

Quantum theory of continuous feedback

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A general theory of quantum-limited feedback for continuously monitored systems is presented. Two approaches are used, one based on quantum measurement theory and one on Hamiltonian system-bath interactions. The former gives rise to a stochastic non-Markovian evolution equation for the density operator, and the latter a non-Markovian quantum Langevin equation. In the limit that the time delay in the feedback loop is negligible, a simple deterministic Markovian master equation can be derived from either approach. Two special cases of interest are treated: feedback mediated by optical homodyne detection and self-excited quantum point processes.

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I. INTRODUCTION

Although feedback is used widely to control noise in open quantum systems (such as lasers), a general theory to determine the quantum-limited behavior of such feedback has not previously existed. There are two main approaches to a quantum theory of feedback. The first is a theory based on quantum Langevin equations (stochastic Heisenberg equations of motion). Over the past decade, a number of authors [1-3] have used this approach to describe particular feedback systems. All of these treatments have used the approximation of linearizing around a constant large mean amplitude to avoid issues of operator ordering. The second main approach is to use quantum trajectories [4-7], which in essence are a consequence of quantum measurement theory applied to continuously monitored systems. Using this technique, it was found possible to derive a master equation describing the effect of the feedback on a cavity in the limit of small time delay [8,9]. For linearized systems, it is possible to solve the stochastic equations, and incorporate arbitrary loop-response functions [9], as is possible with the quantum Langevin approach. However this quantum-trajectory approach was only understood for homodyne measurement (which is equivalent to direct detection with the large-amplitude approximation).

These two approaches to feedback are conceptually quite different. The first treats the feedback current as an operator with quantum fluctuations, whereas the second treats it as a classical quantity with objectively real fluctuations. In this paper, I show that these two pictures are in fact equivalent. Furthermore, the treatment given is exact; it does not rely on any linearization approximation, and it applies to direct detection as well as to homodyne detection. The result of the quantum Langevin approach is a non-Markovian stochastic evolution equation for an arbitrary system operator. The corresponding quantum trajectory is a non-Markovian stochastic evolution equation for the system state matrix. In the limit that the time delay in the loop goes to zero, one obtains a Markovian Langevin equation in the first case, which is equivalent to the master equation derived for the second case.

Irrespective of their origin, the equations are simple to apply. For some applications, either the master equation or quantum Langevin equation form may be more apt, so it is good to have a choice. The theory applies to feedback onto any system that obeys a master equation. It does not apply to the traveling wave problem [1], to which a linearized approximation is the only alternative to a stochastic numerical solution. The most obvious applications are for quantum optical cavities, in which, for example, the driving, or the loss rate, or non-linear coupling strengths could be controlled by feedback. This is possible using various electro-optic devices, such as a current-sensitive birefringent crystal combined with a polarization-dependent beam splitter [10]. The Markovian approximation (which allows the use of a master equation) is generally valid if the time delay in the loop is much smaller than the cavity lifetime. If this is not the case, the only tractable approach is to linearize, using either the quantum trajectory or the quantum Langevin method.

This paper is organized as follows. Section II presents the quantum-trajectory approach to feedback. The general feedback master equation is first derived from principles of measurement theory. It is then rederived in a way that shows its relation to experiments using photocurrents and electro-optic devices. To do this, it is necessary to develop a general stochastic calculus, because of the noisy character of the feedback photocurrent. Next, an approximate master equation including the effect of a small time delay is derived. This immediately gives the condition on the feedback time delay necessary to justify the Markovian approximation of the original master equation. In Sec. III, I present the alternative derivation using quantum Langevin equations and input-output theory [11,12]. To do this, it is first necessary to review quantum stochastic differential calculus, and extend it for interactions such as that describing photon pressure on a mirror. Section IV treats the special case of feedback mediated by homodyne measurement, using both approaches. This case is interesting because, as noted above, linear systems with feedback having an arbitrary time delay can be treated analytically. This enables comparison with the short delay ap-

proximate master equation derived in Sec. IID. In Sec. V the measurement theory approach is used in order to describe self-excited quantum point processes. This is another interesting special case which can be treated with a master equation under some circumstances. Section VI concludes.

II. FEEDBACK FROM QUANTUM TRAJECTORIES

A. Master equation, measurement, and feedback

As the title of this section suggests, the foundation of the feedback theory presented here is the theory of master equations and quantum measurement. A master equation (ME) is a generalization of the Schrödinger equation for open systems interacting with a bath. In deriving a master equation, it is necessary to make a Markovian assumption, which is that the influence of the system on the bath is dissipated so quickly that the change in the system depends only on its present state. This is a good approximation for many open quantum systems, such as a good optical cavity or a free atom. Of course, by putting in a feedback loop, the evolution of the system is deliberately made to depend on its history. However, in the limit of a small delay in the feedback loop, this non-Markovian behavior can be approximated by a Markovian ME.

The most general form of ME for the density operator of an open quantum system is [13,14]

$$\dot{\rho} = -i[H, \rho] + \sum_{\mu} \mathcal{D}[c_{\mu}] \rho, \quad (2.1)$$

where H is a Hermitian operator and \mathcal{D} is a superoperator taking one of the arbitrary operators c_{μ} as its argument, defined by $\mathcal{D}[c] = \mathcal{J}[c] - \mathcal{A}[c]$, where for all ρ ,

$$\mathcal{J}[c]\rho = c\rho c^{\dagger}; \quad \mathcal{A}[c]\rho = \frac{1}{2}[c^{\dagger}c\rho + \rho c^{\dagger}c]. \quad (2.2)$$

The vector of operators c_{μ} is not unique; a unitary transformation in the complex vector space indexed by μ will leave the ME unchanged [6]. For simplicity, consider the case where there is only one source of irreversibility so that

$$\dot{\rho} = -i[H, \rho] + \mathcal{D}[c]\rho. \quad (2.3)$$

Even here, this representation is not unique, for this ME is invariant under the transformation

$$c \rightarrow c + \gamma; \quad H \rightarrow H - i\frac{1}{2}(\gamma^*c - \gamma c^{\dagger}), \quad (2.4)$$

where γ is an arbitrary c number. It is even possible for γ to be an operator on another system, in the quantum theory of cascaded open systems [15,16].

Next, we need the most general formulation of quantum measurement theory. In orthodox quantum mechanics, this is as follows [17,14]. A measurement in the time interval $(t, t + T)$ yields the answer α with probability

$$\text{Prob}(\alpha) = \text{Tr}[\tilde{\rho}_{\alpha}(t + T)], \quad (2.5)$$

where

$$\tilde{\rho}_{\alpha}(t + T) = \sum_{\beta} \Omega_{\alpha, \beta}(T) \rho(t) \Omega_{\alpha, \beta}^{\dagger}(T) \quad (2.6)$$

is the unnormalized state matrix after the measurement. The $\Omega_{\alpha, \beta}(T)$ are arbitrary operators that satisfy the completeness relation

$$\sum_{\alpha, \beta} \Omega_{\alpha, \beta}^{\dagger}(T) \Omega_{\alpha, \beta}(T) = 1, \quad (2.7)$$

where the sum over α is the sum over all possible results of the measurement. The normalized density operator, conditioned on the result α is, of course,

$$\rho_{\alpha}(t + T) = \tilde{\rho}_{\alpha}(t + T) / \text{Prob}(\alpha). \quad (2.8)$$

The presence of the extra parameter β indicates classical ignorance in that the measurement result α does not distinguish results that potentially could be distinguished without altering the nonselective evolution of the system. This nonselective evolution is obtained by averaging over all possible measurement results

$$\begin{aligned} \rho(t + T) &= \sum_{\alpha} \text{Prob}(\alpha) \rho_{\alpha}(t + T), \quad (2.9) \\ &= \sum_{\alpha, \beta} \Omega_{\alpha, \beta}(T) \rho(t) \Omega_{\alpha, \beta}^{\dagger}(T). \quad (2.10) \end{aligned}$$

The general form of the ME (2.3) and the general form of quantum measurement can be put in one-to-one correspondence once a particular representation of the ME has been chosen. That is to say (for the case of one output channel), when a physically meaningful value for the parameter γ has been chosen. For continuous measurement, the appropriate measurement time is the infinitesimal dt . Assuming perfect measurement (as will be done in the remainder of this section), the parameter β is not needed and there are just two measurement operators,

$$\Omega_1(dt) = \sqrt{dt} c, \quad (2.11)$$

$$\Omega_0(dt) = 1 - (iH + \frac{1}{2}c^{\dagger}c) dt. \quad (2.12)$$

It is easy to verify that the nonselective evolution under this measurement

$$\rho(t + dt) = \sum_{\alpha=0,1} \Omega_{\alpha}(dt) \rho(t) \Omega_{\alpha}^{\dagger}(dt). \quad (2.13)$$

is equivalent to the ME (2.3).

We thus see that noninvasive measurements on a Markovian quantum system necessarily yields a measurement record which is a point process. For almost all infinitesimal time intervals, the measurement result is $\alpha = 0$, which is thus regarded as a null result. At randomly determined (but not necessarily Poisson distributed) times, there is a result $\alpha = 1$, which I will call a *detection*. Furthermore, if a detection does not occur, the system changes infinitesimally, but not unitarily, via the operator $\Omega_0(dt)$. On the other hand, the effect of a detection is a finite change in the system, via $\Omega_1(dt)$. This

change can validly be called a *quantum jump* [7]. Real measurements that correspond to this ideal measurement theory are made routinely in experimental quantum optics. If c is the annihilation operator of a cavity, multiplied by the square root of the cavity damping rate, then this theory describes the cavity evolution in terms of photodetections. Because at present the primary application of this measurement theory is quantum optics, the terms photodetection and photocurrent *et cetera* will often be used instead of the more general terminology.

To incorporate feedback into this model is quite straightforward. In order for the feedback to be Markovian, the mechanism must cause an immediate change in the system based only on the result of the measurement in the preceding infinitesimal time interval. Because the null result $\alpha = 0$ occurs almost all of the time, feeding back this information is pointless. The feedback must act immediately after a detection, and cause a finite amount of evolution. Let this finite evolution be effected by the superoperator $e^{\mathcal{K}}$, where \mathcal{K} is a Liouville superoperator [so that $\mathcal{K}\rho$ conforms to the right-hand side of Eq. (2.1)]. Then the unnormalized density operator following a detection at time t is

$$\tilde{\rho}_1(t + dt) = e^{\mathcal{K}}c\rho(t)c^\dagger dt. \quad (2.14)$$

The superoperator acts on the product of all operators to its right. Note that the feedback does not alter the trace of this density operator, as is required by conservation of probability. The nonselective evolution of the system is still given by

$$\rho(t + dt) = \tilde{\rho}_1(t + dt) + \tilde{\rho}_0(t + dt). \quad (2.15)$$

Since $\tilde{\rho}_0(t + dt)$ is unchanged by feedback, we have simply

$$\dot{\rho} = -i[H, \rho] + e^{\mathcal{K}}\mathcal{J}[c]\rho - \mathcal{A}[c]\rho. \quad (2.16)$$

This is the most general form of feedback master equation for perfect detection via a single loss source. If the detection is not perfect, or if there are other loss sources, then the Hamiltonian evolution term must be replaced by a more general Liouville term. It can be shown that Eq. (2.16) does conform to the general form of the ME (2.1). Assuming for simplicity that \mathcal{K} acts as

$$\mathcal{K}\rho = -i[Z, \rho] + \mathcal{D}[b]\rho, \quad (2.17)$$

the master equation (2.16) can be written

$$\begin{aligned} \dot{\rho} = & -i[H, \rho] + \sum_{m=0}^{\infty} \int_0^1 ds_m \int_0^{s_m} ds_{m-1} \cdots \\ & \times \int_0^{s_2} ds_1 \mathcal{D}[h_m(s_m, s_{m-1}, \dots, s_1)c]\rho, \end{aligned} \quad (2.18)$$

where

$$\begin{aligned} h_m(s_m, s_{m-1}, \dots, s_1) = & e^{-(iZ + \frac{1}{2}b^\dagger b)(1-s_m)} \\ & \times b e^{-(iZ + \frac{1}{2}b^\dagger b)(s_m - s_{m-1})} \\ & \times b \cdots e^{-(iZ + \frac{1}{2}b^\dagger b)s_1}. \end{aligned} \quad (2.19)$$

In the special case where $\mathcal{K}\rho = -i[Z, \rho]$, this simplifies greatly to

$$\dot{\rho} = -i[H, \rho] + \mathcal{D}[e^{-iZ}c]\rho. \quad (2.20)$$

B. Stochastic calculus for feedback

The general feedback master equation (2.16) derived in the preceding section is an essential result of this paper. In many ways, however, it is unsatisfying. What is the relation to a real experimental feedback mechanism? How fast must the mechanism respond to justify the Markovian approximation? Is it possible to formulate the problem without a measurement step? In this section, I will attempt to answer the first question. To do this, I define a general stochastic differential calculus which can deal with point processes as well as Gaussian processes. This is necessary to make the connection between a physical feedback mechanism, which functions smoothly in time, and the feedback measurement result, which is a point process, as the preceding section showed. The reason that past feedback theories [1–3] have not been concerned with this distinction is that they used only linear approximations, for which it is not necessary to be careful. The explanation of the general stochastic differential calculus will occupy the greater part of this section.

First, it is useful to explicitly represent the selective evolution of the monitored open systems using the real random variable $dN_c(t)$, rather than the operators $\Omega_c(dt)$. The point process $dN_c(t)$ is the increment (either zero or one) in the photon count $N_c(t)$ (or whatever the detection tally is called) in the time interval $(t, t + dt)$ [7]. It is defined by

$$[dN_c(t)]^2 = dN_c(t), \quad (2.21a)$$

$$E(dN_c(t)) = \text{Tr}[\Omega_1(dt)\rho_c(t)\Omega_1^\dagger(dt)] = \langle c^\dagger c \rangle_c(t)dt, \quad (2.21b)$$

where E denotes expectation value, and the subscript c indicates that the quantity to which it is attached is conditioned on the history of measurements up to that time. It is easy to verify that the normalized conditioned density operator obeys the following nonlinear stochastic evolution equation

$$d\rho_c(t) = \{dN_c(t)\mathcal{G}[c] + dt\mathcal{H}[-iH + \frac{1}{2}c^\dagger c]\} \rho_c(t). \quad (2.22)$$

Here, the nonlinear superoperators \mathcal{G} and \mathcal{H} are defined by

$$\mathcal{G}[a]\rho = \frac{a\rho a^\dagger}{\text{Tr}[a\rho a^\dagger]} - \rho, \quad (2.23)$$

$$\mathcal{H}[a]\rho = a\rho + \rho a^\dagger - \text{Tr}[a\rho + \rho a^\dagger]\rho. \quad (2.24)$$

Because of the assumed perfect detection, the stochastic equation for the state matrix is equivalent to the following stochastic equation for the state vector:

$$d|\psi_c(t)\rangle = \left[dN_c(t) \left(\frac{c}{\sqrt{\langle c^\dagger c \rangle_c(t)}} - 1 \right) + dt \left(\frac{\langle c^\dagger c \rangle_c(t)}{2} - \frac{c^\dagger c}{2} - iH \right) \right] |\psi_c(t)\rangle. \quad (2.25)$$

In practice, experimentalists usually consider a photocurrent $I_c(t)$ rather than a photocount $N_c(t)$. In terms of the photocount, a photocurrent may be defined as

$$I_c(t) = \frac{dN_c(t)}{dt}. \quad (2.26)$$

Note that this mathematical photocurrent is a highly singular object. A typical feedback circuit would use the photocurrent to control an electro-optic or electro-mechanical device which influences the source. For Markovian feedback (which is the desired end), the photocurrent should be fed back unaltered. In reality, a finite time delay in the feedback loop is inevitable. The feedback will approach a Markovian process in the limit that the loop delay τ goes to zero. The simplest assumption is that the feedback adds a time-dependent term to the system Hamiltonian, linear in the photocurrent

$$H_{fb}(t) = I_c(t - \tau)Z, \quad (2.27)$$

where Z is an arbitrary Hermitian operator which is dimensionless (using units such that $\hbar = 1$). This Hamiltonian generates the evolution of the state matrix by

$$[\dot{\rho}_c(t)]_{fb} = -i[H_{fb}(t), \rho_c(t)]. \quad (2.28)$$

Naively, substituting in the expression (2.26) for the current would suggest that the increment to the conditioned density operator would be

$$[d\rho_c(t)]_{fb} = -i[Z, \rho_c(t)]dN_c(t - \tau). \quad (2.29)$$

Unfortunately, this term is arbitrarily large [when $dN_c(t - \tau) = 1$] and so the positivity of the density operator may be not preserved. If one were to naively add the extra Hamiltonian term (2.27) to the stochastic Schrödinger equation (2.25) instead, one would get

$$[d|\psi_c(t)\rangle]_{fb} = -iZ|\psi_c(t)\rangle dN_c(t - \tau). \quad (2.30)$$

Then positivity would be preserved, but trace would not.

The problem lies in attempting to treat equations such as (2.28) by the regular rules of calculus, ignoring the highly singular nature of the mathematical photocurrent dN/dt . Of course a real photocurrent, like any physical classical quantity, is a sufficiently smooth function of time to be treated by standard calculus. Thus, the stochastic equations should be interpreted as if the term dN/dt were a smooth function of time, even though it is not according to the definitions (2.21). It is useful to introduce some terminology. An equation like Eq. (2.22) or Eq. (2.25) is called an *explicit* equation. The increment is explicitly given by the right-hand side of the equation, and will always be indicated so on the left-hand side by $d\rho$ or $d|\psi\rangle$. In contrast, Eq. (2.28) is an *implicit* equation. The quantity on the right-hand side is not equal to the

increment divided by dt , even though the quantity on the left-hand side is $d\rho/dt$. To emphasize this, the left-hand side of an implicit equation will always be written as $\dot{\rho}$ rather than $d\rho/dt$.

The implicit feedback evolution equation (2.28) is complicated by the fact that the stochastic current is already correlated with the system, because it is being fed back with a finite time delay. This complication can be ignored for the present, as it is not essential to the argument. That is, $I(t - \tau)$ is simply to be treated as a stochastic quantity. Now, if the noise in the current were Gaussian white noise, then the explicit (implicit) form discussed in the previous paragraph would be identical to the Itô (Stratonovich) form of stochastic differential calculus [18]. For the one-dimensional case, an equation written like

$$\dot{x} = \alpha(x) + \beta(x)\xi(t) \quad (2.31)$$

would, in my notation, be a Stratonovich equation, where $\xi(t)$ represents Gaussian white noise. On the other hand, one written like

$$dx = a(x)dt + b(x)dW(t), \quad (2.32)$$

would be an Itô equation, with $dW(t) = \xi(t)dt$. The infinitesimal Wiener increment is defined by the Itô rules

$$E(dW(t)) = 0; [dW(t)]^2 = dt. \quad (2.33)$$

The Itô form allows an explicit calculation of the state of the system evolved forward in time by dt . If these equations were meant to be equivalent, then they would be related by [18]

$$\alpha(x) = a(x) - \frac{1}{2}b(x)b'(x), \quad (2.34a)$$

$$\beta(x) = b(x), \quad (2.34b)$$

where the prime denotes differentiation with respect to x . Thus a generalization of this Stratonovich to Itô conversion rule is needed, which would apply to any sort of stochastic evolution, not just to Wiener processes. For the jump processes considered here, the explicit rules of the stochastic calculus analogous to Eqs. (2.33) are Eqs. (2.21). This is the appropriate calculus for the conditioning equation (2.22) because the jumps are not due to any physical process, but rather to a change in the observer's state of knowledge about the system.

The general problem is as follows. The implicit form of a stochastic differential equation is given as

$$\dot{x}(t) = \chi(x(t))\mu(t). \quad (2.35)$$

Here, $\chi(x)$ is an arbitrary function of x and $\mu(t)$ is a possibly stochastic function defined by

$$\mu(t) = \frac{dM}{dt}(t), \quad (2.36)$$

where $dM(t)$ could be dt (deterministic), $dW(t)$ (diffusive), $dN(t)$ (as above), or some other stochastic increment. The normal rules of calculus apply in the implicit form, because $\mu(t)$ is acting as if it were a smooth function of time. Thus, for a suitable function $f(t) = f(x(t))$,

$$\dot{f}(x(t)) = f'(x(t))\chi(x(t))\mu(t) = \phi\{f(x(t))\}\mu(t), \quad (2.37)$$

where $\phi(f)$ is defined here implicitly. However, the implicit equation is difficult to solve because one cannot simply apply the stochastic rules. To do this the equation must be in the explicit form. This procedure is well known for $dM(t) = dW(t)$ but not in general. The approach adopted here is to solve Eq. (2.35) under the rules of regular calculus, to all orders in dM . This can be written formally as

$$x(t + dt) = \exp(dt\partial_s) x(s)|_{s=t} \quad (2.38)$$

$$= \exp[\chi(x)dM(t)\partial_x] x|_{x=x(t)}, \quad (2.39)$$

where $\partial_u \equiv \partial/\partial u$. Here I have used the relation

$$\left\{ \frac{d}{ds} x(s) = \chi(x(s)) \frac{dM(t)}{dt} \right\}_{s=t}, \quad (2.40)$$

which is the explicit meaning of the implicit Eq. (2.35).

The explicit stochastic differential equation (SDE) is then defined to be

$$dx(t) = [\exp\{\chi(x(t))dM(t)\partial_x\} - 1]x(t), \quad (2.41)$$

which means

$$dx(t) = [\exp\{\chi(x)dM(t)\partial_x\} - 1]x|_{x=x(t)}. \quad (2.42)$$

Providing this expression converges (which it will do for $dM = dW$ or $dM = dN$), it is compatible with the requirement on the implicit form [Eq. (2.37)], independent of the nature of the stochasticity. This can be seen by calculating the increment in $f(x)$ using the explicit form

$$df(x(t)) = f(x(t) + dx(t)) - f(x(t)) \quad (2.43a)$$

$$= f(\exp\{\chi(x)dM(t)\partial_x\}x|_{x=x(t)}) - f(x(t)) \quad (2.43b)$$

$$= \exp\{\chi(x)dM(t)\partial_x\}f(x)|_{x=x(t)} - f(x(t)) \quad (2.43c)$$

$$= [\exp\{\phi(f)dM(t)\partial_f\} - 1]f|_{f=f(x(t))}. \quad (2.43d)$$

The final expression here is precisely what would have been obtained by turning the implicit equation (2.37) into an explicit equation. This completes the proof.

For deterministic processes, there is no distinction between the explicit and implicit forms, as only the first-order expansion of the exponential remains with dt infinitesimal. For Gaussian white noise, the formula (2.41) is the usual rule (2.34) for converting from Stratonovich to Itô. That is, if the Stratonovich SDE is Eq. (2.35) with $dM(t) = dW(t)$, then the Itô SDE is

$$dx(t) = \chi(x(t))dW(t) + \frac{1}{2}\chi(x(t))\chi'(x(t))dt. \quad (2.44)$$

Here, the Itô rule $[dW(t)]^2 = dt$ has been used. This rule implies that it is only necessary to expand the exponential to second order. This fact makes the inverse transformation (Itô to Stratonovich) easy [18]. For the jump process $[dM(t) = dN(t)]$, the rule $[dN(t)]^2 = dN(t)$ means that the exponential must be expanded to all orders. This gives

$$dx(t) = dN(t) [\exp\{\chi(x(t))\partial_x\} - 1]x(t). \quad (2.45)$$

In this case, the inverse transformation would not appear to be easy to find, in general. The multidimensional generalization of the above formulas is obvious. If the implicit form is (using the Einstein summation convention)

$$\dot{x}_i(t) = \chi_{ij}(\mathbf{x}(t))\mu_j(t), \quad (2.46)$$

then the explicit form is

$$dx_i(t) = \{\exp[\chi_{kj}(\mathbf{x}(t))dM_j(t)\partial_k] - 1\}x_i(t). \quad (2.47)$$

The quantum mechanical state matrix ρ in general must be specified by a double infinity of real numbers. Fortunately, however, its equations of motion are linear, with

$$\dot{\rho}_c(t) = \mathcal{K}\rho_c(t)I_c(t - \tau), \quad (2.48)$$

where $\mathcal{K}\rho = -i[Z, \rho]$. Since this feedback equation (2.29) must be treated as an implicit equation, the explicit form is

$$[d\rho_c(t)]_{\text{fb}} = dN_c(t - \tau)(e^{\mathcal{K}} - 1)\rho_c(t). \quad (2.49)$$

This is, of course, equivalent to the result which would have been obtained by replacing Eq. (2.30) by the appropriate explicit equation

$$[d|\psi_c(t)\rangle]_{\text{fb}} = dN_c(t - \tau)(e^{-iZ} - 1)|\psi_c(t)\rangle. \quad (2.50)$$

This is guaranteed by the fact that an implicit equation follows the normal rules of calculus. Equations (2.48, 2.49) are in fact valid for any Liouville superoperator \mathcal{K} , not merely the Hamiltonian evolution generator. For the general case, it is necessary to use the state matrix description. The complete selective evolution equation of the system under feedback is thus

$$d\rho_c(t) = \{dN_c(t - \tau)(e^{\mathcal{K}} - 1) + dN_c(t)\mathcal{G}[c] + dt\mathcal{H}[-iH + \frac{1}{2}c^\dagger c]\}\rho_c(t). \quad (2.51)$$

It is not possible to turn this stochastic equation into a master equation by taking an ensemble average, as was possible with Eq. (2.22). This is because the noise term at the later time is not independent of that at the earlier time. Physically, it is not possible to derive a master equation because the feedback is not Markovian.

C. The Markovian limit

In order to make Eq. (2.51) more useful, and to compare it with the result of Sec. II A, it would be desirable to take the Markovian limit $\tau \rightarrow 0$, and hopefully derive a master equation. To understand how this limit can be taken, consider first the one-dimensional case with Gaussian white noise. The deterministic part of the evolution is irrelevant, so it will be ignored. The explicit equation of motion is thus

$$dx(t) = b(x)dW(t). \quad (2.52)$$

Now let the noise interact again with the system at a time τ later, via the implicit feedback equation

$$[\dot{x}(t)]_{\text{fb}} = \gamma(x(t))\xi(t - \tau). \quad (2.53)$$

Turning this into an explicit equation gives the complete evolution

$$dx(t) = b(x(t))dW(t) + \gamma(x(t))dW(t - \tau) + \frac{1}{2}\gamma(x(t))\gamma'(x(t))dt. \quad (2.54)$$

This is the analog of Eq. (2.51). Note that it is an explicit equation, which I have previously stated is an Itô equation for Gaussian white noise. However, in another sense, it is not an Itô equation because the noise term $dW(t - \tau)$ is *not* independent of the system state at the time it acts. This fact is what makes the process feedback. What would be desirable would be to derive a true Itô equation, with all stochastic increments being independent of the state. This would only be possible in the limit $\tau \rightarrow 0$. In taking this limit, it is necessary to remember that the feedback must act after the “measurement” (2.52). Taking this into consideration suggests the following expression derived using the general rules stated above:

$$x(t + dt) = \exp[\gamma(x)dW(t - \tau)\partial_x] \times \{x(t) + b(x(t))dW(t)\}. \quad (2.55)$$

For τ finite, the placement of the feedback after the other dynamics is not important, and expanding the exponential gives the above expression (2.54). However, for $\tau = 0$, the later action of the feedback is essential. In this case, expanding the exponential yields

$$dx = \gamma(x)b'(x)dt + \frac{1}{2}\gamma(x)\gamma'(x)dt + [b(x) + \gamma(x)]dW(t). \quad (2.56)$$

This equation is a true Itô equation, with the noise $dW(t)$ being independent of the state $x(t)$. The first deterministic increment here represents the feedback, while the second is simply a consequence of the noisiness of the feedback quantity.

This approach can be applied to the quantum feedback equation by putting

$$\rho_c(t + dt) = \exp[dN_c(t - \tau)\mathcal{K}]\{1 + dN_c(t)\mathcal{G}[c] + dt\mathcal{H}[-iH - \frac{1}{2}c^\dagger c]\}\rho_c(t). \quad (2.57)$$

For τ finite, this reproduces Eq. (2.51). However, if $\tau = 0$, expanding the exponential gives

$$d\rho_c(t) = \{dN_c(t)e^{\mathcal{K}}\mathcal{G}[c] + dt\mathcal{H}[-iH - \frac{1}{2}c^\dagger c]\}\rho_c(t). \quad (2.58)$$

In this equation, it is possible to take the ensemble average because $dN_c(t)$ can simply be replaced by its expectation value (2.21b), giving

$$\dot{\rho} = \{e^{\mathcal{K}}\mathcal{J}[c] - \mathcal{A}[c]\}\rho - i[H, \rho]. \quad (2.59)$$

This master equation is of course the same as Eq. (2.29) derived from general principles. Now, however, the relation of the superoperator \mathcal{K} to experiment via Eq. (2.48) is known. Although it might seem that the assumption that the feedback is linear in the photocurrent is too restrictive to give a general feedback master equation, this is not the case. Because of the stochastic rule $(dN)^2 = dN$, any function of the instantaneous photocurrent is actually linear.

The general ME (2.59) is most obviously applicable to a quantum optical cavity, as discussed above. Provided the external continuum is in the vacuum state, taking c to be the annihilation operator of the cavity allows an interpretation in terms of direct photodetection. Rewrite Eq. (2.59) as follows:

$$\dot{\rho} = \{\mathcal{L}_0 + \mathcal{D}[c] + (e^{\mathcal{K}} - 1)\mathcal{J}[c]\}\rho \equiv \mathcal{L}\rho, \quad (2.60)$$

where the internal Hamiltonian evolution has been generalized to possibly include irreversibility via the Liouville superoperator \mathcal{L}_0 . If \mathcal{L}_0 and \mathcal{K} are “classical” superoperators, preserving the positivity of the Glauber-Sudarshan P function [14], then it is manifest that the ME (2.60) also describes a classical process. This follows from the fact that the jump operator $\mathcal{J}[c]$ preserves coherent states. That is, feedback based on direct detection cannot *produce* nonclassicality. This shows that earlier models which predicted contrary results [19,20] are flawed. In fact, this result holds for feedback based on any form of extra-cavity detection, as the different forms correspond merely to the transformation (2.4). The transformed jump operator still preserves coherent states, and the transformed Hamiltonian simply has an extra driving term. These transformations will be pursued in Sec. IV.

From the preceding paragraph, it is evident that controlling intracavity classical dynamics by an externally measured photocurrent cannot produce nonclassical (sub-Poissonian) photon statistics as measured by an independent detector. The in-loop detector may record such statistics, however. This does not mean that there is nonclassical light incident on the in-loop detector. If all of the superoperators are classical, then the entire feedback process may be described in terms of coherent states. The explanation for the possibility of sub-Poissonian in-loop statistics is that the two-time correlation function for the current no longer measures normally ordered intensity correlations. Rather, it is easy to show (by the method of Ref. [7]) that

$$E(I_c(t')I_c(t)) = \text{Tr} \left\{ e^{\mathcal{K}}\mathcal{J}[c]e^{\mathcal{L}(t'-t)}e^{\mathcal{K}}\mathcal{J}[c]\rho(t) \right\} + \text{Tr} \{ \mathcal{J}[c]\rho(t) \} \delta(t' - t) \quad (2.61)$$

$$= \text{Tr} \left[c^\dagger c e^{\mathcal{L}(t'-t)}e^{\mathcal{K}}c\rho(t)c^\dagger \right] + \text{Tr} [c^\dagger c\rho(t)] \delta(t' - t), \quad (2.62)$$

where \mathcal{L} is as defined in Eq. (2.60). This expression shows that it is the effect of the feedback specific to the in-loop current via $e^{\mathcal{K}}$, not the overall evolution including feedback via $e^{\mathcal{L}(t'-t)}$, which may cause sub-Poissonian statistics for the fed-back current.

D. The effect of a small time delay

This fairly brief section returns to the question raised at the beginning of the Sec. IIB, how fast must the feedback mechanism respond to justify the Markovian approximation? In answering this question, I derive an approximate master equation which is valid in the limit where the feedback time delay is small but not negligible. The approach used is a perturbative one. That is, it is assumed as a first approximation that the evolution of the system can be described by

$$\rho(t') = e^{\mathcal{L}(t'-t)}\rho(t) \quad (t' > t), \quad (2.63)$$

where \mathcal{L} is the superoperator defining the entire feedback process, as in Eq. (2.60).

Now consider feedback in the selective picture, as described by Eq. (2.51), with a time delay τ . Consider a hypothetical photodetection at time $t - \tau$. The conditioned-state matrix is

$$\rho_c(t - \tau + dt) = \{1 + dN_c(t - \tau)\mathcal{G}[c] + O(dt)\}\rho(t - \tau). \quad (2.64)$$

Here, all of the nonjump evolution has been included in the term of order dt . This includes the feedback from earlier jumps, which will be of order dt on average. This is equivalent to making the Markovian approximation to the feedback, which also allows the nonselective state matrix $\rho(t)$ to be used. Now the secondary effect of this possible detection at time $t - \tau$ on the system due to feedback is delayed by τ . By that time, the state of the system has evolved to

$$\rho_c(t) = e^{\mathcal{L}\tau} \{1 + dN_c(t - \tau)\mathcal{G}[c] + O(dt)\}\rho(t - \tau), \quad (2.65)$$

where the zeroth order approximation to the evolution [Eq.(2.63)] has been used. Over the next infinitesimal time step, the feedback takes effect, so that the conditioned state is

$$\rho_c(t + dt) = (1 + dN_c(t)\mathcal{G}[c] + dt\{\mathcal{L}_0 - \mathcal{A}[c]\})\rho_c(t) + dN_c(t - \tau)(e^{\mathcal{K}} - 1)\rho_c(t). \quad (2.66)$$

The form of this equation has been chosen so that the nonselective equation is simple to see. In taking the ensemble average, the first term simply turns into the usual expression for the state matrix without feedback,

$$(1 + dt\{\mathcal{L}_0 + \mathcal{D}[c]\})\rho(t). \quad (2.67)$$

However, for the second term, it is necessary to use the expression (2.65), because of the feedback correlations, to get

$$E[(e^{\mathcal{K}} - 1)e^{\mathcal{L}\tau}dN_c(t - \tau)\{1 + dN_c(t - \tau)\mathcal{G}[c] + O(dt)\}\rho(t - \tau)]. \quad (2.68)$$

Using the stochastic rules (2.21), this expression becomes

$$(e^{\mathcal{K}} - 1)e^{\mathcal{L}\tau} \{\mathcal{J}[c] + O(dt)\}\rho(t - \tau)dt. \quad (2.69)$$

Adding the two terms together gives the first order approximation to the effect of a finite delay τ ,

$$\dot{\rho}(t) = \{\mathcal{L}_0 + \mathcal{D}[c]\}\rho(t) + (e^{\mathcal{K}} - 1)e^{\mathcal{L}\tau} \mathcal{J}[c]\rho(t - \tau). \quad (2.70)$$

It must be emphasized that this is an approximate solution only. In fact, since it is a solution of first order in τ , it is quite proper to expand its individual terms to first order in τ also. That is, one can approximate $e^{\mathcal{L}\tau}$ by $1 + \mathcal{L}\tau$, and use

$$\rho(t - \tau) \simeq (1 - \mathcal{L}\tau)\rho(t). \quad (2.71)$$

Substituting these into Eq. (2.70) gives

$$\dot{\rho}(t) = [\mathcal{L} + \tau(e^{\mathcal{K}} - 1)\{\mathcal{L}\mathcal{J}[c] - \mathcal{J}[c]\mathcal{L}\}]\rho(t). \quad (2.72)$$

This final approximate master equation is equal to the instantaneous feedback master equation (2.60), plus a correction linear in τ . The condition for this correction to the Markovian feedback master equation to be negligible is obviously

$$\tau\|(e^{\mathcal{K}} - 1)\{\mathcal{L}\mathcal{J}[c] - \mathcal{J}[c]\mathcal{L}\}\rho\| \ll \|\mathcal{L}\rho\|, \quad (2.73)$$

where ρ is a suitable density operator [perhaps the steady state solution of Eq. (2.60)], and the bounds $\|\cdot\|$ indicate a suitable norm. To elucidate this expression, consider a typical quantum optical system, damped to the vacuum at a rate of unity, so that c represents the annihilation operator for the intracavity field. Let the intracavity photon number n have a large mean μ and a relatively small variance $\sim \mu$. In this case, the evolution can be successfully described by a Fokker-Planck equation for a distribution function such as the P or W function [14]. The magnitude of the damping evolution $\|\mathcal{D}[c]\rho\|$ can be seen to be of order $\partial_n n \sim 1$. If the feedback is to be of the same order of magnitude then we require $\mathcal{K} \sim \mu^{-1}$. Now, $[\mathcal{L}, \mathcal{J}[c]] \sim [\partial_n n, n] \sim \mu$. Thus the condition (2.73) simply reduces to $\tau \ll 1$. That is to say, the feedback loop delay must be much less than the cavity lifetime. This is quite feasible, with loop delays of order 10^{-8} s and cavity lifetimes of order 10^{-7} s. It is important to note that it is *not* necessary for τ to be much less than the time between detections, which is of order the cavity lifetime divided by μ . If the latter condition were necessary, then Markovian feedback would probably be quite impractical. In Sec. IV, it is shown that Eq. (2.72) agrees with the exact results for time-delayed feedback which may be obtained from a special case of homodyne-mediated feedback.

III. FEEDBACK WITHOUT MEASUREMENT

This section uses a different approach to feedback from the preceding one. The physical system being modeled is the same as that of the preceding section: an open quantum system continuously monitored by a detector, the output of which is used to control the evolution of the system. However, the theory in this section lacks any measurement step; the entire analysis is undertaken within the framework of unitary quantum mechanics. Unitary evolution of the system plus bath can give rise to non-unitary evolution of the system. The bath carries away information about the system, causing it to change irreversibly. A detector can regain this information, and feed it back into the system. In the approach of the previous system, the information was explicitly realized as a classical measurement result before being fed back. In this section, the information remains in a virtual form, as the entire loop is treated formally as a quantum system. I begin by reviewing and extending the theory of system-bath coupling [11,12].

A. Input-output formalism

The theory presented here describes a system interacting locally with a bath consisting of a continuum of harmonic oscillators. Physically, the system may be an optical cavity, and the bath the external electromagnetic field modes with momentum aligned to the cavity axis. The electric field (or rather, one polarization component) at a particular point in space-time (parametrized by z, t) is represented approximately by the Heisenberg-picture operator [12]

$$E(z, t) = \sqrt{\frac{\hbar k}{2\epsilon_0 A}} [b(z, t) + b^\dagger(z, t)]. \quad (3.1)$$

Here, A is the cross sectional area of the beam, and only frequencies near the central wave number k are assumed to be of interest. The canonical commutation relations for the complex amplitudes $b(z, t)$ are

$$[b(z, t), b^\dagger(z', t)] = c\delta(z - z'), \quad (3.2)$$

where c is the speed of light, and for regions where the field propagates freely,

$$b(z, t + \tau) = b(z - c\tau, t). \quad (3.3)$$

Let the external field be coupled to the cavity by a very good mirror at $z = 0$. The field with $z < 0$ then represents an incoming field and that with $z > 0$ an outgoing one. Assume for now a linear coupling of the form

$$H_1(t) = i\hbar[b^\dagger(0, t)c(t) - c^\dagger(t)b(0, t)], \quad (3.4)$$

where $c(t)$ is the annihilation operator of the cavity (tuned to the frequency ck), multiplied by the square root of the cavity decay rate. Ignoring other dynamics, the evolution of an arbitrary Heisenberg operator $a(t)$ is

$$\dot{a}(t) = -[b^\dagger(0, t)c(t) - c^\dagger(t)b(0, t), a(t)]. \quad (3.5)$$

Now because of the singularity of the canonical commutation relations (3.2), it is necessary to be careful in dealing with this evolution equation. As explained in the preceding section, Eq. (3.5) is an implicit equation, which must be converted to an explicit equation. The noise is of a Gaussian nature, but is complicated by being operator valued. What is required is quantum Itô stochastic differential calculus [11,21]. Define an input field, representing the field just before it interacts with the cavity at time t by

$$b_1(t) = b(0^-, t). \quad (3.6)$$

This can be thought of as a white noise term, independent of the state of the cavity at time t . The analog of the Wiener increment in the Itô calculus is then

$$dB_1(t) = b_1(t)dt, \quad (3.7)$$

which satisfies

$$[dB_1(t), dB_1^\dagger(t)] = dt. \quad (3.8)$$

The evolution of an arbitrary operator is then given explicitly by

$$a(t + dt) = U_1^\dagger(t, t + dt)a(t)U_1(t, t + dt), \quad (3.9)$$

where

$$U_1(t, t + dt) = \exp [dB_1^\dagger(t)c(t) - dB_1(t)c(t)]. \quad (3.10)$$

In Eq. (3.9), the bath operators $dB_1(t)$ and $dB_1^\dagger(t)$ are independent of the system operator $a(t)$, and $U_1(t, t + dt)$ must be expanded to second order. Now if $b_1(t)$ is to be thought of as a bath, it should be specifiable simply by its moments. For simplicity, assume that the bath is in the vacuum state. Then it is completely specified by

$$dB_1(t)dB_1^\dagger(t) = dt, \quad (3.11)$$

with all other first and second order moments vanishing. This nonvanishing second order contribution could be thought of as vacuum noise. Using this relation, the explicit quantum Langevin equation is

$$da = (c^\dagger ac - \frac{1}{2}ac^\dagger c - \frac{1}{2}c^\dagger ca) dt - [dB_1^\dagger c - dB_1 c^\dagger, a]. \quad (3.12)$$

The final, stochastic term in this equation is essential to preserve canonical commutation relations [11]. However, the stochastic terms can be ignored when changing from the Heisenberg to the Schrödinger picture and deriving the evolution of the density operator for the cavity mode alone. This is found from the relation

$$\langle da(t) \rangle = \text{Tr}[d\rho(t) a], \quad (3.13)$$

where the picture (Schrödinger or Heisenberg) is specified by the placement of the time argument. The resulting master equation is

$$\dot{\rho}(t) = \mathcal{D}[c]\rho, \quad (3.14)$$

as assumed in Sec. II.

In order to consider feedback, we are interested in the light which leaves the cavity through the output mirror, as well as the internal state of the cavity. From Eq. (3.3), the expression for the field operator of the light leaving the cavity is evidently given by

$$b_2(t) \equiv b(0^+, t) = U_1^\dagger(t, t + dt)b_1(t)U_1(t, t + dt). \quad (3.15)$$

To lowest order in dt , this is

$$b_2(t) = b_1(t) + c(t). \quad (3.16)$$

Just as $b_1(t)$ is independent of, and so commutes with, an arbitrary system operator $a(t')$ at an earlier time $t' < t$, the output field commutes with all system operators at a later time [11]. This fact will be essential to the feedback theory developed later. The output photon-flux operator is defined by

$$I_2(t) = b_2^\dagger(t)b_2(t). \quad (3.17)$$

Define a photon count increment operator [21]

$$dN_2(t) = I_2(t)dt. \quad (3.18)$$

From the relations (3.11) and (3.16), it is easy to verify that

$$[dN_2(t)]^2 = dN_2, \quad (3.19a)$$

$$\langle dN_2(t) \rangle = \langle c^\dagger c \rangle dt. \quad (3.19b)$$

That is to say, the photon flux output is equivalent to the photocurrent, as would be expected. This is another fact essential to the feedback theory to be developed. It is also useful to list the nonzero products of dN with the quantum Wiener increments:

$$dB(t)dN(t) = dB(t) = [dN(t)dB^\dagger(t)]^\dagger. \quad (3.20)$$

B. Photon flux pressure

The quantum stochastic calculus used in the preceding section assumed a coupling linear in the bath amplitude. To my knowledge, applying the quantum stochastic calculus to a nonlinear coupling has not been attempted before. However, such a nonlinear coupling arises naturally in physics from the light pressure force. This example will be used to illustrate the use of the nonlinear coupling which will be needed for feedback. The classical expression for the pressure due to an axially reflected light beam at position z and time t is [22]

$$P(t) = 2\overline{\mathcal{U}(z, t)}, \quad (3.21)$$

where \mathcal{U} is the energy density of the field, and the bar indicates an average over one optical period. For a freely

propagating field, the energy density is related to the electric field by $\mathcal{U} = \epsilon_0|E|^2$. Using the expression for the electric field (3.1), while remembering that b and b^\dagger rotate oppositely at frequency ck , gives

$$\overline{\mathcal{U}(z, t)} = \frac{\hbar k}{A} b^\dagger(z, t)b(z, t), \quad (3.22)$$

where A is the area of the beam. Now the pressure (3.21) is related to the potential energy V by $PA = \nabla V$. Thus the interaction Hamiltonian is

$$V(z, t) = 2\hbar k b^\dagger(z, t)b(z, t)z, \quad (3.23)$$

where z is the coordinate of the mirror. As promised, this expression is bilinear in the field amplitude.

Now, if the mirror does not move significantly, the argument z for the field operators can be assumed constant (say $z = 0$ as before). Then the evolution of some arbitrary mirror operator is given by

$$\dot{a} = i[2kz b^\dagger(0, t)b(0, t), a]. \quad (3.24)$$

This is an implicit equation, and must be treated using the formalism developed in Sec. II. Define an input photon flux operator

$$I_1(t) = b^\dagger(0^-, t)b(0^-, t). \quad (3.25)$$

This expression will not be well defined if the input bath state is in a finite temperature thermal state, at least if the latter is treated in the white noise approximation which is standard [11]. That is because the white-noise approximation assumes an infinite bandwidth of modes, each with a nonzero occupation number. This adds up to give an infinite photon flux. On the other hand, a zero temperature bath will give a zero photon flux, and have no effect on the mirror at all. However, it is possible for Eq. (3.25) to be well defined and nonzero, if for example the input operator is in a coherent state of amplitude β . This corresponds to replacing b by $\beta + \nu$, where ν represents a vacuum state. Then the operator $dN_1(t) = I_1(t)dt$ satisfies

$$[dN_1(t)]^2 = dN_1(t), \quad (3.26a)$$

$$\langle dN_1(t) \rangle = |\beta|^2 dt. \quad (3.26b)$$

In fact these relations will also hold for an input state in a phase-diffused coherent state, such as that produced by a laser.

The implicit equation (3.24) can be rewritten

$$\dot{a} = -I_1(t)\mathcal{K}a, \quad (3.27)$$

where $\mathcal{K}a = -i[2kz, a]$. The explicit counterpart is then

$$da(t) = \{\exp[-\mathcal{K}dN_1(t)] - 1\}a(t) \quad (3.28)$$

$$= dN_1(t) [U^\dagger a(t)U - a(t)], \quad (3.29)$$

where $U = \exp[-2ikz]$. Just as in Eq. (3.12), the stochastic term is necessary to preserve the commutation relations. This can be seen by calculating the increment in the product of two system operators, using the Itô (explicit calculus) rule

$$d(a_1 a_2) = (da_1)a_2 + a_1(da_2) + (da_1)(da_2). \quad (3.30)$$

The result is

$$d(a_1 a_2) = dN_1(t) [U^\dagger a_1 a_2 U - a_1 a_2], \quad (3.31)$$

as necessary for (3.29) to be a valid quantum Langevin equation.

When turning (3.29) into a master equation for the mirror by the relation (3.13), the noise term $dN_1(t)$ is replaced by its expectation value giving

$$\dot{\rho} = |\beta|^2 (U \rho U^\dagger - \rho) = |\beta|^2 \mathcal{D}[\exp(-2ikz)]\rho. \quad (3.32)$$

This is precisely what could have been predicted straight away from Eq. (3.21), with a photon flux of $|\beta|^2$ and a photon momentum of $\hbar k$. Each photon gives a kick to the momentum of the mirror of magnitude $2\hbar k$ as it is reflected. The effect on the output field is to cause a phase shift

$$b_2(t) \equiv b(0^+, t) = e^{-\mathcal{K}dN_1(t)} b_1(t) = e^{-2ikz} b_1(t), \quad (3.33)$$

due to the shifting of the mirror away from the position $z = 0$. Of course, the photon flux operator is unchanged by the feedback, as

$$\begin{aligned} I_2(t) &= b_2^\dagger(t) b_2(t) = b_1^\dagger(t) e^{2ikz} e^{-2ikz} b_1(t) \\ &= b_1^\dagger(t) b_1(t) = I_1(t). \end{aligned} \quad (3.34)$$

C. Feeding back the output

The point of the preceding section was not to describe the quantum effect of light pressure, but rather to show how an obvious result can be derived using the formalism of implicit and explicit stochastic quantum differential equations. In this section, the formalism will be applied to the problem of feedback. An arbitrary system operator obeys the Langevin equation

$$\begin{aligned} da &= i[H, a]dt + (c^\dagger ac - \frac{1}{2}ac^\dagger c - \frac{1}{2}c^\dagger ca)dt \\ &\quad - [dB_1^\dagger c - dB_1 c^\dagger, a], \end{aligned} \quad (3.35)$$

and the output field is defined as

$$b_2(t) = b_1(t) + c(t), \quad (3.36)$$

where $b_1(t)$ is a vacuum operator. As shown above, the output photon flux operator $I_2(t) = b_2^\dagger(t) b_2(t)$ is equivalent to the photocurrent derived from a perfect detection of that field. This suggests that feedback could be treated in the Heisenberg picture by using the Hamiltonian

$$H_{\text{fb}}(t) = I_2(t - \tau) Z(t), \quad (3.37)$$

where now each of these quantities is an operator (and $\hbar = 1$ again).

It might be thought that there is an ambiguity of operator ordering in this expression, because I_2 contains system operators. In fact, the ordering is not important

because $b_2(t)$ commutes with all system operators at a later time [11], and so $I_2(t)$ does also. Of course, $b_2(t)$ will not commute with system operators for times after $t + \tau$ (when the feedback acts), but $I_2(t)$ still will because it is not changed by the feedback interaction, as shown above. This fact would allow one to use the formalism developed here to treat feedback of a photocurrent smoothed by time averaging. That is to say, there is still no operator ambiguity in the expression

$$H_{\text{fb}}(t) = Z(t) \int_0^\infty h(s) I_2(t - s) ds. \quad (3.38)$$

For a sufficiently broad response function $h(s)$, there is no need to use stochastic calculus for the feedback; the explicit equation of motion due to the feedback would simply be

$$d\rho_c(t) = -i[H_{\text{fb}}(t), \rho_c(t)]dt. \quad (3.39)$$

However, this approach makes the Markovian limit difficult to find. Thus, as in Sec. II, the response function will be assumed to consist of a time delay only, as in Eq. (3.37).

Proceeding as in the photon pressure case, the total quantum Langevin equation including feedback is

$$\begin{aligned} da &= i[H, a]dt + dN_2(t - \tau) (e^{iZ} a e^{-iZ} - a) \\ &\quad + (c^\dagger ac - \frac{1}{2}ac^\dagger c - \frac{1}{2}c^\dagger ca) dt - [dB_1^\dagger c - dB_1 c^\dagger, a]. \end{aligned} \quad (3.40)$$

Here all time arguments are t unless otherwise indicated. This should be compared to Eq. (2.51). The obvious difference is that Eq. (2.51) explicitly describes direct photodetection, followed by feedback, whereas the irreversibility in Eq. (3.35) does not specify that the output has been detected. Indeed, the original Langevin equation (3.35) is unchanged if the output is subject to homodyne, rather than direct, detection. This is the essential difference between the virtual quantum fluctuations of Eq. (3.35) and the fluctuations due to information gathering in Eq. (2.51). Expanding $dN_2(t)$ gives

$$\begin{aligned} da &= i[H, a]dt + [c^\dagger(t - \tau) + b_1^\dagger(t - \tau)] (e^{iZ} a e^{-iZ} - a) \\ &\quad \times [c(t - \tau) + b_1(t - \tau)] dt \\ &\quad + (c^\dagger ac - \frac{1}{2}ac^\dagger c - \frac{1}{2}c^\dagger ca) dt - [dB_1^\dagger c - dB_1 c^\dagger, a]. \end{aligned} \quad (3.41)$$

It can be verified that this is a valid non-Markovian quantum Langevin equation, in the sense explained in the section on photon flux pressure.

In Eq. (3.41), the vacuum field operators $b_1(t)$ have been deliberately moved to the outside [using the fact that $b_2(t - \tau)$ commutes with system operators at time t]. This has been done for convenience, because in this position, they disappear when the trace is taken over the bath density operator. Taking the total trace over system and bath density operators gives

$$\langle da \rangle = \langle i[H, a] + c^\dagger(t - \tau)(e^{iZ} a e^{-iZ} - a)c(t - \tau) + (c^\dagger a c - \frac{1}{2} a c^\dagger c - \frac{1}{2} c^\dagger c a) \rangle dt. \quad (3.42)$$

In the limit $\tau \rightarrow 0$, so that $c(t - \tau)$ differs negligibly from $c(t)$, this gives

$$\langle da \rangle = \langle c^\dagger(e^{iZ} a e^{-iZ} - a)c + i[H, a] + (c^\dagger a c - \frac{1}{2} a c^\dagger c - \frac{1}{2} c^\dagger c a) \rangle dt. \quad (3.43)$$

This is precisely what would have been obtained from the Markovian feedback ME (2.59) for $\mathcal{K}\rho = -i[Z, \rho]$.

Moreover, it is possible to set $\tau = 0$ in Eq. (3.41) and still obtain a valid Langevin equation:

$$da = i[H, a]dt - [a, c^\dagger] \left(\frac{1}{2}c + b_1 \right) dt + \left(\frac{1}{2}c^\dagger + b_1^\dagger \right) [a, c]dt + (c^\dagger + b_1^\dagger) (e^{iZ} a e^{-iZ} - a) (c + b_1) dt. \quad (3.44)$$

This equation is quite different from Eq. (3.41) because it is Markovian. This implies that in this equation, it is no longer possible to freely move $b_2 = (c + b_1)$, as it now has the same time argument as the other operators, rather than an earlier one. In this case, it is b_1 rather than b_2 which commutes with all system operators. This must be borne in mind when proving that Eq. (3.44) is a valid Heisenberg equation of motion. This trick with time arguments and commutation relations enables the correct quantum Langevin equation describing feedback to be derived without worrying about the method of dealing with the $\tau \rightarrow 0$ limit used in the Sec. II C. This method is actually quite difficult to apply in the Heisenberg picture. The subtleties involved will become apparent in Sec. III B, where I will use both methods to treat quadrature feedback in the Heisenberg picture. In any case, there is no disputing that Eq. (3.44) is the correct quantum Langevin equivalent to the feedback master equation,

$$\dot{\rho} = -i[H, \rho] + \mathcal{D}[e^{-iZ} c] \rho. \quad (3.45)$$

IV. QUADRATURE FEEDBACK

A. Quantum trajectories

A special case of the feedback theory presented here is feedback mediated by homodyne detection. This has been considered in detail elsewhere [8,9], so here only a brief account will be given. Homodyne detection involves the addition of a coherent field (called a local oscillator) to the output field before detection, and is equivalent to a transformation of the master equation as in Eq. (2.4) of Sec. II. This can be achieved by putting the output of the cavity through a low-reflectivity beam splitter, where the other input is a very intense coherent local oscillator. The amplitude of the transmitted field is effectively $c + \beta$, where β is a complex number. From Eq. (2.4) and Eq. (2.22), the stochastic evolution of the state matrix of a cavity under homodyne measurement is

$$d\rho_c(t) = \{dN_c(t)\mathcal{G}[c + \beta] + dt\mathcal{H}[-iH + \frac{1}{2}(-c\beta^* + c^\dagger\beta) - \frac{1}{2}(c + \beta)^\dagger(c + \beta)]\}\rho_c(t). \quad (4.1)$$

Let β be real, so that a measurement of the x quadrature of the field can be made. This is evident from the rate of detections at a detector placed at the output of the beam splitter,

$$E\left(\frac{dN_c(t)}{dt}\right) = \text{Tr}\{[\beta^2 + \beta(c + c^\dagger) + c^\dagger c]\rho(t)\}. \quad (4.2)$$

In the limit $\beta \gg 1$, this is equal to a constant, plus a term proportional to $2x = c + c^\dagger$, plus a much smaller term. In this limit, it can be shown [23] that the photocurrent can be approximated by a signal plus Gaussian white noise:

$$I_c^x(t) = \langle c + c^\dagger \rangle + \xi(t), \quad (4.3)$$

where the normalized homodyne photocurrent is defined by

$$I_c^x(t) = \lim_{\beta \rightarrow \infty} \frac{dN_c(t)/dt - \beta^2}{\beta}. \quad (4.4)$$

The conditioning equation for the state matrix is

$$d\rho = -i[H, \rho]dt + \mathcal{D}[c]\rho + dW(t)\mathcal{H}[c]\rho, \quad (4.5)$$

where $dW(t) = \xi(t)dt$.

Now consider feedback. Because the homodyne photocurrent (4.3) has indefinite sign, only Hamiltonian feedback can be considered. That is to say, the feedback must be modeled by

$$H_{fb}(t) = F I_c^x(t), \quad (4.6)$$

where $F = F^\dagger$. Because the noise involved is Gaussian, this equation can be treated simply using the method explained in Sec. II C. In fact, this feedback model was solved before that based on direct detection, precisely because the nature of the noise was better understood. The conditioned equation including feedback is

$$\rho_c(t + dt) = \{1 + \mathcal{K}[(c + c^\dagger)_c(t)dt + dW(t)] + \frac{1}{2}\mathcal{K}^2 dt\} \times \{\rho_c(t) - i[H, \rho_c(t)]dt + \mathcal{D}[c]\rho_c(t)dt + dW(t)\mathcal{H}[c]\rho_c(t)\}, \quad (4.7)$$

where $\mathcal{K}\rho = -i[F, \rho]$. This becomes

$$d\rho_c(t) = dt\{-i[H, \rho_c(t)] + \mathcal{D}[c]\rho_c(t) - i[F, c\rho_c(t) + \rho_c(t)c^\dagger] + \mathcal{D}[F]\rho_c(t)\} + dW(t)\mathcal{H}[c - iF]\rho_c(t), \quad (4.8)$$

which is a true Itô equation with $dW(t)$ independent of $\rho_c(t)$. Thus, the ensemble average evolution is simply

$$\dot{\rho} = -i[H + \frac{1}{2}(c^\dagger F + Fc), \rho] + \mathcal{D}[c - iF]\rho \equiv \mathcal{L}\rho, \quad (4.9)$$

where the terms have been arranged deliberately to conform to the general master equation Eq. (2.1). The effect of the feedback is thus seen to replace c by $c - iF$, and

to add an extra term to the Hamiltonian. The two-time correlation function of the current can be found [9] from Eq. (4.8),

$$E(I_c^x(t')I_c^x(t)) = \text{Tr}\{(c + c^\dagger)e^{\mathcal{L}(t'-t)}[(c - iF)\rho(t) + \rho(t)(c^\dagger + iF)]\} + \delta(\tau). \quad (4.10)$$

Note that the feedback affects the term in square brackets, as well as the evolution by \mathcal{L} for time τ .

All of the above results can be obtained from the formalism of Sec. II, keeping β finite until the last step. Defining the feedback by

$$[\dot{\rho}_c(t)]_{\text{fb}} = -i[F, \rho_c(t)] \frac{dN_c(t)/dt - \beta^2}{\beta}, \quad (4.11)$$

the feedback master equation becomes

$$\dot{\rho} = -i[H + i\frac{1}{2}(-c\beta^* + c^\dagger\beta) - F\beta, \rho] + \mathcal{D}\left[e^{-iF/\beta}(c + \beta)\right]\rho. \quad (4.12)$$

Expanding the exponential to second order in $1/\beta$ and then taking the limit $\beta \rightarrow \infty$ reproduces (4.9). Thus the feedback theory for direct detection includes feedback based on homodyne detection as a special case. In another sense, however, feedback based on homodyne detection is more general. As noted above, direct detection in the presence of thermal (or squeezed) white noise is not well defined, because the photon flux becomes infinite. The ‘‘quadrature flux,’’ on the other hand, remains finite, and the equations [(4.3) and (4.5)] can be generalized to cover this case [24]. The effect is merely to change the coefficient of the Gaussian noise term. The noise will be increased by thermal noise, but may be decreased by suitably squeezed white noise. Physically, the reason that homodyne measurement may be well defined, even though direct detection is not, is that no noise is truly white. For direct detection, broad-band noise is as good

as white for masking the signal. For homodyne detection, the local oscillator amplitude can always be increased indefinitely so that the signal (the slowly varying system quadrature with the same phase as the local oscillator) will be amplified while the rapidly varying (but finite) noise cancels out on average. The equations pertaining to feedback in the presence of white noise are given in Ref. [24] and so will not be reproduced here.

B. Quantum Langevin equation

The quantum Langevin treatment of quadrature flux feedback is relatively straightforward, again because of the Gaussian nature of the noise. The homodyne photocurrent is identified with the quadrature of the outgoing field

$$I^x(t) = b_2(t) + b_2^\dagger(t) = c(t) + c^\dagger(t) + b_1(t) + b_1^\dagger(t). \quad (4.13)$$

The feedback Hamiltonian is defined as

$$H_{\text{fb}}(t) = F(t)I^x(t - \tau). \quad (4.14)$$

The time delay τ ensures that the quadrature output operator $I^x(t)$ commutes with all system operators at the same time. Thus it will commute with $F(t)$ and there is no ambiguity in the operator ordering in Eq. (4.14). Treating the equation of motion generated by this Hamiltonian as an implicit equation, the explicit equation is

$$[da(t)]_{\text{fb}} = i[I^x(t - \tau)dt][F(t), a(t)] - \frac{1}{2}[F(t), [F(t), a(t)]]dt. \quad (4.15)$$

Adding in the the nonfeedback evolution gives the total explicit equation of motion

$$da = i[H, a]dt + i\{[c^\dagger(t - \tau)dt + dB_1^\dagger(t - \tau)][F, a] + [F, a][c(t - \tau)dt + dB_1(t - \tau)]\} - \frac{1}{2}[F, [F, a]]dt + (c^\dagger ac - \frac{1}{2}ac^\dagger c - \frac{1}{2}c^\dagger ca)dt - [dB_1^\dagger c - dB_1 c^\dagger, a]. \quad (4.16)$$

Here, all time arguments are t unless indicated otherwise.

In Eq. (4.16), I have once again used the commutability of the output operators with system operators to place them suitably on the exterior of the feedback expression. This ensures that when an expectation value is taken, the input noise operators annihilate the vacuum and hence give no contribution. This is the same trick as used in Sec. III C, and putting $\tau = 0$ in Eq. (4.16) also gives a valid Heisenberg equation of motion. That equation is the counterpart to the homodyne feedback master equation (4.9). However, this trick will not work if the input field is not in the vacuum state, but is for example in a thermal state. For direct detection, it is impossible to treat feedback in the presence of white noise, so the operator ordering trick is perfectly legitimate. However, for quadrature-based feedback, as explained in Sec. IV A, it

is possible to treat white noise. Thus it is necessary to give a method of treating the Markovian ($\tau \rightarrow 0$) limit in this general case (although I will only give results for a vacuum input). The necessary method is just that explained in Sec. IID. In applying it to Heisenberg equations of motion, it will be seen that one has to be quite careful with operator ordering.

If $\tau = 0$ then the feedback Hamiltonian (4.14) does have an ordering ambiguity. Choose a symmetric ordering as a starting point

$$H_{\text{fb}} = \frac{1}{2}\{F, c + c^\dagger + b_1 + b_1^\dagger\}, \quad (4.17)$$

where the curly brackets denote an anticommutator. Although the time argument of all the operators in this expression is supposedly t , the operator F must actually

be of a slightly later time, after the bath operators b_1 and b_1^\dagger have interacted with the system. That is to say, the actual expression for the feedback Hamiltonian should be

$$H_{\text{fb}} = \frac{1}{2}\{F - dB_1^\dagger[c, F] + dB_1[c^\dagger, F] + O(dt), c + c^\dagger + b_1 + b_1^\dagger\}, \quad (4.18)$$

where here the time arguments really are all t . Using the

$$a(t + dt) = \exp[iH_{\text{fb}}dt]\{a + i[H, a]dt + (c^\dagger ac - \frac{1}{2}ac^\dagger c - \frac{1}{2}c^\dagger ca)dt - [dB_1^\dagger c - dB_1 c^\dagger, a]\} \exp[-iH_{\text{fb}}dt], \quad (4.20)$$

where all time arguments are t . Expanding the exponentials using Eq. (3.11) gives

$$da = i[H, a]dt - [a, c^\dagger] \left(\frac{1}{2}c dt + dB_1\right) + \left(\frac{1}{2}c^\dagger dt + dB_1^\dagger\right) [a, c] + i\{[c^\dagger dt + dB_1^\dagger][F, a] + [F, a][c dt + dB_1]\} - \frac{1}{2}[F, [F, a]]dt. \quad (4.21)$$

This equation is a valid Markovian quantum Langevin equation, equivalent to the homodyne feedback master equation (4.9). It is a true quantum Itô equation, in the sense of Ref. [11].

C. The effect of a finite time delay

In this section, I show that it is possible to solve exactly the problem of non-Markovian feedback for linear systems, and furthermore that the result is in agreement with the approximate master equation derived in Sec. IID for the case of small time delays. By a linear system, I mean one in which the equation of motion for the quadrature operator of interest (x) is linear. The non-Markovian feedback is solved using the quantum Langevin approach, although it is quite possible to use the quantum trajectory approach also [9]. Let the system be a cavity with an output mirror with decay rate unity, allowing a homodyne measurement of the output x quadrature to be made. Also let there be a second loss source, with loss rate l . The cavity could be driven, but this would merely lead to a moving of the equilibrium position away from the vacuum, and so will be ignored. The only other sort of dynamics which gives a linear equation for x is parametric driving with $H = \kappa(yx + xy)$. The total equation of motion for the x operator is then

$$\dot{x} = -\gamma x - \frac{1}{2}\xi_1 - \sqrt{l}\frac{1}{2}\zeta, \quad (4.22)$$

where $\xi_1 = b_1 + b_1^\dagger$ is the input x quadrature noise operator for the first mirror, and ζ is likewise for the second loss source, and $\gamma = (1 + l)/2 - \kappa$ will be assumed positive in order to guarantee stability. This equation is the most general linear equation. The two loss sources are necessary because only one of them will be used for feedback. Although this equation is written as an implicit equation, it is only necessary to multiply both sides by dt to obtain an explicit one, because of the linearity.

Let the feedback be effected by driving the system, with Hamiltonian

rules (3.11) gives

$$H_{\text{fb}}dt = F(dB_1 + dB_1^\dagger) + (c^\dagger F + Fc)dt, \quad (4.19)$$

where the ordering of the first term is unimportant because dB_1 and dB_1^\dagger commute with F at the same time. From Sec. IIC, the total evolution of the system is thus given by

$$H_{\text{fb}}(t) = -\lambda y(t) \int_0^\infty h(s)[2x(t-s) + \xi_1(t-s)]ds, \quad (4.23)$$

where $h(s)$ is an arbitrary response function normalized so that $\int_0^\infty h(s)ds = 1$. For a simple time delay of τ , as previously considered, $h(s) = \delta(\tau - s)$. Because this gives a linear equation for x , it is again unnecessary to use stochastic calculus. The overall result is

$$\dot{x} = -\gamma x - \frac{1}{2}\xi_1 - \sqrt{l}\frac{1}{2}\zeta - \lambda \int_0^\infty h(s)[x(t-s) + \frac{1}{2}\xi_1(t-s)]ds. \quad (4.24)$$

This equation may be solved by Fourier transform, giving

$$\tilde{x}(\omega) = \frac{-\sqrt{l}\tilde{\zeta}(\omega) - [1 + \lambda\tilde{h}(\omega)]\tilde{\xi}_1(\omega)}{2[-i\omega + \gamma + \lambda\tilde{h}(\omega)]}. \quad (4.25)$$

Here the noise terms in the Fourier domain commute and are independent. Their properties (here defined only for ξ_1) are

$$\tilde{\xi}_1(\omega)^* = \tilde{\xi}_1(-\omega), \quad (4.26a)$$

$$\langle \tilde{\xi}_1(\omega)\tilde{\xi}_1(\omega') \rangle = 2\pi\delta(\omega + \omega'). \quad (4.26b)$$

The expression (4.25) may be used to find observable quantities, such as the spectrum of a homodyne measurement of the free output of the cavity, from the second mirror with loss rate l . The normalized spectrum (equal to one at high frequencies) is defined as

$$S(\omega) = \frac{1}{2\pi l} \int d\omega' \langle \tilde{J}^x(\omega)\tilde{J}^x(-\omega') \rangle, \quad (4.27)$$

where the free homodyne current is

$$J^x(t) = 2lx(t) + \sqrt{l}\zeta(t). \quad (4.28)$$

The result is

$$S(\omega) = 1 + l \frac{|1 + \lambda \tilde{h}(\omega)|^2 + l - 2\{\gamma + \lambda \text{Re}[\tilde{h}(\omega)]\}}{|-i\omega + \gamma + \lambda \tilde{h}(\omega)|^2}. \quad (4.29)$$

In the limit of small time delay τ , $\tilde{h}(\omega)$ can be approximated (over the bandwidth of the cavity) by

$$\tilde{h}(\omega) = 1 + i\omega\tau. \quad (4.30)$$

Expanding (4.29) to first order in τ gives

$$S(\omega) = 1 + l \frac{(1 + \lambda)^2 + l - 2(\gamma + \lambda)}{[\omega(1 - \lambda\tau)]^2 + (\gamma + \lambda)^2} \\ = 1 + l \frac{[(1 + \lambda)^2 + l - 2(\gamma + \lambda)](1 + 2\lambda\tau)}{\omega^2 + [(\gamma + \lambda)(1 + \lambda\tau)]^2}. \quad (4.31)$$

Note that this expression is Lorentzian, just as it would be with zero time delay. The effect of the time delay is to broaden the bandwidth (for λ positive). This increases the total amount of noise, which is as expected for a less than optimal feedback loop.

Now consider the other approach to treating a small time delay, the approximate master equation derived in Sec. IID. If the homodyne feedback master equation in the $\tau = 0$ limit is

$$\dot{\rho} = -i[H, \rho] + \mathcal{D}[c]\rho - i[F, c\rho + \rho c^\dagger] + \mathcal{D}[F]\rho \equiv \mathcal{L}\rho, \quad (4.32)$$

then the equation corresponding to Eq. (2.72) is

$$\dot{\rho} = -i[H, \rho] + \mathcal{D}[c]\rho - i[F, c\rho + \rho c^\dagger] + \mathcal{D}[F]\rho \\ + \tau(-i)[F, \mathcal{L}(c\rho + \rho c^\dagger) - c(\mathcal{L}\rho) - (\mathcal{L}\rho)c^\dagger]. \quad (4.33)$$

For the linear case considered above, $F = -\lambda y$ and

$$\mathcal{L}\rho = -i\kappa[yx + xy, \rho] + (1 + l)\mathcal{D}[c]\rho \\ + i\lambda[y, c\rho + \rho c^\dagger] + \mathcal{D}[\lambda y]\rho. \quad (4.34)$$

$$S(\omega) = 1 + l \frac{[(l + 1 + 2\lambda + \lambda^2)(1 + 2\lambda\tau) - 2\lambda\tau(\gamma + \lambda)] - 2(\gamma + \lambda)(1 + \lambda\tau)}{\omega^2 + [(\gamma + \lambda)(1 + \lambda\tau)]^2}. \quad (4.41)$$

This is easily seen to be equivalent to the expression derived above (4.29) from the exact treatment of non-Markovian feedback. This supports the derivation of the approximate equation in Sec. IID. Moreover, it is easy to see from the above expressions that the condition for the time delay in the loop to be negligible is simply

$$\lambda\tau \ll 1. \quad (4.42)$$

Since λ will typically be of order the cavity linewidth, this condition is quite feasible experimentally, as explained in Sec. IID. If this condition does not hold, then it is necessary to use the exact non-Markovian treatment. Of course this is only possible if the dynamics are linear, but for many systems that is a good approximation.

Now an alternative definition for a linear equation for x is that the marginal distribution (the Wigner function) for x obeys an Ornstein-Uhlenbeck equation [18]. That is, it obeys an equation of the form

$$\dot{W}(x) = (k\partial_x x + \frac{1}{2}D\partial_x^2)W(x), \quad (4.35)$$

where k and D are constants. It is easy to verify that the Liouville differential operator corresponding to Eq. (4.34) is

$$\mathcal{L}W = [(\gamma + \lambda)\partial_x x + \frac{1}{8}(l + 1 + 2\lambda + \lambda^2)\partial_x^2]W. \quad (4.36)$$

The correspondence

$$c\rho + \rho c^\dagger \rightarrow (2x + \frac{1}{2}\partial_x)W \quad (4.37)$$

then allows one to write down the equation of motion for W corresponding to the approximate master equation (4.33)

$$\dot{W} = \left(\mathcal{L} + \tau \frac{\lambda}{2} \partial_x [\mathcal{L}, 2x + \frac{1}{2}\partial_x] \right) W. \quad (4.38)$$

Evaluating the commutators of the differential operators yields another Ornstein-Uhlenbeck equation for W , but this time with

$$k = (\gamma + \lambda)(1 + \lambda\tau), \quad (4.39a) \\ D = \frac{1}{4}(l + 1 + 2\lambda + \lambda^2)(1 + 2\lambda\tau) - \frac{1}{2}\lambda\tau(\gamma + \lambda). \quad (4.39b)$$

For linear systems, there is a simple relationship between the output quadrature spectra and the internal dynamics [14]. In terms of the drift and diffusion constants of the general linear Wigner function equation (4.35), the spectrum defined above (4.27) is

$$S(\omega) = 1 + l \frac{4D - 2k}{\omega^2 + k^2}. \quad (4.40)$$

Substituting in the constants (4.39) for the approximate master equation gives

V. SELF-EXCITED QUANTUM POINT PROCESSES

The concept of a quantum point process (QPP) arises naturally from quantum measurement theory in the context of Markovian open systems. That is to say, as shown in Sec. IIA, the result of continuous monitoring is a measurement record consisting of detections. Classically, a Poissonian point process can be made non-Poissonian by making its rate depend on past detections. This is called a self-exciting point process [25]. The quantum analog to this is a source of irreversibility whose strength is controlled by the rate of detections from that source. This can be called a self-excited quantum point process (SEQPP). This sort of feedback has not before been given

a correct quantum treatment. Classically, the spectrum of the self-excited point process is of primary interest [25]. In the quantum case, the effect of the SEQPP on the source is of more interest. The aim of this section is to derive a master equation which describes this effect in the Markovian limit. The concept of a SEQPP is interesting for a number of reasons. Firstly, it is a nontrivial extension of the general theory developed in Sec. II. Secondly, it is an example in which the measurement theory approach is clearly easier to apply than the Langevin approach. Thirdly, a SEQPP was the first feedback system for which a model was attempted using the measurement theory and master equation approach [19,20]. It turns out that this early approach was flawed, which will be seen by comparison with the correct equation. Fourthly, a SEQPP has potential applications in noise reduction, of which one is treated explicitly.

A. Master equation for a SEQPP

Consider a QPP with a time-variable rate $\kappa(t)$, so that the ME is

$$\dot{\rho} = \kappa(t)\mathcal{D}[c]\rho - i[H, \rho]. \quad (5.1)$$

If the QPP is to be a self-exciting QPP, then $\kappa(t)$ becomes $\kappa_c(t)$, conditioned on the photocurrent

$$\kappa_c(t) = 1 + \lambda I_c(t), \quad (5.2)$$

where $I_c(t) = dN_c(t)/dt$ and it is necessary to use the selective ME

$$d\rho_c(t) = \{dN_c(t)\mathcal{G}[c] + dt\mathcal{H}[-iH - \frac{1}{2}\kappa_c(t)c^\dagger c]\}\rho_c(t)\rho_c(t) \quad (5.3)$$

where now

$$E(dN_c(t)) = \kappa_c(t)\langle c^\dagger c \rangle_c(t)dt. \quad (5.4)$$

Note that since $\kappa_c(t)$ must always be positive, λ must always be positive also, hence the self-excitation rather than self-inhibition.

The same considerations of physicality [causality and smoothness of $I_c(t)$] explained in Sec. II also apply to Eq. (5.2). Thus the general theory of Sec. II implies that the nonselective ME must be of the form (2.59). Comparison of Eqs. (5.1) and (5.2) with Eqs. (2.48) would suggest $\mathcal{K} = \lambda\mathcal{D}[c]$. Under more careful consideration, it is obvious that this equation describes feedback controlling a second irreversible coupling, rather than a self-exciting process. Since the original superoperator has been modified from \mathcal{D} to $e^{\mathcal{K}}\mathcal{J} - \mathcal{A}$, the action of the feedback superoperator should be

$$\mathcal{K} = \lambda(e^{\mathcal{K}}\mathcal{J} - \mathcal{A}). \quad (5.5)$$

Here, the argument $[c]$ to the superoperators \mathcal{J} , \mathcal{A} , and \mathcal{D} is being omitted for convenience. The Markovian SEQPP ME can be written

$$\dot{\rho} = \lambda^{-1}\mathcal{K}\rho - i[H, \rho], \quad (5.6)$$

Unfortunately, there is no closed-form solution to the transcendental superoperator equation (5.5).

If λ were small in some sense, then \mathcal{K} could be approximated by iterating Eq. (5.5). However, this is not a very satisfactory solution, as the self-excitation does not even become evident until the second iteration. An alternate approximation may be found by constraining the nature of the system, rather than the strength of self-excitation. Let the state of the system have a well defined value of $c^\dagger c$. That is to say, let the mean μ of $c^\dagger c$ be very large, and the variance be of the same order. Then the superoperator \mathcal{D} is of order 1, but \mathcal{J} is of order μ . For the self-excitation to be of order one requires $\lambda\mu \sim 1$. This implies that λ is small, but the self-excitation strength is not small. Expanding the SEQPP superoperator \mathcal{K} to second order in $1/\mu$ is then a good approximation for the systems under consideration. Effectively, this can be achieved by assuming a solution of the form

$$\mathcal{K} = \lambda\mathcal{D}\mathcal{F} + \lambda^2\mathcal{D}^2\mathcal{S} + O(\lambda^3), \quad (5.7)$$

where \mathcal{F} and \mathcal{S} are superoperators of order 1 to be determined. This approximation also implies that the superoperator ordering in the second order term $\lambda\mathcal{D}^2\mathcal{S}$ is not really important.

Substituting the ansatz (5.7) into Eq. (5.5) and equating powers of λ yields

$$\mathcal{F} = 1 + \mathcal{F}\lambda\mathcal{J}, \quad (5.8)$$

$$\mathcal{S} = \mathcal{S}\lambda\mathcal{J} + \frac{1}{2}\mathcal{F}^2\lambda\mathcal{J}. \quad (5.9)$$

Formally evaluating these gives

$$\mathcal{F} = (1 - \lambda\mathcal{J})^{-1}, \quad (5.10)$$

$$\mathcal{S} = \frac{\lambda\mathcal{J}}{2(1 - \lambda\mathcal{J})^3}. \quad (5.11)$$

Thus the approximate expression for the SEQPP ME is

$$\dot{\rho} = -i[H, \rho] + \mathcal{D}(1 - \lambda\mathcal{J})^{-1}\rho + \lambda\mathcal{D}^2\frac{\lambda\mathcal{J}}{2(1 - \lambda\mathcal{J})^3}\rho. \quad (5.12)$$

This evidently includes terms which indicate an arbitrarily large number of detections within an infinitesimal time interval. This is one reason why the calculation of the spectrum of the SEQPP is too difficult to attempt here.

B. Application to a noisy laser

There are obvious quantum optical applications for which the approximations leading to Eq. (5.12) are valid. Consider a laser cavity with one end mirror of variable transmittivity controlled by a current. If the controlling current comes from a photodetector just outside that end, then the photodetection is a self-exciting quantum point process. Assuming that the time delay in the feedback loop is negligible, the formalism developed above

can be applied. Furthermore, the quantity measured by the point process (photon number) is well defined in a laser, with standard deviation of the order of the square root of the mean (which is much greater than one). Thus it is appropriate to use the approximate expression (5.12)

$$\begin{aligned} \dot{\rho} = & \mu \left(\mathcal{E}[c^\dagger] + \frac{q}{2} \mathcal{E}[c^\dagger]^2 \right) \rho + \left\{ \sum_{m=0}^{\infty} \lambda^m c^{m+1} \rho c^{\dagger m+1} \right\} - \frac{1}{2} \sum_{m=0}^{\infty} \lambda^m (c^\dagger c^{m+1} \rho c^{\dagger m} + c^m \rho c^{\dagger m+1} c) \\ & + \frac{\lambda}{2} \sum_{m=0}^{\infty} \frac{m(m+1)}{2} \lambda^m \left[c^{m+2} \rho c^{\dagger m+2} - c^\dagger c^{m+2} \rho c^{\dagger m+1} - c^{m+1} \rho c^{\dagger m+2} c \right. \\ & \left. + \frac{1}{4} (c^{\dagger 2} c^{m+2} \rho c^{\dagger m} + 2c^\dagger c^{m+1} \rho c^{\dagger m+1} c + c^m \rho c^{\dagger m+2} c^2) \right] \rho, \end{aligned} \quad (5.13)$$

where the operators in the \mathcal{D}^2 term have been put in normal order for simplicity. Here, μ is the mean photon number in the absence of feedback ($\mu \gg 1$ due to the laser pump), and q is a measure of the pump regularity (equal to zero for a Poissonian pump and -1 for a perfectly regular pump). The laser pump superoperator $\mathcal{E}[c^\dagger]$ is defined (for an arbitrary operator a) by [26]

$$\mathcal{E}[a] = \mathcal{J}[a] (\mathcal{A}[a])^{-1} - 1 = \mathcal{D}[a] (\mathcal{A}[a])^{-1}. \quad (5.14)$$

Before continuing, it is interesting to contrast Eq. (5.13) with that from a previous model also intended to apply to the apparatus described above, as an example of a self-exciting quantum point process. The master equation postulated in Ref. [19] was

$$\begin{aligned} \dot{\rho} = & \mu \left(\mathcal{E}[c^\dagger] + \frac{q}{2} \mathcal{E}[c^\dagger]^2 \right) \rho + \left\{ \sum_{m=0}^{\infty} \lambda^m c^{m+1} \rho c^{\dagger m+1} \right\} \\ & - \frac{1}{2} \sum_{m=0}^{\infty} \lambda^m (c^{\dagger m+1} c^{m+1} \rho + \rho c^{\dagger m+1} c^{m+1}). \end{aligned} \quad (5.15)$$

The first feedback term in this equation (enclosed in curly brackets) is identical to the corresponding term in the correct equation (5.13). This part of Eq. (5.15) was in fact the only feedback term which was “derived” in Ref. [19]. The form of the remaining two terms was assumed (wrongly as it turns out) to follow automatically from that of the first term. It was these erroneous terms which lead to the prediction of nonclassical light generation by the feedback loop. In contrast, the corresponding terms in the correct equation (5.13) manifestly cannot lead to nonclassical states. In addition, there are the extra final terms in Eq. (5.13) which increase the photon number variance without significantly affecting the mean.

Because Eq. (5.13) does not produce nonclassical states (unless $q < 0$), it can be solved using the Glauber-Sudarshan $P_{\text{GS}}(\alpha, \alpha^*)$ function. The stationary state will have zero phase information, so only the photon number $n = |\alpha|^2$ need be considered here. The normally ordered statistics for the photon number are given by $P_{\text{GS}}(n)$.

for the SEQPP superoperator.

Measuring time in units of the no-feedback cavity linewidth, the operator c is the annihilation operator for the cavity mode. In terms of c and c^\dagger , the resulting master equation is then

From Eq. (5.13), this obeys

$$\begin{aligned} \dot{P}_{\text{GS}}(n) = & \left[\frac{\partial}{\partial n} \left(\frac{n}{1 - \lambda n} - \mu \right) \right. \\ & \left. + \frac{1}{2} \frac{\partial^2}{\partial n^2} \left(\frac{(\lambda n)^2}{(1 - \lambda n)^3} n + q\mu \right) \right] P_{\text{GS}}(n). \end{aligned} \quad (5.16)$$

The drift term is just what would be expected classically for a self-excited damping process. The steady state photon number is

$$\bar{n} = \mu / (1 + \lambda\mu), \quad (5.17)$$

which always exists. Furthermore, this gives $\lambda\bar{n}$ always less than one, so that the expressions in Eq. (5.16) are well defined, at least for a linearized solution. Linearizing around this steady state gives the Ornstein-Uhlenbeck equation

$$\begin{aligned} \dot{P}_{\text{GS}}(n) = & \left[\frac{\partial}{\partial n} (1 + \lambda\mu)^2 (n - \bar{n}) \right. \\ & \left. + \frac{1}{2} [q + (\lambda\mu)^2] (1 + \lambda\mu) \bar{n} \frac{\partial^2}{\partial n^2} \right] P_{\text{GS}}(n). \end{aligned} \quad (5.18)$$

Since all of the output light is used in the feedback loop, the scheme as stated is only useful in reducing the intracavity photon number variance. This is measured by the Mandel Q parameter [27], defined by

$$Q = \bar{n}^{-1} \int_0^\infty dn (n - \bar{n})^2 P_{\text{GS}}(n), \quad (5.19)$$

which at steady state equals

$$Q_\lambda = \frac{1}{2} [q + (\lambda\mu)^2] / (1 + \lambda\mu). \quad (5.20)$$

Without feedback, $Q = Q_0 = q/2$, while with optimal feedback $\lambda = \lambda_{\text{opt}} = [-1 + \sqrt{1 + q}] / \mu$,

$$Q_{\text{opt}} = -1 + \sqrt{1 + q}. \quad (5.21)$$

This shows that the intracavity variance can always be reduced by the self-exciting loss source, unless $Q_0 = 0$. However, the nature of the induced nonlinearity is such

that it cannot produce a nonclassical state ($Q_{\text{opt}} < 0$) from a no-feedback classical state ($Q_0 \geq 0$). For a comparison with a non-SEQPP type of feedback, see Ref. [9], and for a comparison with an intrinsically nonlinear loss source, see Ref. [28].

VI. CONCLUSION

In this paper I have attempted to give the framework in which a complete quantum theory of feedback can be constructed. The central results for this framework were derived in Secs. II and III. In Sec. II, a general algorithm for dealing with stochastic differential equations with arbitrary kinds of noise was derived, and applied to the problem of quantum feedback. This is the first treatment of feedback which does not rely on some linearizing assumption. The second major achievement was Sec. III, which showed that the measurement theory approach of Sec. II is equivalent to an approach based on quantum Langevin equations. This established for the first time a link between the early work on quantum feedback using Langevin equations [1-3] and the more recent theory using measurement theory [8,9]. The remaining sections of the paper consisted of extending the basic formalism to two special cases, homodyne-detection-based feedback and self-exciting quantum point processes. For all of the feedback schemes, a master equation could be derived in the limit of zero time delay in the feedback loop. This is an important simplification with many applications. It is important to note, however, that the Markovian assumption is not a necessary part of the general theory.

It is perhaps tempting to think of the two approaches to feedback (based on quantum trajectories and quantum Langevin equations) as being simply the Schrödinger or Heisenberg picture equivalents of the same process. However, this is not the case. The distinction really is one between a measurement and a no-measurement approach. It would be possible, but unwieldy, to do the quantum Langevin calculations in the Schrödinger picture, using the state matrix for the system and bath. The quantum trajectory method must use a state matrix, because quantum measurement theory is defined that way. The stochastic quantum Langevin equation in the absence of feedback is independent of any measurement process, whereas the form of stochastic quantum trajectory in the absence of feedback is defined by the choice of measurement scheme. Thus the equivalence of the two methods is not a trivial occurrence. From a philosophical point

of view, it is desirable that the equivalence exists. This is because it is a fundamental tenet of quantum mechanics that any closed system can be described from within the framework of unitary quantum mechanics. Measurements are not part of that framework. In practice, measurement theory is a convenient way to describe a macroscopic feedback loop, and it makes no difference whether or not the experimenter chooses to become aware of the information being conveyed by the feedback loop. However, in principle it should be possible to give a formal description of the feedback loop without invoking any measurement step. In essence, that is what the Langevin equation approach to feedback is.

Finally, it should be noted that there are some feedback equations more general than those presented here. They are detailed elsewhere [24]. The first generalization is to consider feedback in the presence of white noise. As explained in Sec. IV, this is only possible with homodyne detection (or effective homodyne detection if the system has a very large well-defined amplitude). It does not lead to any radically different sort of behavior, and can be treated using the same method as used in this paper. The more important generalization is feedback which cannot be produced by using the results of a measurement. In the context of this paper, such a concept is a contradiction in terms. However, it arises naturally from a consideration of all-optical (no electronic devices) feedback [24]. Some all-optical feedback schemes reproduce the behavior of electro-optical schemes (which is what this paper analyzes). Others produce behavior with no electro-optic equivalent. It is thus a matter of definition whether the theory presented in this paper is considered *the* quantum theory of continuous feedback, or merely the special case of feedback which can be produced by measurements. In practice, the all-optical schemes are complicated and difficult to achieve, whereas electro-optic feedback is already in wide use. The theory presented here should have an increasingly wide field of applicability as more devices are becoming quantum limited, and controlling noise becomes important.

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