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Quantization of constraint solutions

E. Kapuscik* and C. A. Uzes†
Department of Physics and Astronomy, University of Georgia, Athens, Georgia 30602

A. O. Barut
Physics Department, University of Colorado, Boulder, Colorado 80309
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The problem of connecting systems with different numbers of degrees of freedom is discussed. Constraints appropriate for "Bose" and "Fermi" quantization are used to construct algebras of Dirac brackets associated with special solutions of the nonlinear complex oscillator. The constraints are shown to provide a basis for characterizing the elementary excitations of the oscillators. An alternative notion of quantization through a correspondence with an enveloping subalgebra of the Dirac brackets is introduced, a notion which simplifies the operator-ordering problem implied by the original Dirac brackets. The infinite- and the two-dimensional representations of the subalgebra are utilized to illustrate the quantization technique.

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I. INTRODUCTION

In order to avoid the inherent difficulties involved in the quantization of a nonlinear relativistic field theory, it has been postulated [1,2] that the spectrum of the quanta of these equations can be obtained through the quantization of the elementary excitations of the nonlinear field rather than the quantization of the field itself. In this procedure some ansatz for simple, special solutions (such as the traveling-wave motion) is employed to reduce the system with an infinite number of degrees of freedom to one of finite number, and to use the canonical formalism for the latter as a basis for the quantization. One outstanding problem is to establish a direct connection between the canonical formalisms of the two systems so obtained which have different degrees of freedom. This is our main goal and motivation.

On the other hand, Dirac's second-class constraints have been (artificially) imposed on a canonical system in order to obtain a canonical description of a system with fewer degrees of freedom [3]. In a sense the notion of constraint solutions could serve as a more precise definition of the notion of "elementary excitations," namely, as the complete set of solutions associated with embedded constrained systems having fewer degrees of freedom, if possible one, then two, and then in increasing order.

The constraints and their associated Dirac brackets are not viewed as necessarily following from the definition of the action, but are imposed as additional conditions with the requirement that the result be self-consistent. Thus the Dirac bracket here is considered as a new Poisson bracket generated by certain constraints. The self-consistency stems from the requirement that the solutions to the equations of motion for the constrained system be simultaneously solutions to the equations of motion of the original unconstrained system [3]. The converse is, of course, not true.

After the classical Dirac bracket appropriate to a given set of constraints has been constructed, there still remains the difficulty of its quantization. This is because the Dirac bracket itself is usually a function of the canonical variables of the constrained system and one has the problem of ordering these variables in the passage to the quantum theory. For situations where this is difficult it may be possible that elements of the enveloping algebra of the Dirac bracket define a simpler subalgebra through which the quantization can be performed unambiguously. It has been shown that for some nonlinear Hamiltonians the requirement of a dynamical-symmetry algebra allows one to choose a particular ordering for which the problem is exactly soluble [4].

The purpose of the present work is to establish the unification of the concepts of the elementary excitations, constraint embedding, and quantization. The theory is developed below in the context of the "Bose" and "Fermi" quantizations of a class of constrained, complex, nonlinear oscillators. The generalizations to field theory will be reported separately. However, it should be pointed out that both of the above quantizations have their analogs and utilization in field theory and their elaboration here is more than an exercise. There has been increased activity and vast literature in the quantization of constrained systems in recent years, chiefly in connection with reducing the gauge degrees of freedom in gauge theories [5]. However, since both our method and goals are different we do not give extensive references to these methods.

*Present address: Institue of Physics and Informatics, Krakow Pedagogical University and Institute of Nuclear Physics, Krakow, Poland.

II. EMBEDDING AND QUANTIZATION PROCEDURE

Consider a Hamiltonian system described by the Hamiltonian

$$H=H(q,p)\ , \tag{1}$$

where q and p are the system's canonical coordinates and momenta. Associated with the Hamiltonian (1) are the canonical equations of motion

$$\frac{dq}{dt}=[q,H]',\quad \frac{dp}{dt}=[p,H]'\ , \tag{2}$$

where $[\ ,\ ]'$ refers to the usual Poisson bracket.

Now consider the imposition of constraint functions

$$\phi_i(q,p)=0,\quad i=1,2,\ldots,2r\ .$$

We want these constraints to be second class in the sense that we can construct a "new" Poisson bracket according to the rule of Dirac [6]. This is possible provided

$$[H,\phi_i]'=\sum_j^{2r}\alpha_{ij}(q,p)\phi_j\ , \tag{3}$$

and provided the determinant $|\Delta(q,p)_{ij}|$ of the matrix formed from the Poisson brackets $[\phi_i,\phi_j]'$ of the constraints nonvanishing. The "Dirac bracket" is then defined in the usual way [6],

$$[f,g]_D=[f,g]'-\sum_{ij}^{2r}[f,\phi_i]'c_{ij}[\phi_j,g]'\ , \tag{4}$$

where the $c_{ij}=-c_{ji}$ are the elements of the inverse of the matrix Δ. Note that as a consequence of its definition the Dirac bracket has the property that it is antisymmetric with respect to interchange of its arguments.

We assume here that there are only second-class constraints present. Then associated with the new bracket, $[\ ,\ ]_D$, are the equations of motion

$$\frac{dg}{dt}=[q,H_\tau]_D,\quad \frac{dp}{dt}=[p,H_r]_D\ , \tag{5}$$

where

$$H_r=H(q,p)|_{\Phi_i=0}$$

is the Hamiltonian of the reduced constrained system. Here the self-consistency of the elementary excitations is equivalent to the requirement that solutions of (5) also be solutions of (2).

We wish now to quantize the constrained system (5). The main difficulty lies in the ordering problem. The set of fundamental Dirac brackets is of the form,

$$[\xi,\xi']_D=f(\xi,\xi')\ , \tag{6}$$

where ξ and ξ' are the new canonical coordinates and momenta whose Dirac bracket determines all other Dirac brackets of the system (5). The problem of quantization is then to give meaning to the correspondence of (5) and (6) as operator expressions with an implied solution to the ordering problem.

In what follows we carry out this program for two choices of constraints on the nonlinear complex oscillator, corresponding to the "Fermi" and "Bose" quantizations. For both types of quantization we consider first the construction of elements of an enveloping subalgebra of the algebra of the Dirac brackets. This subalgebra is then used to define a Lie algebra in terms of which the constrained Hamiltonian and all essential variables can be evaluated. Then we show how this carries over to quantization and leads us to solve the operator ordering problem of (5) and (6).

III. CLASSICAL CONSTRAINED COMPLEX OSCILLATORS

We consider the class of nonlinear complex oscillators described by the Hamiltonian

$$H=p^*p+\lambda\frac{(qq^*)^{N+1}}{N+1} \tag{7}$$

(N is an integer), and the consequent equations of motion

$$\ddot q+\lambda(q^*q)^Nq=0,\quad \ddot q^*+\lambda(q^*q)^Nq^*=0\ . \tag{8}$$

With this system one should keep in mind that

$$p^*=\dot q,\quad p=\dot q^*\ . \tag{9}$$

The fundamental brackets are

$$\begin{gathered}[q,p]'=[q^*,p^*]'=1\ ,\\ [q,q^*]'=[q^*,q]'=[q,p^*]'=[q^*,p]'=0\ .\end{gathered} \tag{10}$$

There are two classes of constraints that are important to consider, those typified by "Bose" and "Fermi" quantization.

A. "Bose" constraints

Let

$$\begin{gathered}qp+p^*q^*=\phi_1\ ,\\ -\lambda(q^*q)^{N+1}+pp^*=\phi_2\ .\end{gathered} \tag{11}$$

Then using (10) one finds that the constraints satisfy the conditions required of second-class constraints relative to the Hamiltonian, i.e.,

$$[H,\phi_1]'=-2\phi_2,\quad [H,\phi_2]'=\lambda(q^*q)^N\phi_1\ ,$$

while the matrix C of (4) has the form

$$C=\frac{1}{2(N+2)p^*p}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\ . \tag{12}$$

Using this C matrix and the definition (4) one can calculate the fundamental Dirac brackets. These are

$$\begin{gathered}[q,q^*]_D=\frac{q^*}{(N+2)p},\quad [q,p]_D=\tfrac{1}{2}\ ,\\ [q,p^*]_D=\frac{-(N+1)q^2}{2(N+2)qq^*}-\frac{p^*}{2(N+2)p}\ ,\\ [p,p^*]_D=-\frac{(N+1)}{(N+2)}\frac{qp}{q^*q}\ .\end{gathered} \tag{13}$$

Using these (and their complex conjugates) and the antisymmetric property of the Dirac brackets one easily finds that

$$[qq^*,qp]_D=0 \tag{14}$$

and that qp and qq^* are both constants of the motion of the constraint system,

$$[H_r,qp]_D=[H_r,qq^*]_D=0 \ , \tag{15}$$

since

$$H_r=\frac{N+2}{N+1}\lambda(qq^*)^{N+1} \ . \tag{16}$$

It is not difficult to ascertain from (5) that the canonical variables q and q^* satisfy the first-order equations

$$\dot q=\frac{p^*q^*}{qq^*}q, \quad \dot q^*=\frac{qp}{qq^*}q^* \ . \tag{17}$$

These equations can be viewed classically in two ways, either as definitions of p and p^*, or, in view of (15), as conditions on the solutions for q and q^*. With the latter view it is easily seen that solutions of (17) are automatically also solutions of (8).

We now consider an interesting relation that will be used in the quantization of the system described by (17). Consider the following elements in the enveloping algebra of the p's and q's:

$$\begin{aligned} x&=\sqrt{2}q[\lambda(p^*q^*)^N]^{1/2(N+2)} \ , \\ x^*&=\sqrt{2}[\lambda(qp)^N]^{1/2(N+2)}q^* \ . \end{aligned} \tag{18}$$

These variables are conjugate relative to the Dirac bracket (13),

$$[x,x^*]_D=-i \ . \tag{19}$$

Moreover, the reduced Hamiltonian can be expressed as a function of xx^* alone,

$$H=\frac{(N+2)}{(N+1)}\lambda^{1/(N+2)}\left[\frac{xx^*}{2}\right]^{2(N+1)/(N+2)} \ . \tag{20}$$

Thus since

$$[xx^*,x]_D=ix, \quad [xx^*,x^*]_D=-ix^* \tag{21}$$

the variables x, x^*, I, and xx^* define a Lie subalgebra, isomorphic to a complex boson algebra, and the constrained Hamiltonian can be expressed solely in terms of these variables.

B. "Fermi" constraints

In this case we consider the new constraints

$$\begin{aligned} p^*+i\sqrt{\lambda}q(qq^*)^{N/2}&=\phi^+ \ , \\ p-i\sqrt{\lambda}(qq^*)^{N/2}q^*&=\phi^- \ . \end{aligned} \tag{22}$$

These constraints are associated with the factorization of (8), using (9) and (10),

$$\left[\frac{d}{dt}+i\sqrt{\lambda}(qq^*)^{N/2}\right]\left[\frac{d}{dt}-i\sqrt{\lambda}(qq^*)^{N/2}\right]q=0 \ , \tag{23}$$

which is the simplest analog of the field-theoretic treatment of obtaining the first-order Dirac equation through factorization of the Klein-Gordon equation. Factorization may be viewed as a special case of constraint dynamics.

The constraints (22) are again of second class with

$$\begin{aligned}[H,\phi^\pm]'=&\mp\frac{N}{2}i\sqrt{\lambda}\frac{q^2}{qq^*}(q^*q)^{N/2}\phi^\mp \\ &\mp i\sqrt{\lambda}\frac{N}{2}(qq^*)^{N/2}\phi^\pm \ , \end{aligned} \tag{24}$$

while

$$[\phi^+,\phi^-]'=-(N+2)i\sqrt{\lambda}(qq^*)^{N/2} \ .$$

From this last equation we construct the Dirac C matrix

$$C=\frac{1}{(N+2)i(qq^*)^{N/2}\sqrt{\lambda}}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} \ , \tag{25}$$

which enters into the definition (4) of the Dirac bracket.

Using (4) and (10), the fundamental Dirac bracket for the Fermi constrained system is then found to be

$$[q,q^*]_D=\frac{1}{(N+2)i\lambda^{1/2}(qq^*)^{N/2}} \ , \tag{26}$$

while the reduced Hamiltonian in the presence of these constraints ($\phi^\pm=0$) has again precisely the same form (16). The resulting equations of motion are

$$\dot q+i\sqrt{\lambda}q(qq^*)^{N/2}=0 \ , \quad \dot q^*-i\sqrt{\lambda}q^*(qq^*)^{N/2}=0 \ . \tag{27}$$

Since qq^* is clearly a constant of the motion, it is easy to verify that the solutions of (27) are automatically also solutions of (8).

Now consider the bracket (26). The new variables

$$x=\sqrt{2\lambda^{1/2}}(qq^*)^{N/4}q, \quad x^*=\sqrt{2\lambda^{1/2}}q^*(qq^*)^{N/4} \tag{28}$$

have again the property that

$$[x,x^*]_D=-i \ , \tag{29}$$

and we have again a subalgebra with elements x, x^*, I, and xx^*. In terms of these variables

$$H_r=\frac{N+2}{N+1}\frac{1}{\lambda^{1/(N+2)}}\left[\frac{xx^*}{2}\right]^{2(N+1)/(N+2)} \ , \tag{30}$$

which is formally the same as that of the classical "Bose" expression (20).

Finally it should be pointed out that the form of the solutions to (17) and (27) are identical in this classical situation [use of the Bose system constraints (11) takes (18) into (28)]. Thus in either case the solutions have the form

$$q=\alpha e^{-i\sqrt{\lambda}t|\alpha|^N}, \quad q^*=\alpha^*e^{i\sqrt{\lambda}t|\alpha|^N} \ . \tag{31}$$

However, from an algebraic point of view the two systems differ. This difference clearly manifests itself in the quantum theory.

IV. "BOSE" AND "FERMI" CONSTRAINT QUANTIZATIONS

In both the "Bose" and "Fermi" constrained systems the reduced Hamiltonian can be expressed solely in terms of the product xx^*, where x and x^* satisfy (19) or (29). Thus it seems reasonable to attempt to carry out the "quantization" in terms of the operator algebras containing the operators X and X^* corresponding to the reduced classical algebras containing x and x^*. By quantization we mean to construct representations of the Lie algebras of the Dirac brackets for both the Bose and Fermi constrained systems in such a way that the equations of motion and operator ordering are consistently defined according to a correspondence principle. Both the Bose and Fermi quantizations given below are carried out in the Heisenberg picture and in representations in which the Hamiltonians are diagonal.

A. "Bose" quantization

Let X, X^* be the boson creation and annihilation operators with quantum brackets,

$$[X,X^*]=I,\quad [XX^*,X]=-X,\quad [XX^*,X^*]=X^* , \tag{32}$$

hence generating an infinite dimensional representation of the Hilbert space of the harmonic oscillator. In addition, the number operator XX^* is diagonal and invertible. The matrix elements in question are

$$\begin{aligned} X_{kl}&=\sqrt{k+1}\delta_{k,l-1}\exp(-i\omega_{kl}t) ,\\ X_{kl}&=\sqrt{k}\,\delta_{k,l-1}\exp(-i\omega_{kl}t) , \end{aligned} \tag{33}$$

where $k,l=0,1,\ldots,\infty$, while clearly

$$(XX^*)_{kl}=(k+1)\delta_{kl} . \tag{34}$$

The ordering problem is facilitated by arranging factors such that only the roots of diagonal operators need to be taken.

We wish now to quantize the commuting reduced system (15) and (16). The ordering suggested in (7) and (11) will be assumed to be correct for the quantum operators Q,Q^*,P,P^* corresponding to the reduced classical variables q,q^*,p,p^*, i.e.,

$$H=P^*P+\lambda\frac{(QQ^*)^{N+1}}{N+1} , \tag{35}$$

$$QP+P^*Q^*=0 , \tag{36}$$

$$-\lambda(Q^*Q)^{N+1}+PP^*=0 . \tag{37}$$

As was the case in the classical corresponding system, H, QP, and P^*P can all be expressed in terms of the diagonal operator XX^*, provided one starts with the identification

$$QQ^*=\left[\frac{XX^*}{2\sqrt{\lambda}}\right]^{2/(N+2)} , \tag{38}$$

which is suggested by the classical identification (18). Here the nth root of a diagonal operator is defined as the diagonal operator whose diagonal elements are the nth roots of the corresponding diagonal elements of the original.

The reduced constraint relations (36) and (37) then lead to

$$(QPP^*Q^*)=\lambda(QQ^*)^{N+2} , \tag{39}$$

$$QP=i\sqrt{\lambda}(QQ^*)^{(N+2)/2} , \tag{40}$$

provided N is an integer. Hence using (36)–(38) one has

$$QP=\frac{i}{2}(XX^*) , \tag{41}$$

$$P^*P=\lambda^{1/(N+2)}\left[\frac{XX^*}{2}\right]^{2(N+1)/(N+2)} , \tag{42}$$

and

$$H=\frac{(N+2)}{(N+1)}\lambda^{1/(N+2)}\left[\frac{XX^*}{2}\right]^{2(N+1)/(N+2)} . \tag{43}$$

Thus the classical expression (20) for H carries over into the quantum theory and the operators (QQ^*), (P^*P), (QP), and (P^*Q^*) are well defined.

It remains to specify Q, Q^*, P, P^*, and Q^*Q in a consistent way. The task is to be able to express in a consistent fashion these fundamental dynamical variables in terms of the boson operators $I,X,X^*,XX^*(X^*X=1+XX^*)$. Since the operator XX^* and its roots are invertible, this is accomplished by giving a proper ordering [consistent with (38)] to the classical correspondences (18),

$$\begin{aligned} Q&=(2\sqrt{\lambda}i^{N/2})^{-1/(N+2)}(XX^*)^{-N/2(N+2)}X ,\\ Q^*&=(2\sqrt{\lambda}i^{N/2})^{-1/N+2}X^*(XX^*)^{-N/2(N+2)} , \end{aligned}$$

which from (34) results in

$$\begin{aligned} Q_{kl}&=\left[\frac{k+1}{2\sqrt{\lambda}i^{N/2}}\right]^{1/N+2}\delta_{k,l-1}e^{-i\omega_{kl}t} ,\\ Q^*_{kl}&=\left[\frac{k}{2\sqrt{\lambda}(-i)^{N/2}}\right]^{1/(N+2)}\delta_{k,l+1}e^{i\omega_{kl}t} . \end{aligned} \tag{44}$$

Multiplication of (37) first from the right by Q and then from the left by Q^* then leads to the relations

$$P^*=\lambda\frac{1}{(QP)}Q(Q^*Q)^{N+1} , \tag{45}$$

$$P=\lambda(Q^*Q)^{N+1}Q^*\frac{1}{P^*Q^*} . \tag{46}$$

But from (44) one can see that

$$Q^*Q=\left[\frac{X^*X}{2\sqrt{\lambda}}\right]^{2/N+2} . \tag{47}$$

Thus (38), (41), and its Hermitian conjugate, and (47) shows that P^* and P can also be consistently expressed as well-defined functions of X, X^*, and (XX^*) or (X^*X), completing the requirement of defining the quantum realizations of the relevant variables.

For complete consistency it is now necessary to show in what sense this quantized system corresponds to the classically constrained system. This correspondence takes place through the traditional classical limit of the large quantum number.

For example, we consider the correspondence of the fundamental Dirac brackets. Using (33) and (37), we have

$$PP^*-P^*P=\lambda[(Q^*Q)^{N+1}-(QQ^*)^{N+1}] .$$

Thus using (34) and the explicit representation (44) we find that in the limit of very large k,

$$\langle k|[P,P^*]|k'\rangle \to -\frac{(N+1)}{(N+2)}\frac{qp}{q^*q}\delta^{kk'} ,$$

where

$$\langle k|Q|k+1\rangle \to q, \quad \langle k|P|k-1\rangle \to p .$$

Similarly,

$$[Q,Q^*]=\left[\frac{XX^*}{2\sqrt{\lambda}}\right]^{2/(N+1)}-\left[\frac{X^*X}{2\sqrt{\lambda}}\right]^{2/(N+2)}$$

so that

$$\langle k|[Q,Q^*]|k'\rangle \to \frac{p^*q^*}{(N+2)pp^*}\delta_{kk'} . \tag{48}$$

One can also verify in the limit of large k that the classical equations of motion (8) and (17) follow from Heisenberg's equation of motion

$$\dot{F}=i[H,F] .$$

B. "Fermi" quantization

Here we use the anticommutator algebra

$$[X,X]_+=0, \quad [X^*,X^*]_+=0, \quad [X,X^*]_+=i[x,x^*]_D I , \tag{49}$$

and consequent relations

$$[XX^*,X]=-X, \quad [XX^*,X^*]=X^* . \tag{50}$$

The reversal of order in the fundamental Dirac bracket corresponds to Hermitian conjugation in the quantum theory. The fundamental Dirac bracket is here interpreted as providing a normalization.

Since canonical anticommutation relations only have finite-dimensional representations, we look for a realization of the two-dimensional representation

$$X=\beta\begin{bmatrix}0&1\\0&0\end{bmatrix}e^{-i\omega t}, \quad X^*=\beta^*\begin{bmatrix}0&0\\1&0\end{bmatrix}e^{i\omega t} , \tag{51}$$

such that Q and Q^* can be obtained from the analog of (28). Classically this would require that

$$x=\sqrt{2}\lambda^{1/2}\left[\frac{xx^*}{\sqrt{2}\lambda^{1/2}}\right]^{N/2(N+2)} q . \tag{52}$$

Since XX^* is diagonal its nth root is defined by the diagonal matrix constructed from the nth root of its diagonal elements. Since here the operator correspondence of xx^* is not invertible, a direct quantum analog is consistent only with the requirement that

$$XX^*Q=Q, \quad XX^*Q^*=0 \quad (X^*XQ^*=Q^*, \ X^*XQ=0) .$$

Thus the quantum analog of q is

$$Q=\left[\frac{\beta}{\sqrt{2}\lambda^{1/2}}\right]^{2/N+2}\begin{bmatrix}0&1\\0&0\end{bmatrix}e^{-i\omega t} ,$$
$$Q^*=\left[\frac{\beta}{\sqrt{2}\lambda^{1/2}}\right]^{2/(N+2)}\begin{bmatrix}0&0\\1&0\end{bmatrix}e^{i\omega t} . \tag{53}$$

The quantization is completed through the requirement that (53) be consistent with the statement that Heisenberg's equations of motion be equivalent to the constraints (27).

In the classical theory, the equations of motion of the constrained system are given by (27). Since QQ^* is a function of XX^* we choose the ordering such that the quantum analog of (27) is

$$\dot{Q}+i\sqrt{\lambda}(QQ^*)^{N/2}Q=0 , \quad \dot{Q}^*-i\sqrt{\lambda}Q^*(QQ^*)^{N/2}=0 , \tag{54}$$

and that the classical Hamiltonian carries over directly into the quantum theory without ordering changes. Consistency of (53) and (54) with Heisenberg's equations implies finally

$$H=\left|\frac{N+2}{N+1}\right|^{N/N+2}\lambda^{1/(N+2)}\begin{bmatrix}0&0\\0&1\end{bmatrix} . \tag{55}$$

V. CONCLUSION

In this work we have defined elementary excitations of the nonlinear complex oscillator by second-class constraints leading to new Poisson brackets. We then required that the solutions to the equations of motion of the constrained system be also solutions to the equations of motion of the original oscillator. It was found that the "Dirac bracket" algebras of both the "Bose" and "Fermi" systems contained a dynamical subalgebra, the well known harmonic-oscillator algebra, and that all the relevant dynamical variables could be expressed in terms of properly ordered elements of the latter, which one may call the "dressed oscillator." Thus the basic notion is that a subalgebra defined by the constraints and their consequent Dirac brackets defines both the elementary excitations and the quantization in this simple model, because the quantization is defined by the representations isomorphic to this subalgebra. The operator-ordering problem was solved by expressing the products of operators in terms of the diagonal elements of this algebra wherever possible. The proper classical correspondence of the equations of motion were also found. The generalization of this procedure to the field-theoretic context will be discussed elsewhere [7].

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