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q-deformed binomial state

Jing Si-cong
Department of Modern Physics, China University of Science and Technology, Hefei, Anhui 230026, People's Republic of China

Fan Hong-yi
*China Center of Advanced Science and Technology (World Laboratory), P.O. Box 8730, Beijing, 100080, People's Republic of China and Department of Material Science and Engineering, China University of Science and Technology, Hefei, Anhui 230026, People's Republic of China**

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On the basis of the q-binomial theorem we construct the q-binomial state for a q-deformed boson which interpolates between the q-number state and the q-coherent state. The properties of the q-binomial state are also discussed.

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I. INTRODUCTION

Recently much attention has been paid to the q-deformed boson oscillator due to its possible applications in various branches of physics [1]. Some important physical concepts, such as the number state and coherent state for the ordinary harmonic oscillator and radiation field, have been extended to the q-deformed case. Besides the well-studied ordinary number state and coherent state [2] there also exists the binomial state in some radiation fields [3], which interpolates between them. In the appropriate limit the binomial state reduces to the coherent state, which corresponds to the fact that the Poisson distribution is a sort of limit of the binomial distribution [4]. From the point of view of the photon-counting statistics, the binomial state produces light that is of antibunched, sub-Poissonian, and squeezed for certain parameter ranges. An interesting question thus naturally arises: how to correctly define a q-deformed binomial state which in some limit can reduce to a q-deformed coherent state. According to mathematics literature there are three kinds of q deformation for Poisson distribution, known as the Euler, pseudo-Euler, and Heine distributions depending on the range of the parameter q [5]. The key point to solving the question is to set up a q-deformed binomial distribution which can naturally approach to the Euler (not the Heine) distribution in some definite limit. This work is organized as follows: In Sec. II we propose the normalized q binomial state which is based on the new q-deformed binomial distribution. We show that the q coherent state is some limit of the q binomial state in Sec. III, and point out that the q binomial state exhibits, an antibunched property in Sec. IV.

II. q BINOMIAL THEOREM AND q BINOMIAL STATE

Let $a^\dagger$ (a) denote the q creation (annihilation) operator which satisfies the commutator

$$aa^\dagger - qa^\dagger a = 1 \ . \tag{1}$$

The q number operator N is defined through

$$[a,N]=a, \quad [a^\dagger,N]=-a^\dagger \ . \tag{2}$$

By introducing

$$[x]=\frac{1-q^x}{1-q} \tag{3}$$

we know [6] $[N]=a^\dagger a$, which possesses the q-deformed eigenstate

$$|n\rangle=\frac{a^{\dagger n}}{\sqrt{[n]!}}|0\rangle \ , \tag{4}$$

where the q factorial $[n]!=[n][n-1]\cdots[1]$. With the use of $|n\rangle$ the q coherent state is constructed as [7,1]

$$|\alpha\rangle=(e_q^{|\alpha|^2|})^{-1/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{[n]!}}|n\rangle \ . \tag{5}$$

To define a q-deformed binomial state we should have a q binomial theorem in advance. To begin with, we define the q binomial expression $(x+y)_q^n$ as

$$(x+y)_q^n=(x+y)(x+qy)\cdots(x+q^{n-1}y)$$
$$=\prod_{k=0}^{n-1}(x+q^k y) \quad (n=1,2,3,\ldots) \ ,$$
$$(x+y)_q^0=1 \ , \tag{6}$$

which implies that the q binomial $(x+y)_q^n$ does not obey the multiplication rule of the ordinary binomials. In fact, we have

$$(x+y)_q^{n+m}=(x+y)_q^m(x+q^m y)_q^n$$
$$=(x+y)_q^n(x+q^n y)_q^m \ , \tag{7}$$

where m and n are both non-negative integers. Correspondingly, we can express the q binomial theorem [8] as [9]

*Mailing address.

$$(x+y)_q^n=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2}x^{n-k}y^k , \tag{8}$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_q=\frac{[n]!}{[k]![n-k]!} . \tag{9}$$

Based on this, we are able to define the q binomial state for $0<q<1$ as

$$|\tau,m\rangle=\sum_{n=0}^{m}\sqrt{b_q(n;m,\tau)}|n\rangle \quad (0<\tau<1) , \tag{10}$$

where

$$b_q(n;m,\tau)=\begin{bmatrix} m \\ n \end{bmatrix}_q \tau^n(1-\tau)_q^{m-n} \tag{11}$$

is the new q-deformed binomial distribution. It is worth pointing out that the expression $(1-\tau)_q^{m-n}$ in Eq. (11) must be understood as (6).

As a result of (10) we can prove that the state $|\tau,m\rangle$ is normalized, e.g.,

$$\begin{aligned}\langle\tau,m|\tau,m\rangle&=\sum_{n=0}^{m}b_q(n,m,\tau)=\sum_{n=0}^{m}\sum_{k=0}^{m-n}\begin{bmatrix} m \\ n \end{bmatrix}_q\begin{bmatrix} m-n \\ k \end{bmatrix}_q q^{k(k-1)/2}(-)^k\tau^{n+k}=\sum_{l=0}^{m}\sum_{k=0}^{l}\frac{(-1)^k[m]!}{[m-l]![k]![l-k]!}q^{k(k-1)/2}\tau^l\\&=\sum_{l=0}^{m}\begin{bmatrix} m \\ l \end{bmatrix}_q\tau^l\sum_{k=0}^{l}\begin{bmatrix} l \\ k \end{bmatrix}_q(-)^kq^{k(k-1)/2}=\sum_{l=0}^{m}\begin{bmatrix} m \\ l \end{bmatrix}_q\tau^l(1-1)_q^l=\begin{bmatrix} m \\ 0 \end{bmatrix}_q=1 .\end{aligned} \tag{12}$$

III. ASYMPTOTIC BEHAVIOR OF THE q BINOMIAL STATE

In this section we explain that the q binomial state interpolates between the q number state and the q coherent state. From (10) is is easily seen that for $\tau=0$ and $\tau=1$ the q binomial state $|\tau,m\rangle$ reduces to the q number state $|n=0\rangle$ and $|n=m\rangle$, respectively. On the other hand, in the limit $m\to\infty$ we shall show that $|\tau,m\rangle$ reduces to a q-deformed coherent state. This statement will be apparent by virtue of the q analog of the limitation expression of the ordinary exponential

$$\lim_{n\to\infty}\left[1+\frac{x}{n}\right]^n=e^x ,$$

namely,

$$\lim_{n\to\infty}\left[1+\frac{x}{[n]}\right]_q^n=e_{1/q}^x , \tag{13}$$

where

$$\begin{aligned}e_{1/q}^x&=\sum_{n=0}^{\infty}\frac{x^n}{[n]_{q^{-1}}!} ,\\ [n]_{q^{-1}}&=\frac{1-q^{-n}}{1-q^{-1}}=q^{1-n}[n] .\end{aligned} \tag{14}$$

In fact, by using (6) and noticing that $\lim_{n\to\infty}[n]=(1-q)^{-1}$, we have

$$\begin{aligned}\lim_{n\to\infty}\left[1+\frac{x}{[n]}\right]_q^n&=\lim_{n\to\infty}\prod_{k=0}^{n-1}\left[1+q^k\frac{x}{[n]}\right]\\&=\prod_{k=0}^{\infty}\{1+(1-q)q^kx\}\\&=(e_q^{-x})^{-1} .\end{aligned} \tag{15}$$

Because $(e_q^x)^{-1}=e_{1/q}^{-x}$ (see [8]), so Eq. (13) is true. Further, by introducing λ through the relation

$$\lambda=\lim_{m\to\infty}\tau[m]=\frac{\tau}{1-q} , \tag{16}$$

and using Eqs. (11) and (15), we obtain

$$\begin{aligned}\lim_{m\to\infty}b_q(0;m,\tau)&=\lim_{m\to\infty}(1-\tau)_q^m\\&=\prod_{k=0}^{\infty}\{1-(1-q)q^k\lambda\}=(e_q^\lambda)^{-1} .\end{aligned} \tag{17}$$

It also follows from (11) and (3) that

$$\begin{aligned}\lim_{m\to\infty}\frac{b_q(n;m,\tau)}{b_q(n-1;m,\tau)}&=\lim_{m\to\infty}\frac{[m-n+1]\tau}{[n](1-q^{m-n}\tau)}\\&=\lim_{m\to\infty}\frac{\tau-q^{m-n+1}\tau}{[n](1-q)(1-q^{m-n}\tau)}\\&=\frac{\tau}{[n](1-q)}=\frac{\lambda}{[n]} .\end{aligned} \tag{18}$$

Combining the results (17) and (18), we arrive at

$$\lim_{m\to\infty}b_q(n;m,\tau)=\frac{\lambda^n}{[n]!}(e_q^\lambda)^{-1} , \tag{19}$$

which is just the q Poisson (or Euler) distribution, as it is the q analog of the ordinary Poisson distribution. Equation (19) also means that the q binomial distribution (11) takes the Euler (not Heine) distribution as its limiting form. Using Eqs. (10) and (18) we have

$$\lim_{m\to\infty}|\tau,m\rangle=(e_q^\lambda)^{-1/2}\sum_{n=0}^{\infty}\frac{\lambda^{n/2}}{\sqrt{[n]!}}|n\rangle , \tag{20}$$

which tells us that in the limit $m\to\infty$ the state $|\tau,m\rangle$ approaches to the q coherent state $|\alpha=\sqrt{\lambda}\rangle$.

IV. SOME PROPERTIES OF $|\tau,m\rangle$

As concluded in Ref. [3], the ordinary binomial state may exhibit antibunched, sub-Poissonian behavior, so it is interesting to examine how the q binomial state behaves. Acting the annihilator a on $|\tau,m\rangle$ and using the identity

$$[n]b_q(n;m,\tau)=[m]\tau b_q(n-1;m-1,\tau)\ , \qquad (21)$$

we have

$$\begin{aligned} a|\tau,m\rangle &= \sum_{n=1}^{m} \sqrt{[n]b_q(n;m,\tau)}|n-1\rangle \\ &= \sqrt{[m]\tau}\sum_{n=0}^{m-1}\sqrt{b_q(n;m-1,\tau)}|n\rangle \\ &= \sqrt{[m]\tau}|\tau,m-1\rangle\ . \end{aligned} \qquad (22)$$

Hence, the q operator a acts as a sort of lowering operator for the state $|\tau,m\rangle$. On the other hand, we obtain

$$a^\dagger|\tau,m\rangle=\frac{1}{\sqrt{[m+1]\tau}}[N]|\tau,m+1\rangle\ . \qquad (23)$$

It then follows that

$$a^n|\tau,m\rangle=\left[\frac{\tau^n[m]!}{[m-n]!}\right]^{1/2}|\tau,m-n\rangle\ , \qquad (24)$$

$$a^{\dagger n}|\tau,m\rangle=\left[\frac{\tau^{-n}[m]!}{[m+n]!}\right]^{1/2}[N][N-1]\cdots[N-n+1]|\tau,m+n\rangle=\left[\frac{\tau^{-n}[m]!}{[m+n]!}\right]^{1/2}a^{\dagger n}a^n|\tau,m+n\rangle \qquad (25)$$

and the expectation values of a^n and $a^{\dagger n}$ in the state $|\tau,m\rangle$ are

$$\langle a^n\rangle=\left[\frac{\tau^n[m]!}{[m-n]!}\right]^{1/2}\langle\tau,m|\tau,m-n\rangle\ , \qquad (26)$$

$$\langle a^{\dagger n}\rangle=\left[\frac{\tau^n[m]!}{[m-n]!}\right]^{1/2}\langle\tau,m-n|\tau,m\rangle\ , \qquad (27)$$

respectively. Using the definition (10) it is also straightforward to calculate

$$\begin{aligned}\langle[N]\rangle &= \sum_{n=0}^{m}[n]b_q(n;m,\tau) \\ &= \sum_{n=1}^{m}\frac{[m]!}{[n-1]![m-n]!}\tau^n(1-\tau)_q^{m-n} \\ &= [m]\tau\ ,\end{aligned} \qquad (28)$$

$$\langle[N]^2\rangle=[m]\tau+q[m][m-1]\tau^2\ , \qquad (29)$$

where the equality $[n+1]=1+q[n]$ was used. From (28) and (29) we obtain the variance

$$\langle([N]-\langle[N]\rangle)^2\rangle=[m]\tau(1-\tau)\ . \qquad (30)$$

Equations (28) and (29) are the q analog of the ordinary binomial probability distribution, so we say that the q binomial state yields a q-deformed distribution for the q boson. Moreover, from (29) and (30) we know

$$\frac{\langle([N]-\langle[N]\rangle)^2\rangle}{\langle[N]\rangle}=1-\tau\ , \qquad (31)$$

which exhibits the sub-Poissonian nature of the q binomial distribution. With use of (24) and (25) we can also calculate the second-order correlation function for $|\tau,m\rangle$,

$$G^{(2)}=\frac{\langle a^{\dagger 2}a^2\rangle}{\langle a^\dagger a\rangle^2}=\frac{[m-1]}{[m]}=q^{-1}\left[1-\frac{1}{[m]}\right]<1\ . \qquad (32)$$

We can conclude that the q binomial state is antibunched.

In summary, we have extended the conception of the binomial state of a radiation field to q-deformed case which is definitely a supplement to the q-deformed number state and q-deformed coherent state. The relation between the q binomial distribution and the q-Poisson distribution has been revealed and the q antibunched property of the q binomial state been investigated.

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