

## Separation of variables and exactly soluble time-dependent potentials in quantum mechanics

Costas John Efthimiou<sup>1,\*</sup> and Donald Spector<sup>2,†</sup>

<sup>1</sup>*Newman Laboratory of Nuclear Studies, Cornell University, Ithaca, New York 14853*

<sup>2</sup>*Department of Physics, Eaton Hall, Hobart and William Smith Colleges, Geneva, New York 14456*

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We use separation of variables as a tool to identify and to analyze exactly soluble time-dependent quantum-mechanical potentials. By considering the most general possible time-dependent redefinition of the spatial coordinate, as well as general transformations on the wave functions, we show that separation of variables applies and exact solubility occurs only in a very restricted class of time-dependent models. We consider the formal structure underlying our findings, and the relationship between our results and other work on time-dependent potentials. As an application of our methods, we apply our results to the calculation of propagators.

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### I. INTRODUCTION

Quantum-mechanical systems with explicit time dependence have been studied by a wide variety of techniques; a representative, though by no means exhaustive, list of such work is [1–11]. In this paper, we will add to this collection of results, with a focus on the use of coordinate transformations and separation of variables to study time-dependent systems, and on the application of such methods to physical calculations in time-dependent models.

This paper should be thought of in particular as a follow-up to [6]. In that paper, the authors identified changes of variables in a limited class of time-dependent systems that allowed those systems to be analyzed via separation of variables. (We will use the term *separable* to describe the situation in which separation of variables can be applied usefully.) The applicability of separation of variables in these models means that for these time-dependent systems, there exists a mathematical structure and formalism analogous to that of the time-independent Schrödinger equation. As an application of their approach, the authors used separation of variables, in combination with both algebraic and analytic techniques, to solve exactly the harmonic oscillator with inverse frequency linear in time.

That paper left open some important questions. First, can the technique of [6] be used to identify exactly soluble time-dependent generalizations of other exactly soluble time-independent systems? Second, how special was the change of variables that paper used to find separable models? Would other changes of variables have yielded other models? Third, can we use this method to simplify or make possible other exact calculations in

time-dependent quantum-mechanical theories? And finally, what connections can be drawn between the use of separation of variables and other approaches to time-dependent quantum-mechanical systems? This paper addresses these questions.

Our central findings are as follows. First, we show explicitly how to construct exactly soluble time-dependent generalizations of any exactly soluble time-independent model. We do this using a time-dependent redefinition of the spatial coordinate which is linear in the spatial variable. Second, we show that this linear change of variables, although apparently quite special, is actually quite general (at least in the case of one spatial dimension): any time-dependent model which is separable following some coordinate transformation is in fact separable under the simple linear transformation already considered. Including arbitrary multiplicative transformations of the wave functions, we are able to obtain a larger family of separable and exactly soluble models. We find that separability singles out a unique possible wave-function transformation, and makes possible a larger (but still specified) set of useful changes of variables. Third, we find these exactly soluble models again by an additional method: we derive the quantum generalization of the work of [12] on classical Hamiltonians with an invariant quadratic in momentum; strikingly, we find that the potentials with a quadratic invariant quantum mechanically are exactly those that have one classically. Finally, as an application of our method, we present exact calculations of propagators in exactly soluble time-dependent models, combining our techniques with those of shape invariance.

One of the key results of this paper is that we reduce the determination of whether a time-dependent potential is exactly soluble to the determination of whether an associated time-independent potential is exactly soluble. We do this through the use of separation of variables as the central aspect of our approach. As is well known, this method reduces a multidimensional problem to two or more lower-dimensional ones. The method of separation of variables is very powerful, and so has been applied

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\*Electronic address: costas@hepht.cornell.edu

†Electronic address: spectator@hws.bitnet

broadly. A summary of such applications is presented in [13], with a particular focus on the relation between Lie groups and the method of separation of variables. In [4], Reid has classified all possible coordinate systems for which the potential-free Schrödinger equation is separable. More recently, Sklyanin has introduced the so-called functional Bethe ansatz as an alternative to the algebraic Bethe ansatz, in order to remove the restrictions inherent in the second approach; this new method is a hybrid of the method of separation of variables and the quantum inverse scattering method [14].

As we mentioned earlier, the existing literature on time-dependent quantum mechanics already covers a variety of other approaches, as seen, for example, by surveying the articles [1–11], as well as the references therein. The articles [2, 3, 7, 9, 11] provide a variety of approaches to time-dependent quantum mechanics, for the most part with a focus on exact solubility from the start. In this way our work is somewhat different; we take separation of variables as our central criterion, and study its implications. A similar approach has been adopted in [1, 4, 5], although the first two of these references only study special cases. In [5], the problem of  $R$  separability is treated in general terms. In that paper, the focus is on the rigorous mathematical structure underlying separability. In our paper, on the other hand, our focus is on the actual transformations that achieve separability, along with the practical applications thereof. Thus these two papers are complementary. If we had to provide a rough characterization of the distinction between these two papers, it would be that [5] presents a mathematician’s approach to separability, and our paper presents a physicist’s approach, and thus these papers obtain distinct but parallel insights. For example, the actual extremely limited set of transformations useful for separability or the framework for forging connections to specific potentials with shape invariance, things which we identify, are not obvious in [5], and the implications for simplifying calculations are only implicit in their work; and, on the other hand, in our work, it is not obvious how to treat general manifolds on a common footing, something which is done quite nicely in [5]. (A specific technical difference is that [5] works in  $d$  dimensions, whereas we focus on the case of one spatial dimension. This makes sense given the other distinctions between our papers: we are focusing on the *process* of changing variables, and since this process can be implemented in essentially the same way in any number of dimensions, we concentrate on presenting this process in the one-dimensional case. In [5], by contrast, the focus is on the general mathematical structure, and so this work is more naturally presented in arbitrary dimension.)

Our interest in separation of variables is twofold. First, as we said above, separability implies the existence of a mathematical structure in a theory analogous to the structure associated with the ordinary time-independent Schrödinger equations; and second, as we see from our work, particularly as it stands in relation to the work in [2, 7, 9, 11] (our methods identify all the exactly soluble models found with other approaches in those references), separation of variables also appears to be an appropriate stepping stone in at least one path toward uncov-

ering and classifying the exactly soluble time-dependent models. The agreement among these various papers (including ours) where they overlap suggests the existence of some fundamental principles that connect their various approaches. We will comment on the connections between our work and that of these other papers where relevant.

For the sake of simplicity, we will, as we indicated above, restrict our attention in this paper to systems with only one spatial dimension, but it will be readily apparent that our techniques can be applied without modification to higher-dimensional models as well (see also [5, 9]). Although we will give some multidimensional results where appropriate or useful, we intend this work primarily to be an exploration of coordinate transformations and separability as elucidated in the context of one-dimensional quantum mechanics.

Overall, we feel time-dependent quantum mechanics is still inadequately understood. For example, an explicitly time-dependent property analogous to shape invariance — which has greatly enriched our understanding of exact solubility in the time-independent case — is still elusive. We believe the restrictive nature of the results we have obtained using changes of variables with separation of variables is significant, because this helps us to determine what the actual range of exactly soluble time-dependent models is; this can serve to focus the search for a fundamental explanation of exact solubility in the time-dependent case.

## II. EXACTLY SOLUBLE TIME-DEPENDENT POTENTIALS

To begin, we consider nonrelativistic quantum mechanics in the Schrödinger picture. The Schrödinger equation in position space reads

$$\mathcal{H}\Psi(x, t) = i\hbar \frac{\partial \Psi(x, t)}{\partial t}, \quad (2.1)$$

where the Hamiltonian, with time-dependent potential, is

$$\mathcal{H} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x, t). \quad (2.2)$$

Our goal is to identify time-dependent potentials  $V(x, t)$  such that the Schrödinger equation (2.1) is exactly soluble. Obviously, separation of variables in terms of  $x$  and  $t$  will not be helpful in this case, so we attempt to find a change of variables for which separation of variables will be useful. To this end, we define a new variable  $y = f(t)x$ . In order that this change of variables be productive, we require that the potential factorizes in terms of these new coordinates, that is, we require that

$$V(x(y, t), t) = g(t)\tilde{V}(y) + h(t). \quad (2.3)$$

The time-dependent Schrödinger equation now reads

$$\begin{aligned}
& -\frac{\hbar^2}{2m} f^2(t) \frac{\partial^2}{\partial y^2} \tilde{\Psi}(y, t) + [g(t) \tilde{V}(y) + h(t)] \tilde{\Psi}(y, t) \\
& = i\hbar \frac{\dot{f}(t)}{f(t)} y \frac{\partial}{\partial y} \tilde{\Psi}(y, t) + i\hbar \frac{\partial \tilde{\Psi}(y, t)}{\partial t}. \quad (2.4)
\end{aligned}$$

We use  $\tilde{\Psi}$  to denote the wave function as a function of  $y$  and  $t$ ; thus  $\Psi(x, t) = \tilde{\Psi}(y(x, t), t)$ . Note that in (2.4), the partial derivatives with respect to time are now taken with  $y$  fixed rather than with  $x$  fixed.

Inspecting (2.4), it is easy to see that separation of variables in  $y$  and  $t$  will work provided that

$$f(t) = (pt + q)^{-1/2}, \quad (2.5)$$

and that

$$g(t) = a f^2(t) = a (pt + q)^{-1}, \quad (2.6)$$

where  $a$ ,  $p$ , and  $q$  are constants. The constant  $a$  may be absorbed in  $\tilde{V}(y)$ ; thus, without loss of generality, we take  $a = 1$ . From now on, we take the separation of variable conditions (2.3), (2.5), and (2.6) to hold. Now by considering solutions to the Schrödinger equation (2.4) of the form  $\tilde{\Psi}(y) = T(t)Y(y)$ , we see that the general solution to the Schrödinger equation can be written as

$$\tilde{\Psi}(y, t) = \sum_k c_k T_k(t) Y_k(y), \quad (2.7)$$

where the  $c_k$  are arbitrary constants, and where  $Y_k$  and  $T_k$  are the solutions of, respectively,

$$-\frac{\hbar^2}{2m} \frac{d^2 Y_k}{dy^2} + i \frac{\hbar p}{2} y \frac{dY_k}{dy} + \tilde{V}(y) Y_k = \gamma_k Y_k \quad (2.8)$$

and

$$i\hbar (pt + q) \frac{dT_k}{dt} - (pt + q)h(t)T_k = \gamma_k T_k, \quad (2.9)$$

with  $\gamma_k$  a complex constant. As we show below,  $\text{Im}(\gamma_k) = -(\hbar p/4)$ . The differential equation for  $T_k$  can be solved exactly,

$$T_k(t) = e^{-\frac{i}{\hbar} \eta(t)} (q + pt)^{-(i\gamma_k/\hbar p)}, \quad (2.10)$$

where

$$\eta(t) = \int_0^t dt h(t). \quad (2.11)$$

Thus, for the above time-dependent Hamiltonians, one recovers a mathematical structure formally analogous to the one one finds for time-independent systems. Defining the (non-Hermitian) pseudo-Hamiltonian

$$\tilde{\mathcal{H}} = -\frac{\hbar^2}{2m} \frac{d^2}{dy^2} + i \frac{\hbar p}{2} y \frac{d}{dy} + \tilde{V}(y), \quad (2.12)$$

we see that the problem of solving the time-dependent Schrödinger equation is equivalent to finding the eigenfunctions and (complex) eigenvalues of  $\tilde{\mathcal{H}}$ . (Note that in

the limit that  $p = 0$ , the pseudo-Hamiltonian becomes the ordinary Hamiltonian, as it must.)

We are now in a position to determine the exactly soluble potentials as follows. Defining the wave function

$$\tilde{Y}_k(y) = e^{-impy^2/4\hbar} Y_k(y), \quad (2.13)$$

we find that  $\tilde{Y}_k$  satisfies an *ordinary* (time-independent) Schrödinger equation

$$-\frac{\hbar^2}{2m} \frac{d^2 \tilde{Y}_k}{dy^2} + \tilde{V}_-(y) \tilde{Y}_k = \epsilon_k \tilde{Y}_k, \quad (2.14)$$

where

$$\tilde{V}_-(y) \equiv \tilde{V}(y) - \frac{p^2 m}{8} y^2, \quad (2.15)$$

and where  $\epsilon_k$  is a real constant given by

$$\epsilon_k \equiv \gamma_k + i \frac{\hbar p}{4}. \quad (2.16)$$

Thus, whenever  $\tilde{V}_-(y)$  is an exactly soluble potential of ordinary time-independent quantum mechanics, then the original time-dependent system is itself exactly soluble. Hence whenever

$$\begin{aligned}
V(x, t) &= f^2(t) \left[ \tilde{V}_-[f(t)x] + \frac{p^2 m}{8} f^2(t) x^2 \right] + h(t) \\
&= \frac{1}{pt + q} \left[ \tilde{V}_-\left(\frac{x}{\sqrt{pt + q}}\right) + \frac{p^2 m}{8} \frac{x^2}{pt + q} \right] \\
&\quad + h(t), \quad (2.17)
\end{aligned}$$

and  $\tilde{V}_-(y)$  is an exactly soluble potential for the ordinary time-independent Schrödinger equation, the time-dependent Hamiltonian of the Schrödinger equation (2.2) is exactly soluble.

In this way, we have identified a family of exactly soluble time-dependent potentials to generalize any exactly soluble time-independent potential. Of course, much work has gone into the study of exactly soluble time-independent potentials; most recent work has focused on the uses of shape invariance [15]. In fact, all the well-known exactly soluble potentials of quantum mechanics can be understood to derive their exact solubility ultimately from the property of shape invariance. If a potential  $\tilde{V}_-(y)$  is shape invariant, this means that

$$\tilde{V}_-(y) = W^2(y, \alpha) - \frac{\hbar}{\sqrt{2m}} W'(y, \alpha), \quad (2.18)$$

and  $W(y, \alpha)$  satisfies the condition

$$\begin{aligned}
W^2(y, \alpha) + \frac{\hbar}{\sqrt{2m}} W'(y, \alpha) \\
= W^2(y, \sigma(\alpha)) - \frac{\hbar}{\sqrt{2m}} W'(y, \sigma(\alpha)) + R(\alpha)
\end{aligned} \quad (2.19)$$

for some functions  $\sigma(\alpha)$  and  $R(\alpha)$ . Using the theory of supersymmetric quantum mechanics [16], the “levels”  $\gamma_k$

are given by

$$\gamma_0 = -i \frac{\hbar p}{4}, \quad (2.20a)$$

$$\gamma_k = -i \frac{\hbar p}{4} + \sum_{j=1}^{k-1} R(\alpha_j), \quad k \neq 0 \quad (2.20b)$$

where  $\alpha_j = \sigma(\alpha_{j-1})$  and  $\alpha_0 = \alpha$ .

One of the intriguing features of the exactly soluble time-dependent potentials we have identified is that, in addition to containing a term of essentially the same form as the original potential, they all contain an additional time-dependent harmonic-oscillator term. We give a simple way to interpret this term below. But first let us point out that because of the appearance of this term, for the harmonic oscillator, and only for the harmonic oscillator, will the time-dependent generalization simply involve introducing a time dependence into the original coefficient of the potential. It is for this reason that [6] discovered no exactly soluble systems other than the time-dependent harmonic oscillator.

As promised, we give here an instructive interpretation of the harmonic-oscillator term that appears in (2.17). Using the definition of  $\tilde{V}_-(y)$  in (2.15), one can write the Schrödinger equation in  $y$  space (2.8) in the following form:

$$\frac{1}{2m} \left( -i\hbar \frac{d}{dy} - \frac{pm}{2} y \right)^2 Y_k + \tilde{V}_-(y) Y_k = \epsilon_k Y_k. \quad (2.21)$$

In other words, we can view (2.8) as describing the coupling of a quantum-mechanical particle of mass  $m$  to a background electromagnetic field described by the two-potential  $(\varphi(y), A(y)) = (\tilde{V}_-(y), (pm/2)y)$ , and this necessitates the harmonic-oscillator-type term. Invoking the gauge transformation  $A(y) \rightarrow A(y) - \beta y$ , one can obtain the eigenfunctions of the gauge-transformed Hamiltonian from those of the original Hamiltonian by observing that under this transformation, the eigenfunctions transform by

$$Y_k(y) \rightarrow \exp\left(\frac{i\beta^2}{\hbar} y\right) Y_k(y). \quad (2.22)$$

In this way, then, the solutions to the eigenfunction equation (2.21) can be obtained from those of the corresponding equation with an ordinary kinetic term. We point this out because we suspect that proper use of such a gauge picture may yield a more fundamental way to understand and to classify the exactly soluble time-dependent models.

### III. CHANGES OF VARIABLES: THE POTENTIAL

The above set of exactly soluble models with the potentials of (2.17) arises from a consideration of the particularly simple change of variables  $y = f(t)x$ . In order to begin to develop a classification of separable and exactly soluble time-dependent models, it is clearly essential to

consider other transformations, to see what separable and what exactly soluble models these produce. We have done this, and the answer we find is surprising. Consider any change of variables of the form  $y = P(x, t)$ , while keeping  $t$  as the second coordinate. Then, neglecting Galilean transformations (as these are essentially trivial), we find that such a transformation can produce separation of variables only when the function  $P$  is in fact a function not of  $x$  and  $t$  arbitrarily, but only of the ratio  $x/\sqrt{pt+q}$ , that is,  $P(x/\sqrt{pt+q})$ . (Of course, the potential must also separate appropriately.) Furthermore, the only exactly soluble models uncovered by such transformations are exactly the exactly soluble models uncovered in (2.17) by the very simplest transformation we considered.

We now offer a proof of this result. The argument is rather technical, and the reader interested primarily in our results might comfortably omit this section on a first reading.

We will build towards the general proof by first giving three preliminary results. First, note that making a transformation  $y = P(x/\sqrt{pt+q})$  for any function  $P$ , will, given an appropriate potential, lead to separation of variables in terms of  $y$  and  $t$ . Moreover, when the time-dependent potentials are determined which are exactly soluble using separation of variables and this transformation, one finds that they are exactly the same as the potentials already uncovered in the preceding section by the linear transformation. The calculation proceeds exactly as in the preceding section; in the interest of brevity, we will not present the details here, as they are straightforward to obtain. [It is helpful in this calculation to invert the transformation formally via  $x = (\sqrt{pt+q})P^{-1}(y)$ .]

Second, consider a transformation slightly more general than the linear transformation of the preceding section, namely, the redefinition of variables

$$y(x, t) = f(t)[x - u(t)]. \quad (3.1)$$

Going through the same calculation as we did for the case  $u(t) = 0$ , we find that separation of variables is only possible for  $\dot{f} \neq 0$  when, as before,

$$f(t) = \frac{1}{\sqrt{pt+q}}, \quad (3.2)$$

and when

$$u(t) = a\sqrt{pt+q} + c. \quad (3.3)$$

(When  $\dot{f} = 0$ , the only solution is a Galilean transformation; since this is a trivial possibility, we ignore this case.) The additional parameters  $a$  and  $c$  are easy to understand if we rewrite (3.1) using the results for  $f(t)$  and  $u(t)$ , obtaining

$$y = f(t)(x - c) - a. \quad (3.4)$$

We see that the parameter  $a$  corresponds to a shift in the origin of  $y$ , while  $c$  corresponds to a shift in the origin of  $x$ . Thus these parameters give us no new models. Any potential for which separation of variables obtains when  $a = 0$  yields separation of variables for any value of  $a$ ,

and so allowing the parameter  $a$  adds no new models to our list (2.17). The parameter  $c$  also gives us no new models. If separation of variables works for  $V(x, t)$ , it will not generally work for  $V(x - c, t)$  in terms of the same separating variables. However,  $V(x, t)$  and  $V(x - c, t)$  correspond to the same physical model. Including the parameter  $c$  thus allows us to use separation of variables directly no matter what origin one uses for  $x$ ; the earlier transformation found all the same models, but entailed choosing a particular origin for  $x$ .

The next result is a slight generalization of the preceding one (3.1). Consider now a transformation of the form

$$y = \varphi(t)[x - u(t)]^n . \tag{3.5}$$

Such a transformation can lead to separation of variables only when  $\varphi(t) = (pt + q)^{-n/2}$  (i.e.,  $[f(t)]^n$ , where  $f(t)$  is as above) and  $u(t) = a\sqrt{pt + q} + c$ , as before. Rather than going through the same type of calculation again, one can obtain this result from the preceding one. If separation of variables holds in terms of  $y$  and  $t$ , then it also holds in terms of  $z = y^{1/n}$  and  $t$ . In this way, we can reduce the problem of finding useful transformations of the type in (3.5) to finding useful transformations of the type in (3.1), which we already know.

We are now in a position to obtain our general result, namely, that any transformation of the form  $y = P(x, t)$  leads to exactly the same separable and exactly soluble models as we have already obtained. To begin, we will write the function  $y = P(x, t)$  as a product,

$$y = G(t) \prod_{j=1}^k [x - u_j(t)]^{n_j} . \tag{3.6}$$

For the purposes of our discussion, we take the  $n_j$  to be positive but not necessarily integral, although the restriction to positive  $n_j$  is not actually necessary. The  $u_j$  must all be distinct. For convenience, we define  $N = \sum_{j=1}^k n_j$ . We note that any  $P$  which is polynomial in the variable  $x$  can be written in this form. Since our proof will hold for all polynomials of any order, it will also hold in the limit of an infinite but converging power series, which can be understood as the limit of such polynomials. The set of transformations which we can handle based on (3.6) is thus a superset of those functions which have a power series in  $x$ . In fact, the form of (3.6) shows us that our argument covers all functions except those with extreme singularities.

The essential idea behind our proof is to take the requirement that separation of variables holds under this transformation, and look at the implications of this requirement in different regions of space and time. This will enable us to place restrictions on the possible forms of  $G(t)$  and the  $u_j(t)$ .

First, consider this transformation at large values of  $x$ , when  $x \gg u_j(t)$  for all  $j$ . In this region, the transformation is essentially

$$y \approx G(t)x^N . \tag{3.7}$$

From our consideration of (3.5), we know that separa-

tion of variables here requires that  $G(t) \propto (pt + q)^{-N/2}$ . Without loss of generality, we may rescale the transformation so that

$$G(t) = \frac{1}{(\sqrt{t + q})^N} . \tag{3.8}$$

Next we consider the case that  $x$  is in the neighborhood of  $u_j(t)$ . Here  $x - u_j(t)$  is a small parameter. Expanding to lowest nontrivial order in this parameter, we see that in this neighborhood

$$y \approx G_j(t)[x - u_j(t)]^{n_j} , \tag{3.9}$$

where  $G_j(t)$  is

$$G_j(t) = G(t) \prod_{i=1, i \neq j}^k [u_j(t) - u_i(t)]^{n_i} . \tag{3.10}$$

In this neighborhood, the transformation takes the form of (3.5), and so we see immediately that we must be able to write each  $u_j$  as

$$u_j(t) = a_j\sqrt{t + q_j} + c_j . \tag{3.11}$$

Also, we see that  $G_j$  is

$$G_j(t) = \left( \frac{b_j}{\sqrt{t + q_j}} \right)^{n_j} . \tag{3.12}$$

Thus

$$\prod_{i=1, i \neq j}^k [u_j(t) - u_i(t)]^{n_i} = \left( \frac{b_j}{\sqrt{t + q_j}} \right)^{n_j} \left( \sqrt{t + q} \right)^N . \tag{3.13}$$

Since we know the general form of the  $u_j$ 's, we know that the left side of the preceding equation (3.13) has no divergences, and thus the right side must have no divergences. This requires that  $q_j = q$  in the expression for  $G_j$ , which in turn dictates that  $q_j = q$  in each of the  $u_j$ , since, as we saw back in (3.5), the square root that appears in  $u(t)$  is the same square root as appears in  $\varphi(t)$ .

Putting all these results together, Eq. (3.13) now reads

$$\prod_{i=1, i \neq j}^k [(a_j - a_k)\sqrt{t + q} + (c_j - c_k)]^{n_i} = (b_j)^{n_j} (\sqrt{t + q})^{N - n_j} . \tag{3.14}$$

In order that the multiplicity of zeroes at  $t = -q$  be the same on both sides of the equation (and that there be no zeroes at any other value on the left side, since there are none on the right side), we must have  $c_j = c_k$  for all values of  $j$  and  $k$ , and  $a_j \neq a_k$  for  $j \neq k$ . (This is consistent with our requirement that all the  $u_j$ 's be distinct.) We define  $c$  by  $c_j = c$  for all  $j$ .

We now see that in order for separation of variables to work, the transformation in (3.6) cannot be the most general product possible, but must necessarily be of the form

$$y = \frac{1}{(\sqrt{t+q})^N} \prod_{j=1}^k (x - a_j \sqrt{t+q} - c)^{n_j}. \quad (3.15)$$

We are free to redefine the origin along the  $x$  axis; this does not change the physical model. We therefore choose the origin such that  $c = 0$ . Factoring out the square roots (and recalling that  $N = \sum_{j=1}^k n_j$ ), we see that the most general possible transformation which has the possibility of producing separation of variables is of the form

$$y = \prod_{j=1}^k \left( \frac{x}{\sqrt{t+q}} - a_j \right)^{n_j}. \quad (3.16)$$

Such a transformation is not a function of both  $x$  and  $t$  separately, but rather is a function only of the combination  $x/\sqrt{t+q}$ . As we discussed above, the separable and exactly soluble models that are identified via separation of variables following a transformation  $y = P(x/\sqrt{t+q})$  are exactly the separable and exactly soluble models found using our original linear transformation.

Thus by considering an arbitrary coordinate transformation which leaves time as the second coordinate, we obtain no models that we did not already identify by considering the simple linear transformation. This result is quite restrictive, and it suggests that there is a rich mathematical structure underlying both the applicability of separation of variables to and the occurrence of exact solubility in time-dependent quantum mechanics. We would like to mention that we have tested our general proof by examining a number of specific changes of variables. In every case we have checked [which includes a number of polynomial and rational functions for which  $y(x)$  can be inverted in closed form], we have verified that the only changes of variables that can produce separability are the ones identified by our general argument.

Thus we have shown that separation of variables only arises for transformations  $y = P(x, t)$  if  $P$  is a function of  $x/\sqrt{pt+q}$ . Let us reemphasize here the twofold nature of the consequences of this result: (1) there are no models that admit separation of variables using an arbitrary transformation for which a linear transformation does not suffice; (2) there are no exactly soluble models that we find using separation of variables and an arbitrary coordinate transform that we did not find using separation of variables and the linear coordinate transform. Thus, unless we consider changes of variables in which the time variable is redefined (a possibility we do

not explore in this paper), we have, by using the simplest time-dependent transformation, already identified the most general exactly soluble time-dependent generalizations of the exactly soluble time-independent models which are accessible by means of separation of variables while using the original wave function. In the next section we include a multiplicative transformation on the wave function along with changes of variables, which enables us to find a somewhat larger class of separable models. Our results here (and in the next section) demonstrate that the space of time-dependent exactly soluble models is extremely constrained, and thus suggests that separability may be a useful tool in classifying such models and in identifying the structure underlying their solubility.

#### IV. CHANGES OF VARIABLES: THE WAVE FUNCTION

The fact that the series of potentials found by the method described above closely resembles the series of potentials discussed in [7] is very intriguing. It suggests that there should be a modification of the above prescription such that the potentials of [7] — and perhaps more — are exactly reproduced. Here we present such a modification, which also serves to help us ascertain the scope of models one can address via separation of variables. To this end, we define a new variable

$$y = f(t) [x - \alpha(t)], \quad (4.1)$$

but also include now, in addition, a transformation of the wave function

$$\tilde{\Psi}(y, t) = \Xi(y, t) e^{\Phi(y, t)}, \quad (4.2)$$

where  $\tilde{\Psi}$  denotes the wave function as a function of  $y$  and  $t$ , i.e.,  $\Psi(x, t) = \tilde{\Psi}(y(x, t), t)$ . [This ansatz is motivated in part by an attempt to generalize the transformation in (2.22).] Without loss of generality we write the potential in the form

$$V(x(y, t), t) = g(t) \tilde{V}(y) + U(y, t) + g_0(t). \quad (4.3)$$

Furthermore, since we are interested in finding new models, we only consider the case that  $\tilde{V}(y)$  is not exclusively a quadratic polynomial in  $y$ .

Using the above definitions, the Schrödinger equation becomes

$$\begin{aligned} & -\frac{\hbar^2}{2m} \frac{\partial^2 \Xi}{\partial y^2} + \frac{\partial \Xi}{\partial y} \left[ -\frac{\hbar^2}{m} \frac{\partial \Phi}{\partial y} - i\hbar \frac{\dot{f}(t)}{f^3(t)} y + i\hbar \frac{\dot{\alpha}}{f} \right] \\ & + \Xi \left[ -\frac{\hbar^2}{2m} \frac{\partial^2 \Phi}{\partial y^2} - \frac{\hbar^2}{2m} \left( \frac{\partial \Phi}{\partial y} \right)^2 + i\hbar \left( \frac{\dot{\alpha}}{f} - \frac{\dot{f}(t)}{f^3(t)} y \right) \frac{\partial \Phi}{\partial y} - \frac{i\hbar}{f^2(t)} \frac{\partial \Phi}{\partial t} \right] \\ & + \Xi \frac{1}{f^2(t)} [g(t) \tilde{V}(y) + U(y, t)] = -\frac{g_0(t)}{f^2(t)} \Xi + \frac{i\hbar}{f^2(t)} \frac{\partial \Xi}{\partial t}. \end{aligned} \quad (4.4)$$

In order that the coefficient of  $\partial \Xi / \partial y$  be a function only of  $y$  (which is necessary for separability), we must have

$$\Phi(y, t) = a(t) + b(t) \frac{y^2}{2} + c(t)y + d(y), \quad (4.5)$$

with

$$b(t) = -\frac{im}{\hbar} \frac{\dot{f}}{f^3}, \quad (4.6)$$

$$c(t) = \frac{im}{\hbar} \frac{\dot{\alpha}}{f}. \quad (4.7)$$

Shifts of  $b(t)$  and  $c(t)$  by a constant from these values are allowed consistent with separation of variables, but such constants can just be absorbed into redefinitions of  $a(t)$  and  $d(y)$ .

Since our goal in (4.2) is to make separation of variables possible where it previously was not, we see that the choices one makes for  $a(t)$  and  $d(y)$  are arbitrary, as these will not change whether the model is separable. Hence, without loss of generality, we make the convenient choices  $d(y) = 0$  and

$$a(t) = \frac{1}{2} \ln f + \frac{im}{2\hbar} \int_0^t ds \dot{\alpha}^2(s) - \frac{i}{\hbar} \int_0^t ds g_0(s). \quad (4.8)$$

Thus we see that given a transformation to new variables (4.1), there is an essentially unique  $\Phi(y, t)$  which has the potential to be useful in transforming the wave function to obtain separability. Using the value of  $\Phi(y, t)$  we have now determined, we see that the Schrödinger equation (4.4) simplifies to

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \Xi}{\partial y^2} + \Xi \left[ m \frac{\ddot{\alpha}}{f^3} y + \frac{m}{2} \frac{2\dot{f}^2 - f\ddot{f}}{f^6} y^2 \right] + \Xi \frac{1}{f^2(t)} [g(t)\tilde{V}(y) + U(y, t)] = \frac{i\hbar}{f^2(t)} \frac{\partial \Xi}{\partial t}. \quad (4.9)$$

Since  $\tilde{V}(y)$  is not simply a quadratic polynomial, in order for separation of variables in  $y$  and  $t$  to work, the function  $g(t)$  must satisfy

$$g(t) = f^2(t). \quad (4.10)$$

We now only have left to consider  $f(t)$  and  $\alpha(t)$ . Note first that if the potential factorizes in the sense of (2.3), so that  $U(y, t) = 0$ , separability will only obtain provided that

$$f(t) = \frac{1}{\sqrt{\lambda t^2 + \mu t + \nu}}, \quad (4.11)$$

and with  $\alpha$  fixed by requiring that the coefficient of the term linear in  $y$  in (4.9) be independent of  $t$ . This case makes contact with and generalizes the results of the previous sections; more significantly, it reproduces exactly the set of models investigated in [7]. So in the special case  $U = 0$ , there are unique transformations (which we have found) of the spatial coordinate and the wave function which can produce separation of variables.

In the more general case that  $U \neq 0$ , there is a somewhat larger family of separable (and, in turn, a larger family of exactly soluble) models. Separability requires that  $U(y, t)$  exactly cancel the  $y$  and  $y^2$  terms, leading to

$$U(y, t) = -m \frac{\ddot{\alpha}}{f^3} y - \frac{m}{2} \frac{2\dot{f}^2 - f\ddot{f}}{f^3} y^2. \quad (4.12)$$

[We can shift  $U(y, t)$  by any function of the form  $U_0(y)f^2(t)$  or  $\Gamma(t)$  consistent with separability, but these can just be absorbed into redefinitions of  $\tilde{V}(y)$  and  $g_0(t)$ , respectively.]

Now by considering solutions to the Schrödinger equation (4.9) of the form  $\Xi(y, t) = T(t)Y(y)$ , we see that, with  $U(y, t)$  as in (4.12), the general solution to the Schrödinger equation can be written as

$$\Xi(y, t) = \sum_k c_k T_k(t) Y_k(y), \quad (4.13)$$

where the  $c_k$  are arbitrary constants, and where  $Y_k$  and  $T_k$  are the solutions of, respectively,

$$-\frac{\hbar^2}{2m} \frac{d^2 Y_k}{dy^2} + \tilde{V}(y) Y_k = \epsilon_k Y_k \quad (4.14)$$

and

$$i\hbar \frac{\dot{T}_k}{T_k} = \epsilon_k f^2(t). \quad (4.15)$$

The differential equation for  $T_k$  can be solved exactly,

$$T_k(t) = e^{-\frac{i}{\hbar} \epsilon_k \tau(t)}, \quad (4.16)$$

where

$$\tau(t) = \int_0^t dt f^2(t). \quad (4.17)$$

Thus, whenever  $\tilde{V}(y)$  is an exactly soluble potential of ordinary time-independent quantum mechanics, then the original time-dependent system is itself exactly soluble. Grouping together several terms by defining

$$h(t) = -m \left( \frac{2\dot{f}^2 - f\ddot{f}}{2f^2} \alpha^2 - \ddot{\alpha} \alpha \right) + g_0(t), \quad (4.18)$$

we see that whenever

$$V(x, t) = f^2(t) \tilde{V}\{f(t)[x - \alpha(t)]\} - m \left( \ddot{\alpha} - \alpha \frac{2\dot{f}^2 - f\ddot{f}}{f^2} \right) x - \frac{m}{2} \frac{2\dot{f}^2 - f\ddot{f}}{f^2} x^2 + h(t), \quad (4.19)$$

the model is separable, and that when, in addition,  $\tilde{V}(y)$  is a standard exactly soluble potential for the time-independent Schrödinger equation, the time-dependent

Hamiltonian of the Schrödinger equation (2.2) is exactly soluble. In this way, we have identified a more general family of exactly soluble time-dependent potentials.

As we have already said, our technique for identifying

exactly soluble models is by no means restricted to the one-dimensional case. In the simplest generalization of our work, one can easily show that the time-dependent potentials

$$V(\vec{x}, t) = \frac{1}{\rho^2(t)} \tilde{V} \left( \frac{\vec{x} - \vec{\alpha}(t)}{\rho(t)} \right) - m \left( \ddot{\vec{\alpha}} - \dot{\vec{\alpha}} \frac{\dot{\rho}(t)}{\rho(t)} \right) \cdot \dot{\vec{x}} - \frac{m}{2} \frac{\ddot{\rho}(t)}{\rho(t)} \vec{x}^2 + h(t) \quad (4.20)$$

are exactly soluble in  $d$  dimensions. This result is of course in agreement with [1, 5].

## V. THE LEWIS-LEACH APPROACH

The series of potentials (4.19) that we obtained has been found before, by Lewis and Leach in a different but related context [12]. They found that a *classical* Hamiltonian with explicit time dependence has an invariant quadratic in the momentum if and only if the potential is of the form (4.19). [Their notation is related to ours via the substitution  $\rho(t) = 1/f(t)$ .] Lewis and Leach examined a purely classical problem; our investigations, however, are explicitly quantum mechanical. In this section, we show how to generalize [12] to the corresponding quantum-mechanical problem. Remarkably, we find that a quantum-mechanical Hamiltonian will have an invariant quadratic in the momentum in exactly the same cases that the classical Hamiltonian will, and that these are the potentials we have identified by the condition of separability.

Any operator in quantum mechanics satisfies Heisenberg's equation

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + \frac{i}{\hbar} [H, I]. \quad (5.1)$$

We are interested in the case for which there is an invariant  $I$  which is quadratic in the momentum. Because of the identity

$$[p, A(x)] = -i\hbar \frac{\partial A(x)}{\partial x}, \quad (5.2)$$

we can always write  $I$  in the form

$$I = f_2(x, t) p^2 + f_1(x, t) p + f_0(x, t). \quad (5.3)$$

Then one can easily derive that the functions  $f_0(x, t)$ ,  $f_1(x, t)$ , and  $f_2(x, t)$  satisfy the following differential equations:

$$\frac{\partial f_2}{\partial x} = 0, \quad (5.4)$$

$$\frac{\partial f_2}{\partial t} - \frac{i\hbar}{2m} \frac{\partial^2 f_2}{\partial x^2} + \frac{1}{m} \frac{\partial f_1}{\partial x} = 0, \quad (5.5)$$

$$\frac{\partial f_1}{\partial t} - \frac{i\hbar}{2m} \frac{\partial^2 f_1}{\partial x^2} + \frac{1}{m} \frac{\partial f_0}{\partial x} - 2f_2 \frac{\partial V}{\partial x} = 0, \quad (5.6)$$

$$\frac{\partial f_0}{\partial t} - \frac{i\hbar}{2m} \frac{\partial^2 f_0}{\partial x^2} + i\hbar f_2 \frac{\partial^2 V}{\partial x^2} - f_1 \frac{\partial V}{\partial x} = 0. \quad (5.7)$$

Notice that in the limit  $\hbar \rightarrow 0$  we recover the classical

equations of Lewis and Leach [12] as we expect. The above system of differential equations for the quantum case can be solved in the same manner as the classical equations. Remarkably, the solution to the system of quantum-mechanical equations is virtually identical to the solution to the classical equations. In particular, we see that  $f_2$  is independent of  $x$ , and that if we define  $\tilde{f}_0(x, t) = f_0(x, t) - i\hbar m \dot{f}_2(t)/2$ , then  $\tilde{f}_0$ ,  $f_1$ , and  $f_2$  satisfy exactly the classical Lewis-Leach equations. We can use this to find easily that the form of all possible time-dependent potentials that satisfy the quantum Lewis-Leach equations is given by (4.19), and the associated integral of motion is

$$I = \frac{1}{2} m \left[ \rho \left( \frac{p}{m} - \dot{\alpha} \right) - \dot{\rho} (x - \alpha) \right]^2 + \tilde{V} \left( \frac{x - \alpha}{\rho} \right), \quad (5.8)$$

with  $\rho(t) = 1/f(t)$ , where  $f(t)$  is the function from the coordinate transformation (4.1). Note that this invariant is exactly the pseudo-Hamiltonian defined earlier in this paper. It is worth pointing out that in the  $d$ -dimensional case given in (4.20), the corresponding integral of motion is, quite clearly,

$$I = \frac{1}{2} m \left[ \rho \left( \frac{\vec{p}}{m} - \dot{\vec{\alpha}} \right) - \dot{\rho} (\vec{x} - \vec{\alpha}) \right]^2 + \tilde{V} \left( \frac{\vec{x} - \vec{\alpha}}{\rho} \right). \quad (5.9)$$

Of course, the existence of an integral of motion does not guarantee exact solubility; rather, it appears as the analogue of separability. Exact solubility will occur only for  $\tilde{V}(y)$  of certain functional forms. The shape invariance condition we used earlier in this paper, for example, does guarantee exact solubility of the corresponding time-dependent model.

## VI. PROPAGATORS

The propagators for the potentials (4.19) can be given, of course, in terms of either the  $x$  or the  $y$  variables. The respective expressions

$$K(x, x'; t, t') = \sum_k \Psi_k(x, t) \Psi_k^*(x', t') \quad (6.1)$$

and

$$\tilde{K}(y, y'; t, t') = \sum_k \tilde{\Psi}_k(y, t) \tilde{\Psi}_k^*(y', t') \quad (6.2)$$



are obviously related by

$$K(x, x'; t - t') = \tilde{K} \left( \frac{x - \alpha(t)}{\rho(t)}, \frac{x' - \alpha(t')}{\rho(t')}; t - t' \right), \tag{6.3}$$

since  $y = [x - \alpha(t)]/\rho(t)$ . [We find it useful to work in terms of  $\rho(t) = 1/f(t)$  in this section.] Using the wave-function transformation (4.2) and substituting in the propagator expression (6.2), we find

$$\begin{aligned} \tilde{K}(y, y'; t, t') &= [\rho(t)\rho(t')]^{-1/2} e^{\Phi(y,t)} e^{\Phi^*(y',t')} \sum_k \Xi_k(y, t) \Xi_k^*(y', t') \\ &= [\rho(t)\rho(t')]^{-1/2} e^{\Phi(y,t)} e^{\Phi^*(y',t')} \sum_k Y_k(y) Y_k^*(y') e^{-\frac{i}{\hbar} \epsilon_k [\tau(t) - \tau(t')]} . \end{aligned} \tag{6.4}$$

Noting that for the ordinary Schrödinger equation (4.14) which arises for  $\Xi(y, t)$ , the propagator is

$$K_0(y, y'; t, t') = \sum_k Y_k(y) Y_k^*(y') e^{-\frac{i}{\hbar} \epsilon_k (t - t')}, \tag{6.5}$$

we see that

$$\tilde{K}(y, y'; t, t') = [\rho(t)\rho(t')]^{-1/2} e^{\Phi(y,t)} e^{\Phi^*(y',t')} K_0(y, y'; \tau(t) - \tau(t')), \tag{6.6}$$

where  $\tau(t)$  is as defined in (4.17). Returning to the original variable  $x$  by using Eq. (6.3), and the results for  $\Phi(y, t)$  contained in (4.6)–(4.8), we find, after some calculation, that

$$\begin{aligned} K(x, x'; t, t') &= [\rho(t)\rho(t')]^{-1/2} \exp \left( -\frac{i}{\hbar} \int_{t'}^t ds h(s) \right) \exp \left[ -\frac{im}{2\hbar} \int_{t'}^t ds \left( \dot{\alpha}(s) - \frac{\dot{\rho}(s)}{\rho(s)} \alpha(s) \right) \right] \\ &\times \exp \left[ \frac{im}{2\hbar} \left( \frac{\dot{\rho}(t)}{\rho(t)} x^2 - \frac{\dot{\rho}(t')}{\rho(t')} x'^2 \right) \right] \\ &\times \exp \left[ \frac{im}{\hbar} \left( \dot{\alpha}(t) - \frac{\dot{\rho}(t)}{\rho(t)} \alpha(t) \right) x - \frac{im}{\hbar} \left( \dot{\alpha}(t') - \frac{\dot{\rho}(t')}{\rho(t')} \alpha(t') \right) x' \right] \\ &\times K_0 \left( \frac{x - \alpha(t)}{\rho(t)}, \frac{x' - \alpha(t')}{\rho(t')}; \tau(t) - \tau(t') \right) . \end{aligned} \tag{6.7}$$

Despite its length, this result is relatively elementary in nature. One should also be able to derive it in a straightforward manner in the path integral formulation [8, 10, 11], although we have not attempted to do so. One can easily check that this propagator gives the correct result in the special cases  $\alpha(t) \neq 0$ ,  $\rho(t) = 1$  and  $\alpha(t) = 0$ ,  $\rho(t) = \sqrt{rt^2 + 2pt + q}$ , by comparison with [7, 8, 11], where these special cases are considered.

We now give several examples based on the general formula (6.7). We will identify models in which our method in combination with other techniques makes the calculation of propagators relatively easy. In some cases we will actually display the propagators, while in other cases we will give the reader the necessary ingredients, but, in the interests of space and clarity, not present the full form of the propagator.

As a preliminary, we rewrite the potentials (4.19) in the form

$$V(x, t) = \frac{1}{\rho^2(t)} \tilde{V} \left( \frac{x - \alpha(t)}{\rho(t)} \right) - F(t) x + \frac{1}{2} m \omega^2(t) x^2, \tag{6.8}$$

where the functions  $\omega(t)$  and  $F(t)$  are solutions to

$$\ddot{\rho} + \omega^2(t) \rho = 0, \tag{6.9}$$

$$\ddot{\alpha} + \omega^2(t) \alpha = \frac{1}{m} F(t). \tag{6.10}$$

We will use  $\omega$  and  $F$  in lieu of  $\alpha$  and  $\rho$  as convenient.

The first example we consider is the free particle of time-independent quantum mechanics, i.e.,  $\tilde{V}(y) = 0$ , which is of course an exactly soluble model. This implies that the forced oscillator with a time-dependent frequency

$$V(x, t) = -F(t) x + \frac{1}{2} m \omega^2(t) x^2 \tag{6.11}$$

is also an exactly soluble model.

We can arrive at the same model by starting with a simple harmonic oscillator  $\tilde{V}(y) = m\omega_0^2 y^2/2$ . Then by taking  $h(t) = -m\omega_0^2 \alpha^2/2\rho^4$  we find exactly the potential (6.11) with one minor modification. Now,  $\rho(t)$  is not a solution of (6.9), but rather of

$$\ddot{\rho} + \omega^2(t) \rho = \frac{\omega_0^2}{\rho^3}. \tag{6.12}$$

Equation (6.10) remains unchanged. This model is still separable and exactly soluble, even though  $\rho$  satisfies (6.12) rather than (6.9), as the effect of the extra term in (6.12) can be absorbed into a redefinition of  $\tilde{V}(y)$  [17].

It is known that the solutions of Eqs. (6.9) and (6.12) are related. In particular, if  $\rho_0(t)$  is a solution of Eq. (6.12), then  $\rho_1(t) = \rho_0(t) \sin \omega_0 \tau_0(t)$  and  $\rho_2(t) = \rho_0(t) \cos \omega_0 \tau_0(t)$ , where  $\tau_0(t) = \int_0^t ds \, 1/\rho_0^2(s)$ , are solutions of Eq. (6.9) [18]. Also, a particular solution for Eq. (6.10) can be found using the theory of Green's functions:

$$\begin{aligned} \alpha(t) &= \int_{t_0}^t ds \frac{F(s)}{m\omega_0} \rho_0(t)\rho_0(s) \sin\{\omega_0[\tau_0(t) - \tau_0(s)]\} \\ &= \int_{t_0}^t ds \frac{F(s)}{m\omega_0} [\rho_1(t)\rho_2(s) - \rho_1(s)\rho_2(t)]. \end{aligned} \quad (6.13)$$

Alternatively, the above expression can be verified by direct substitution in Eq. (6.10).

Thus, given  $\rho_0(t)$ , we can use either the propagator of the free particle or the propagator of the simple harmonic oscillator with constant frequency to obtain the propagator of the potential (6.11). Since the resulting expression is rather long and complicated, we will not write down the result here, as its evaluation is entirely straightforward.

As a by-product, notice that the propagator of the time-independent oscillator of quantum mechanics can be derived by the one of the free particle as had been no-

ticed in [19]. In this case,  $F(t) = 0$ ,  $\omega(t) = \omega_0 = \text{const}$ , and it is enough to take  $\alpha(t) = 0$ ,  $\rho(t) = \cos \omega_0 t$ ; thus  $\tau(t) = (1/\omega_0) \tan(\omega_0 t)$ . Using the propagator of the free particle

$$K_0(y, y'; t, t') = \sqrt{\frac{m}{2\pi i \hbar (t' - t)}} \exp \left[ \frac{im}{2\hbar} \frac{(y - y')^2}{t - t'} \right], \quad (6.14)$$

we find the well-known expression

$$\begin{aligned} K(x, x'; t, t') &= \sqrt{\frac{m}{2\pi i \hbar \sin \omega_0 (t' - t)}} \\ &\times \exp \left\{ \frac{im\omega_0}{2\hbar} \left[ (x^2 + x'^2) \cot \omega_0 (t - t') \right. \right. \\ &\quad \left. \left. - \frac{2xx'}{\sin \omega_0 (t - t')} \right] \right\}. \end{aligned} \quad (6.15)$$

We now consider a related model, the free particle in

$$\tilde{V}(y) = \begin{cases} 0 & \text{if } 0 < y < L_0 \\ +\infty & \text{if } y \leq 0 \text{ or } y \geq L_0. \end{cases} \quad (6.16)$$

The exact solubility of this model implies that the infinite well with moving boundaries

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$$V(x, t) = \begin{cases} -F(t)x + \frac{1}{2} m \omega^2(t) x^2 & \text{if } \alpha(t) < x < \rho(t) L_0 + \alpha(t) \\ +\infty & \text{if } x \leq \alpha(t), x \geq \rho(t) L_0 + \alpha(t) \end{cases} \quad (6.17)$$

is also exactly soluble. The propagator of the time-independent infinite well can be written down effortlessly using the energy eigenvalues  $E_n = n^2 \pi^2 \hbar^2 / 2mL_0^2$  and eigenfunctions  $Y_n(y) = \sqrt{2/L_0} \sin(n\pi y/L_0)$ , giving

$$\begin{aligned} K_0(y, y'; t, t') &= \frac{2}{L_0} \sum_{n=1}^{+\infty} \exp \left( -\frac{i}{\hbar} \frac{n^2 \pi^2 \hbar^2}{2mL_0^2} (t - t') \right) \\ &\times \sin \frac{n\pi y}{L_0} \sin \frac{n\pi y'}{L_0}. \end{aligned} \quad (6.18)$$

This can be written in terms of the Jacobi  $\theta_3$  function,

$$\theta_3(u|z) \equiv 1 + 2 \sum_{n=1}^{+\infty} e^{in^2 \pi z} \cos(2nu), \quad (6.19)$$

giving

$$\begin{aligned} K_0(y, y'; t, t') &= \frac{1}{2L_0} \left[ \theta_3 \left( \frac{\pi(y - y')}{2L_0} \middle| -\frac{\pi \hbar (t - t')}{2mL_0^2} \right) \right. \\ &\quad \left. - \theta_3 \left( \frac{\pi(y + y')}{2L_0} \middle| -\frac{\pi \hbar (t - t')}{2mL_0^2} \right) \right]. \end{aligned} \quad (6.20)$$

It is now an easy exercise again to write down the propagator for the infinite well with moving boundaries in  $(x, t)$  space.

Using the theory of supersymmetric quantum mechanics, Das and Huang [20] have developed a very elegant way of calculating the propagators of the time-independent exactly soluble potentials of quantum mechanics. For example, using this method one can obtain the exact propagator for

$$\tilde{V}(y) = \frac{\hbar^2}{2m} \frac{n(n+1)}{y^2}, \quad n = 1, 2, 3, \dots \quad (6.21)$$

which is [19–21]

$$\begin{aligned} K_0(y, y'; t) &= \frac{m\omega_0 \sqrt{yy'}}{\hbar \sin(\omega_0 t)} i^{-(n+\frac{3}{2})} J_{n+1/2} \\ &\times \left( \frac{m\omega_0 yy'}{\hbar \sin(\omega_0 t)} \right) e^{i \frac{m}{2t} (y^2 + y'^2)}. \end{aligned} \quad (6.22)$$

Our results, then, show that not only is the time-dependent potential

$$V(x, t) = -F(t)x + \frac{1}{2} m \omega^2(t) x^2 + \frac{\hbar^2}{2m} \frac{n(n+1)}{[x - \alpha(t)]^2} \quad (6.23)$$

exactly soluble, but also that its propagator can easily be obtained from the propagator (6.22) and our fundamental propagator formula (6.7).

As a final example, we note that in [20], the propagator is found for the exactly soluble potential

$$\tilde{V}(y) = \frac{\beta^2 \hbar^2}{m} \left( \frac{-1}{\cosh^2(\beta y)} + \frac{1}{2} \right). \quad (6.24)$$

Putting the results of this paper together with those of [20], then, we can easily find the propagator for the potential

$$V(x, t) = -\frac{\beta^2 \hbar^2}{m} \frac{1}{\rho^2(t)} \cosh^{-2} \left( \beta \frac{x - \alpha(t)}{\rho(t)} \right) - F(t)x + \frac{1}{2} m \omega^2(t) x^2. \quad (6.25)$$

The final expression for the propagator is lengthy, but simple to obtain.

Needless to say, one can in the same way find the propagators for the time-dependent generalizations (4.19) of the other known exactly soluble time-independent potentials of quantum mechanics. The examples above have been selected as a sampler of typical results and techniques.

## VII. CONCLUSION

In this paper, we have explored the applicability of separation of variables to quantum-mechanical systems with explicitly time-dependent potentials. We have presented this work in one spatial dimension, but clearly the same methods can be applied in higher dimensions as well. We have found that only a very limited set of changes of variables and transformations of the wave functions can produce separability. The uniqueness we find is remarkable, and it indicates that the underlying principles that allow separation of variables are quite restrictive.

Using as our central technique separation of variables, we have also constructed exactly soluble time-dependent generalizations of any exact soluble time-independent model. In fact, our method reduces the identification of exactly soluble time-dependent models to the identification of exactly soluble time-independent models. This allows us to make contact with the very powerful insights and techniques of shape invariance, and promises

a deeper understanding of exact solubility in the time-dependent case.

Our work also makes significant contact with other work on time-dependent Hamiltonians. For example, we have used separation of variables to find the exactly soluble models discussed by [7]. We have also worked out the quantum generalization of [12], and shown that the existence of an invariant quadratic in the momentum occurs for exactly the same time-dependent Hamiltonians as separability does.

Of course, we cannot claim to have identified or classified all exactly soluble time-dependent potentials. However, the unique time-dependent generalizations produced by *any* useful changes of variables suggests that there is indeed a very intricate structure underlying the exact solubility of these models. To identify this structure — ideally, to do so directly in ordinary position space — would be a significant contribution to our understanding of exact solubility and to a possible classification of exactly soluble time-dependent models.

Furthermore, the results of this paper coincide at various points with the results of the references cited on exactly soluble time-dependent quantum mechanics [2, 7, 9, 11], despite the differing approaches of all these papers (including ours). This suggests that separation of variables is a sound guide to finding exactly soluble time-dependent potentials. Given the variety of techniques that have been applied to time-dependent quantum mechanics, all of them of course producing mutually consistent results, there is probably much progress to be made by using these various approaches in tandem.

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