Minimum-uncertainty states for noncanonical operators

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The usual uncertainty relation $\Delta A^2 \Delta B^2 \geq \langle C\rangle^2/4$ between two Hermitian operators A and B satisfying the noncanonical commutation relation $[A, B] = iC$, where C is not a constant multiple of the unit operator, fails to give a nontrivial lower bound on the product of the variances ΔA^2 and ΔB^2 when $\langle C\rangle = 0$. For those operators, therefore, the general uncertainty relation $\Delta A^2\Delta B^2 \geq [\langle C\rangle^2 + \langle F\rangle^2]/4$ where $\langle F\rangle = \langle AB+BA\rangle - 2\langle A\rangle\langle B\rangle$ is better suited to determine the lower bound on $\Delta A^2 \Delta B^2$. The implications of the general uncertainty relation and the properties of the minimum-uncertainty states, i.e., the states for which the general uncertainty relation is satisfied with equality, are discussed. The minimum-uncertainty states are found to fall in two different classes. One is the usually studied class for which $\langle F\rangle = 0$, that is, the case when the usual uncertainty relation holds. The other is the hitherto unnoticed class for $\langle C\rangle = 0$, that is, when the usual uncertainty relation is redundant. The squeezing properties of the present class of minimum-uncertainty states is discussed by defining squeezing in the light of the general uncertainty relation.

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The variances ΔA^2 and ΔB^2 in the measurement of the observables associated with the Hermitian operators A and B satisfying the commutation relation

$$[A, B] = iC, \tag{1}$$

where C is Hermitian, are related by the uncertainty relation [1]

$$\Delta A^2 \Delta B^2 \geq \frac{\langle C\rangle^2 + \langle F\rangle^2}{4}, \tag{2}$$

where

$$\langle F\rangle = \langle AB + BA\rangle - 2\langle A\rangle\langle B\rangle, \tag{3}$$

is a measure of correlations between A and B. However, it is the special form

$$\Delta A^2\Delta B^2 \geq \frac{\langle C\rangle^2}{4} \tag{4}$$

of the general uncertainty relation (2) that is widely used. The special form (4) is, of course, identical with the general form (2) if A and B are uncorrelated, i.e., if $\langle F\rangle = 0$. The uncertainty relation in its special form (4) becomes redundant; in other words, it does not determine a nontrivial lower bound on the product of the uncertainties in the measurement of two noncommuting operators if $\langle C\rangle = 0$. For canonical operators, i.e., the operators for which C is a constant multiple of the unit operator $\langle C\rangle$ is, of course, never zero and hence, for those operators, (4) is never redundant. The general uncertainty relation (2) is therefore better suited to determine the lower bound on the product of variances in the measurement of observables corresponding to noncanonical operators. Here we study the implications of that relation. First, we identify the class of states for which even the general relation (2) is redundant. Next, we find the states for which (2) is satisfied with equality. Those states are the solutions of the eigenvalue equation

$$[A - \langle A\rangle]|\psi\rangle = -i\lambda[B - \langle B\rangle]|\psi\rangle. \tag{5}$$

We show that, for noncanonical operators, the states $|\psi\rangle$ in (5) fall into two classes. One is the class of states for which there is no correlation between the obsevables A and B, i.e., $\langle F\rangle = 0$, so that for them the special and the general uncertainty relations are equivalent. They solve (5) for λ real. That is the only known class of minimum uncertainty states for noncanonical operators. The other is the class of states for which $\langle C\rangle = 0$, i.e., the states for which the special uncertainty relation becomes redundant. That class of states solves (5) for λ imaginary. For those states $\langle F\rangle$ is finite, i.e., the observables A and B are correlated in those states.

The issue of squeezing should also be reexamined in light of the general uncertainty relation. Recall that if the variances of the operators A and B satisfy the inequality (4) then A is said to be squeezed if $\Delta A^2 < |\langle C\rangle|/2$, with a similar condition for the squeezing of B. Generalizing that definition to the case of the variances satisfying the general inequality (2), we say that A is squeezed if

$$\Delta A^2 < \frac{\sqrt{\langle C\rangle^2 + \langle F\rangle^2}}{2}, \tag{6}$$

with a similar condition for the squeezing of B. The usual squeezing condition follows if $\langle F\rangle = 0$. Clearly, if the usual condition for squeezing in the variance of an operator A is satisfied, then so is the general condition but not necessarily vice versa. However, if $\langle C\rangle = 0$, then the usual condition for squeezing can never be satisfied. In that case recourse must be taken to the general condition (6).

Let us first determine the conditions under which even the general uncertainty relation (2) fails to determine a nontrivial lower bound on the product of variances, i.e., the conditions under which the right hand side of (2) is zero but the left hand side is not. The right hand side of

1050-2947/94/49(3)/2178(3)/$06.00

(2) is zero if $\langle C\rangle = \langle F\rangle = 0$. If $\langle C\rangle = 0$, then, following the commutation relation (1), $\langle AB\rangle = \langle BA\rangle$. It then follows that $\langle F\rangle$ defined in (3) will simultaneously be zero if

$$\langle AB\rangle = \langle BA\rangle = \langle A\rangle\langle B\rangle. \tag{7}$$

Thus, if (7) is satisfied, then the right hand side of (2) is zero. The condition (7) holds, in particular, for a state which is an eigenstate of A or of B. For those states either $\Delta A = 0$ or $\Delta B = 0$, so that then the two sides of (2) vanish simultaneously, i.e., (2) is then a nonredundant relation. However, if a state satisfies (7) so that the right hand side of (2) is zero, but if it is not an eigenstate of A or of B, then the variances ΔA and ΔB and consequently the left hand side of (2) remain finite. Hence, if (7) is satisfied by the states which are not eigenstates of A or of B, then even the general uncertainty relation (2) fails to give a nontrivial lower bound on $\Delta A\Delta B$.

Consider now the eigenvalue equation (5) which determines the states $|\psi\rangle$ for which the general uncertainty relation is satisfied with equality. It readily follows from (5) that

$$\Delta A^2 - \lambda^2\Delta B^2 = -i\lambda\langle F\rangle, \tag{8}$$

$$\Delta A^2 + \lambda^2\Delta B^2 = \lambda\langle C\rangle. \tag{9}$$

Equations (8) and (9) are solved by $\lambda = [\langle C\rangle + i\langle F\rangle]/2\Delta B^2$. The class of states for λ real, i.e., $\langle F\rangle = 0$, have been widely studied for the canonical commutators [2] and also for the noncanonical operators satisfying the SU(2) [3–5] or SU(1,1) [3] commutation relations. That is the case for which the special and the general uncertainty relations are equivalent. The minimum uncertainty states for the λ complex in the case of canonical commutation relations have been discussed by Dodonov *et al.* [6].

The special uncertainty relation, on the other hand, is redundant if $\langle C\rangle = 0$. The minimum uncertainty states in the case $\langle C\rangle = 0$ are the solutions of (8) and (9) for λ pure imaginary. Since $\langle C\rangle$ can vanish only for noncanonical operators, the states for λ imaginary constitute a different class of minimum uncertainty states for noncanonical operators. That is the class for which the usual uncertainty relation is redundant. Here we discuss some properties of the states belonging to that class. Clearly, λ in that case will be positive or negative imaginary depending on whether $\langle F\rangle$ is positive or negative. It will be useful for further discussion to define the parameter $\lambda = i\tan(\phi)$. It then follows from (8) and (9) that

$$\Delta A^2 = \frac{\tan(\phi)}{2}\langle F\rangle; \qquad \Delta B^2 = \frac{\langle F\rangle}{2\tan(\phi)}. \tag{10}$$

Hence it is clear that

$$\Delta A^2 < \frac{|\langle F\rangle|}{2} \qquad \text{if} \qquad |\tan(\phi)| < 1, \tag{11}$$

$$\Delta B^2 < \frac{|\langle F\rangle|}{2} \qquad \text{if} \qquad |\tan(\phi)| > 1. \tag{12}$$

Comparing (11) with (6) it follows that, since $\langle C\rangle = 0$, therefore A is squeezed if $|\tan(\phi)| < 1$. Similarly, B is squeezed if $|\tan(\phi)| > 1$.

Let us now apply the preceding considerations to the angular momentum operators S_μ, $\mu = x, y, z$ obeying the SU(2) commutation relations

$$[S_x, S_y] = iS_z, \qquad [S_z, S_x] = iS_y, \qquad [S_y, S_z] = iS_x. \tag{13}$$

The total angular momentum operator $S^2 = S_x^2 + S_y^2 + S_z^2$ commutes with all S_μ, $\mu = x, y, z$. Hence the angular momentum operators couple only those states which have the same eigenvalue for S^2. The eigenvalues of S^2 are known to be given by $S(S+1)$, where S can be an integer or half-odd integer. A particularly useful complete set of angular momentum states is $|S,m\rangle$, where $|S,m\rangle$ is an eigenstate of S_z with eigenvalue m, where $m = -S, -S+1, \ldots S$. For the sake of definiteness let $A = S_x$, $B = S_y$, so that $C = S_z$. It can be proved easily that in the case of $S = 1/2$, the pure states for which the right hand side of the general uncertainty relation (2) is zero are also the eigenstates of S_x or of S_y, i.e., the states for which the left hand side of (2) also vanishes. Therefore, in the case of spin-1/2, the general uncertainty relation is nontrivial for all the pure states. For mixed states, however, the right hand side of (2) may vanish but not necessarily the left hand side, even for a spin-1/2 system. An example of such a mixed state is $I/(2S+1)$, where I is a $(2S+1)\times(2S+1)$ identity matrix. However, for $S \neq 1/2$, the two sides of the general uncertainty relation (2) may not vanish simultaneously even for a pure state.

Next, we derive explicit expressions for the states satisfying (5) for the angular momentum operators. The properties of those states for λ real have been discussed in Refs. [3–5]. Here, we solve (5) for imaginary $\lambda = i\tan(\phi)$. By defining $S_\pm = S_x \pm iS_y$ and $\beta = [\exp(i\phi)\langle S_+\rangle + \exp(-i\phi)\langle S_-\rangle]/2$, Eq. (5) can be written in the form

$$[\exp(i\phi)S_+ + \exp(-i\phi)S_-]|\psi\rangle = 2\beta|\psi\rangle. \tag{14}$$

Introducing the unitary operator

$$U = \exp(i\phi S_z)\exp\left(-i\frac{\pi}{2}S_y\right), \tag{15}$$

Eq. (14) can be rewritten as

$$US_zU^\dagger|\psi\rangle = \beta|\psi\rangle. \tag{16}$$

Hence,

$$\begin{aligned}|\psi_m\rangle &= U|S,m\rangle \\ &\equiv \exp(im\phi)\exp\Bigg(-\frac{\pi}{4}[\exp(i\phi)S_+ \\ &\quad - \exp(-i\phi)S_-]\Bigg)|S,m\rangle \end{aligned} \tag{17}$$

solves (16) with $\beta = m$. The set $|\psi_m\rangle$ of states is complete, as those states are obtained as a result of a unitary transformation on the complete set $|S,m\rangle$. Note that the state $|\psi_{-S}\rangle$ is the same as an atomic coherent state [3]. It is straightforward to show that, in the state $|\psi_m\rangle$,

$$\langle S_x\rangle = m\cos(\phi), \qquad \langle S_y\rangle = -m\sin(\phi),$$

$$\Delta S_x^2 = \tfrac{1}{2}[S(S+1)-m^2]\sin^2(\phi),$$

$$\Delta S_y^2 = \tfrac{1}{2}[S(S+1)-m^2]\cos^2(\phi),$$

$$\begin{aligned}\langle F\rangle &\equiv \langle S_xS_y+S_yS_x\rangle - 2\langle S_x\rangle\langle S_y\rangle \\ &= \frac{\sin(2\phi)}{2}[S(S+1)-m^2]. \end{aligned} \tag{18}$$

Thus $\langle F\rangle = 0$ for $\phi = 0, \pi/2, \pi$, and $3\pi/2$, and for those values of ϕ, either ΔS_x or ΔS_y is zero so that then both the sides of the uncertainty relation reach their minima.

The states $|\psi_m\rangle$ can be generated, for example, in the interaction of N identical two-level atoms with laser field. If the field frequency is equal to the atomic transition frequency, then the Hamiltonian in the interaction picture has the form [3]

$$H = \alpha S_+ + \alpha^* S_-, \tag{19}$$

where α is related to the field strength and the atomic dipole moment. Since the state of the atomic system at time t is given by $\exp(-iHt)|0\rangle$, where $|0\rangle$ is the state at time $t=0$, therefore, by preparing the atoms initially in the state $|S,m\rangle$ by an application of a suitable e.m. field and by an appropriate choice of time and α, the states $|\psi_m\rangle$ defined in (17) can evidently be generated.

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