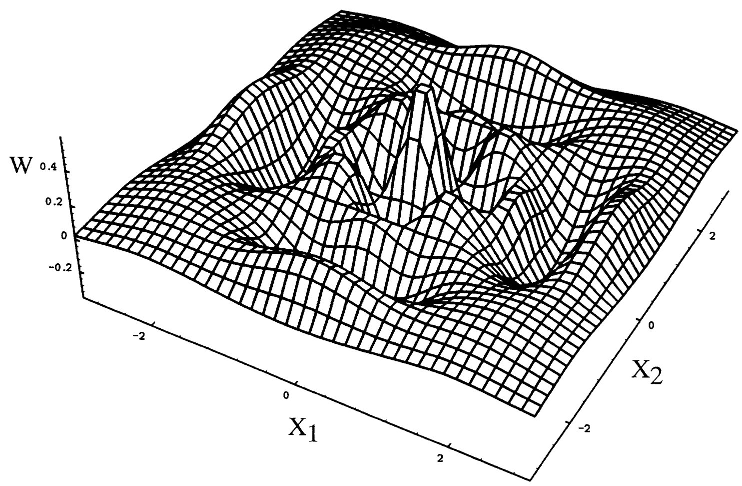


FIG. 1. Phase-space plots of the Wigner function $W(\alpha)$ vs the scaled variables $X_1 = \mathrm{Re}(\alpha)/\sqrt{2}$, $X_2 = \mathrm{Im}(\alpha)/\sqrt{2}$, where β is taken to have the form $\beta = (b + ib)/\sqrt{2}$ thus satisfying the necessary condition for minimal simultaneous-quadrature fourth-order squeezing. (a) $b = 0$, a degenerate case where the orthogonal-even coherent state goes over to the vacuum; (b) $b = 0.671$, found by numerical calculations to produce minimal simultaneous-quadrature fourth-order squeezing. A slight "tucking in" of the Wigner function along the X_1, X_2 axes is just noticeable; (c) $b = 0.794$. The "tucking in" is more pronounced, but here one is at the end of the range of fourth-order squeezing; (d),(e) $b = 1.0, 2.0$. The plots exhibit complex features due to "interference in phase space."



(d)



(e)

FIG. 1. (*Continued*).

in the coherent state (0.75). It is surprising and quite unexpected that this minimal value is quite close to the best value found in numerical calculations [5], $S_{\text{num}}(\min)\approx 0.6984$, utilizing a more general superposition of number states in a variational calculation.

An interesting feature, which is the point of this paper, is found upon using the prescription for obtaining the corresponding result for X_2, i.e., letting $\alpha\rightarrow i\alpha$, or $\phi\rightarrow\phi+\pi/2$. Because ϕ only appears in Eq. (10) as $\cos 4\phi$, the resulting expression for $\langle\langle\phi|(\Delta X_2)^4|\phi\rangle\rangle$ is identical to that obtained for X_1, i.e., simultaneous-quadrature fourth-order squeezing.

It is entirely possible that for $N=6,8,\ldots$, the orthogonal-even coherent states might exhibit simultaneous-quadrature higher-order squeezing similar to that found in variational calculations [5]. This question has not been pursued because already in fourth-order the calculations are extremely laborious.

VI. WIGNER FUNCTIONS

For a system with density operator ρ, the Wigner function $W(\alpha)$ is defined as [18]

$$W(\alpha)=\frac{1}{\pi^2}\int d^2\lambda\, e^{-\lambda\alpha^*+\lambda^*\alpha}\,\mathrm{Tr}\{\rho e^{\lambda a^\dagger-\lambda^* a}\}\ . \tag{11}$$

Here the integration is taken over the entire complex λ plane. For the orthogonal-even coherent state,

$$\rho=|\phi\rangle\rangle\langle\langle\phi|=AB\sum_{i,j=1}^{4}|\alpha_i\rangle\langle\alpha_j|\ , \tag{12a}$$

where

$$\alpha_k\equiv i^{k-1}\beta,\quad k=1,2,3,4\ . \tag{12b}$$

The Wigner function then takes the form

$$W(\alpha)=AB\sum_{i,j=1}^{4}w_{ij}(\alpha;\alpha_i,\alpha_j)\ , \tag{13a}$$

where using Eq. (11) it can be shown that

$$w_{ij}(\alpha;\alpha_i,\alpha_j)=\frac{2}{\pi}e^{-(1/2)|\alpha_i|^2-(1/2)|\alpha_j|^2}e^{\alpha_i\alpha_j^*-2(\alpha-\alpha_i)(\alpha^*-\alpha_j^*)}\ . \tag{13b}$$

We now choose β to have the form $\beta=(b+ib)/\sqrt{2}$, so that $\arg(\beta)=\pi/4$, the necessary condition for minimal simultaneous-quadrature fourth-order squeezing. Figure 1 shows phase-space plots of $W(\alpha)$ vs the (scalar) variables $X_1=\mathrm{Re}(\alpha)/\sqrt{2}$, $X_2=\mathrm{Im}(\alpha)/\sqrt{2}$. It is seen that the plot with $b=0$ (a degenerate case for which the orthogonal-even coherent state goes over to the vacuum) can scarcely be distinguished from that belonging to the value $b=0.671$, the case of minimal simultaneous-quadrature fourth-order squeezing. There is perhaps a hint of "tucking in" along the X_1,X_2 axes, which is much more exaggerated and easily seen in the cases $b=0.794$, 1.0. (However, recall that the value $b=0.794$ marks the upper end of the range of values for which such simultaneous squeezing is observed.) For $b=2.0$ the plot of the Wigner function takes on highly complex features. Even if the very slight "tucking in" along X_1 and X_2 can only be conjectured to represent the source of simultaneous-quadrature squeezing, it is clear from the symmetry of these plots that no second-order squeezing might be expected, as is found to be the case by calculation.

Finally, compare the Fig. 1 plots to those in Fig. 2, which depict the Wigner function of a statistical mixture of the $|\alpha_k\rangle$ states,

$$\rho_M=\frac{1}{4}\sum_{k=1}^{4}|\alpha_k\rangle\langle\alpha_k| . \tag{14}$$

The lack of cross terms of the form $|\alpha_i\rangle\langle\alpha_j|$ in ρ_M eliminates "interference in phase space" [10,14] and simply yields a broad peak for $b=0.671$, which then goes over to four separate peaks in the case $b=2.0$. This confirms the central role of wave-function superposition in quantum mechanics for obtaining such nonclassical effects such as squeezing, in the "ordinary" (second) as well as higher-order forms.

VII. SUMMARY AND CONCLUSIONS

It has been shown that a particular combination of even coherent states, dubbed the *orthogonal-even coherent state,* can furnish near-optimal simultaneous-quadrature

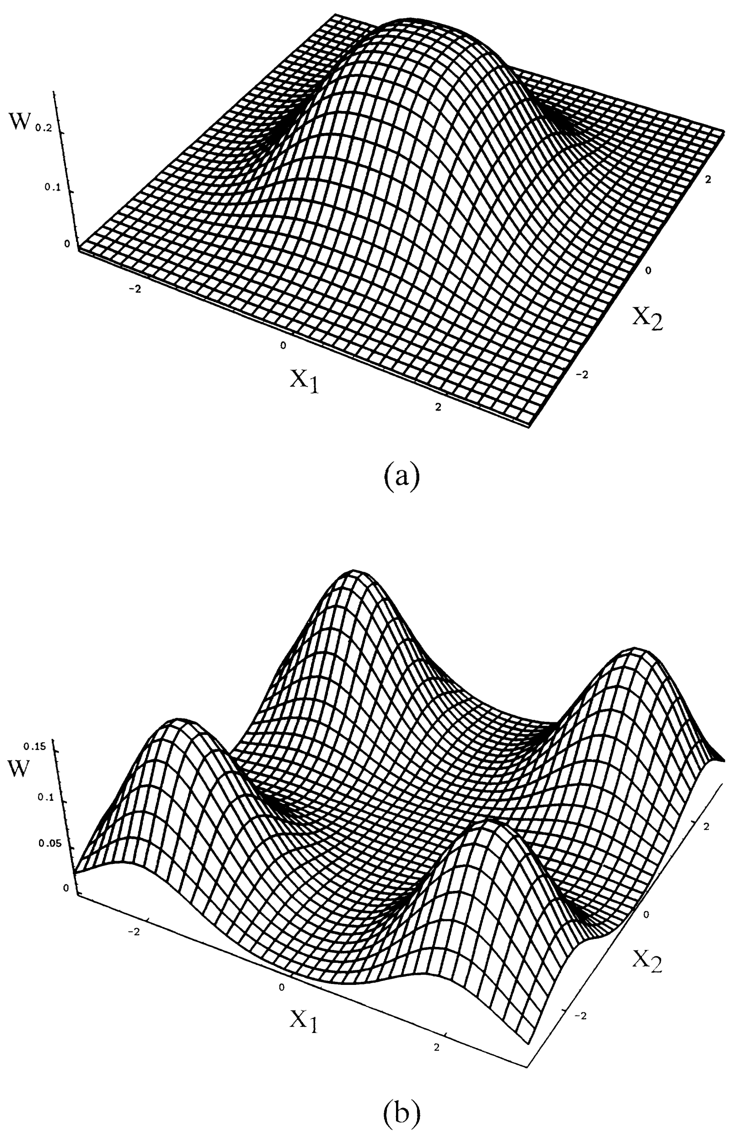


FIG. 2. Repeat of phase-space plots, with the same parametrization as Fig. 1, but that of the Wigner function $W(\alpha)$ resulting from the density matrix given by Eq. (14), representing a *statistical mixture* of the four states $|\alpha_k\rangle$ entering into the orthogonal-even coherent state. (a) $b=0.671$; the lack of cross terms in the density matrix yields a simple broad peak, due to the lack of "interference in phase space"; (b) $b=2.0$. Compare the separate peaks obtained in this statistical mixture to the complex features of the corresponding Fig. 1 plot.

fourth-order squeezing. Yurke and Stoler [19] have shown that even coherent states can be experimentally produced via propagation of coherent light in an amplitude-dispersive medium. Recently and more to the point, Jex and Bužek [16] have briefly mentioned a mechanism for production of the orthogonal-even coherent state itself, via a resonant two-photon Jaynes-Cummings model with the cavity field initially in an even coherent state. In view of the attraction these days towards nonclassical optical states it would be interesting to witness experimental generation of the orthogonal-even coherent states and subsequent study of their higher-order squeezing properties.

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