Propagation of quantum fluctuations in single-pass second-harmonic generation for arbitrary interaction length

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By solving linearized wave equations of nonlinear polarization for a medium, we derive the input-output relationship for the propagation of the quantum fluctuations in single-pass second-harmonic generation with arbitrary interaction length. Spectra of quadrature-phase squeezing are calculated from the input-output relations. It is found that the intensity fluctuations of both harmonic and fundamental fields are squeezed below the vacuum fluctuation level. For large interaction length, an arbitrary amount of squeezing can be achieved for the fundamental field whereas only 50% squeezing is possible for the harmonic field. The bandwidth of squeezing is determined by the phase-matching condition and can be very broad in this case. The possibility of a quantum-nondemolition measurement in the process is discussed. Some interesting features in quantum fluctuations of the fields are presented.

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I. INTRODUCTION

The discovery of second-harmonic generation (SHG) by Franken *et al.* [1] marked the beginning of the field of nonlinear optics. Because of its simplicity in the nonlinear coupling, SHG is among the first few problems that were studied in the early development of nonlinear optics [2] and has now become a typical example in most nonlinear optics textbooks [3]. High conversion efficiency from the fundamental to the harmonic field has been routinely achieved in laboratories and can be described quite well by the theory. Because of the nonlinear nature of the process, high-intensity short pulses are commonly used in single-pass SHG [4]. Continuous-wave (cw) SHG with high conversion efficiency has also been achieved with the aid of a resonant cavity (multiple pass) [5]. Recent development in waveguide nonlinear conversion [6] and availability of crystals with large nonlinearity such as $KNbO_3$ have made efficient conversion possible in cw single-pass harmonic generation.

Quantum fluctuations in second-harmonic generation have been studied by many people [7–11] in the past. Most of them start with a Hamiltonian that describes the nonlinear interaction between the fundamental and harmonic fields and then derive the equations of motion for the fields from the Hamiltonian. Although this model is simple and can give some general pictures about the quantum fluctuations in the process, it is an oversimplified model. Later, Drummond, McNeil, and Walls [12] considered a more realistic model of harmonic generation inside a resonator with dissipation included. This model predicted [13] that the quantum noise of both the harmonic and fundamental fields is squeezed under certain conditions. This was later confirmed experimentally by Pereira *et al.* [14]. On the other hand, for the problem of propagation of quantum fluctuations in the nonlinear medium (single-pass case), the treatment with the Hamiltonian is not appropriate because spatial propagation of the fields needs to be taken into consideration. So far there has not been a general treatment for arbitrary interaction length and conversion efficiency in the propagation of quantum fluctuations in the nonlinear medium. Generally speaking, when the conversion from the fundamental to the harmonic field is high, the equations of evolution for the amplitudes of the fields become nonlinear and hard to solve. It becomes even more complicated when we consider multifrequency components, where different frequency components are coupled through frequency summation processes. As for the problem of squeezing in the single-pass case, even in the model with a simple Hamiltonian, there only exist short time solutions up to the second order in interaction length of the nonlinear evolution [9–11]. It was found that there are quadrature-phase squeezing effects in both fundamental and harmonic fields. Therefore it is necessary to explore the problem even further into the regime in which the conversion is large and higher orders in interaction length are involved. Furthermore, the coupling between the fundamental and harmonic fields in the process suggests possible quantum correlations between the two fields as they propagate in the nonlinear medium. Indeed, such correlations are found to exist in cw harmonic generation with a resonator and quantum-nondemolition (QND) measurement is possible under certain conditions [15].

In this paper, we will start from wave equations with nonlinear polarizations for the medium and derive the nonlinear coupling equations for the propagation of different frequency components of the amplitudes of the fields in the nonlinear medium. To solve these equations, we linearize in Sec. III the quantum fluctuations around the large coherent components of the fields and derive a set of linear equations for the quantum fluctuations of the fields. We then solve these equations and find the transformation relationship between the input and output of the quantum fluctuations as the fields propagate inside the nonlinear medium. From this relationship, we will analyze in Sec. IV quadrature-phase squeezing in both harmonic and fundamental fields and will discuss correlations between the outputs as well as between the input and output of the fields which lead to realization of QND measurement and quantum-optical tap in the process.

II. EQUATIONS FOR THE PROPAGATION OF FIELDS IN HARMONIC GENERATION

In order to include the relevant frequency components, let us consider the time-dependent polarization

$$\mathbf{P}(\mathbf{r},t)=\mathbf{P}^{(L)}(\mathbf{r},t)+\mathbf{P}^{(\mathrm{nl})}(\mathbf{r},t)$$

of the medium. It consists of two parts, the linear polarization $\mathbf{P}^{(L)}(\mathbf{r},t)$ and the nonlinear polarization $\mathbf{P}^{(\mathrm{nl})}(\mathbf{r},t)$. The linear polarization $\mathbf{P}^{(L)}(\mathbf{r},t)$ is a linear function of the electric field $\mathbf{E}(\mathbf{r},t)$:

$$\begin{aligned} P_i^{(L)}(\mathbf{r},t) &= \int_{-\infty}^{\infty} d\tau \chi_{ij}^{(1)}(t-\tau)E_j(\mathbf{r},\tau) \\ &= \int d\omega\, e^{-i\omega t}\chi_{ij}^{(1)}(\omega)E_j(\mathbf{r},\omega) , \end{aligned} \tag{1a}$$

where $\chi_{ij}^{(1)}(\tau)$ is the linear susceptibility of the medium and $\chi_{ij}^{(1)}(\omega)$ is its Fourier transformation. $E_j(\mathbf{r},\omega)$ is the Fourier component of the electric field $\mathbf{E}(\mathbf{r},t)$ [see below for its explicit form in Eq. (2)]. For second-harmonic generation, the nonlinear polarization $\mathbf{P}^{(\mathrm{nl})}(\mathbf{r},t)$ is related to the electric field $\mathbf{E}(\mathbf{r},t)$ through the second-order nonlinear susceptibility $\chi_{ijk}^{(2)}$ in a general quadratic form [16]

$$\begin{aligned} P_i^{(\mathrm{nl})}(\mathbf{r},t) = \int_{-\infty}^{\infty} & \chi_{ijk}^{(2)}(t-t_1,t-t_2)E_j(\mathbf{r},t_1) \\ & \times E_k(\mathbf{r},t_2)dt_1dt_2 . \end{aligned} \tag{1b}$$

Here in Eqs. (1), we have assumed that the response of the medium is local so that there is no integration over the spatial variables. Because of causality, it is obvious that $\chi_{ij}^{(1)}(\tau)=0$ for $\tau<0$ and $\chi_{ijk}^{(2)}(\tau_1,\tau_2)=0$ for $\tau_1<0$ or $\tau_2<0$. i,j,k denote the polarizations of the fields. In second-harmonic generation, because of phase-matching condition for bulk material, the oscillation polarizations for the relevant fields are fixed in certain directions so that we can drop the indices i,j,k in Eqs. (1). The linear part of polarization will give rise to a frequency-dependent index of refraction $n(\omega)=\sqrt{1+4\pi\chi^{(1)}(\omega)}$ for the medium [see below in Eq. (9)], which causes the dispersion effect of the medium.

For quantized fields, Eqs. (1) still stand except that P and E are operators and the electric field operator $\hat{E}(t)$ has the form of

$$\hat{E}(\mathbf{r},t)=\hat{E}^{(-)}(\mathbf{r},t)+\hat{E}^{(+)}(\mathbf{r},t) , \tag{2a}$$

with

$$\begin{aligned} [\hat{E}^{(-)}(\mathbf{r},t)]^\dagger &= \hat{E}^{(+)}(\mathbf{r},t) \\ &= \int_0^\infty d\omega\, i\left[\frac{\hbar\omega}{cn(\omega)}\right]^{1/2}\hat{a}(\omega)e^{i(kz-\omega t)} , \end{aligned} \tag{2b}$$

where $\hat{a}(\omega)$ is the annihilation operator of the field and satisfies $[\hat{a}(\omega),\hat{a}^\dagger(\omega')]=\delta(\omega-\omega')$. A one-dimensional plane wave is used to describe the spatial mode and the field propagates along z direction with wave vector $k(\omega)=\omega n(\omega)/c$. The Fourier transformation of the nonlinear polarization $P^{(\mathrm{nl})}(t)$ can then be calculated from Eqs. (1) and (2) as

$$\begin{aligned} P^{(\mathrm{nl})}(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, P^{(\mathrm{nl})}(t)e^{i\omega t} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt_1dt_2\chi^{(2)}(t-t_1,t-t_2)[E^{(-)}(t_1)E^{(-)}(t_2)+E^{(-)}(t_1)E^{(+)}(t_2) \\ &\qquad + E^{(+)}(t_1)E^{(-)}(t_2)+E^{(+)}(t_1)E^{(+)}(t_2)]e^{i\omega t} . \end{aligned} \tag{3}$$

With Eq. (2b), the first term in Eq. (3) is calculated as

$$\begin{aligned} \{E^{(-)}E^{(-)}\} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, dt_1dt_2\chi^{(2)}(t-t_1,t-t_2)e^{i\omega t}\int d\omega' d\omega'' e^{i\omega' t_1}A^*(z,\omega')e^{i\omega'' t_2}A^*(z,\omega'') \\ &= \int d\omega' d\omega''\chi^{(2)}(-\omega',-\omega''))\delta(\omega'+\omega''+\omega)A^*(z,\omega')A^*(z,\omega'') \\ &= \int d\omega'\chi^{(2)}(-\omega',\omega'+\omega)A^*(z,\omega')A^*(z,-\omega-\omega') , \end{aligned} \tag{4}$$

with

$$\chi^{(2)}(\omega',\omega'')\equiv\int_0^\infty d\tau_1d\tau_2\chi^{(2)}(\tau_1,\tau_2)e^{i\omega'\tau_1+i\omega''\tau_2}$$

and

$$A(z,\omega)=i\left[\frac{\hbar\omega}{cn(\omega)}\right]^{1/2}a(z,\omega)e^{ikz} , \tag{5}$$

where the amplitude $a(z,\omega)$ is treated as a c number and is a slowly varying function of z as compared to e^{ikz}. Similarly, for the other terms in Eq. (3), we have

$$\{E^{(-)}E^{(+)}\}=\int d\omega'\chi^{(2)}(-\omega',\omega+\omega')A^*(z,\omega')A(z,\omega'+\omega) , \tag{4'}$$

$$\{E^{(+)}E^{(-)}\}=\int d\omega'\chi^{(2)}(\omega',\omega-\omega')A(z,\omega')A^*(z,\omega'-\omega) , \tag{4''}$$

$$\{E^{(+)}E^{(+)}\} = \int d\omega' \chi^{(2)}(\omega', \omega-\omega') A(z,\omega') A(z,\omega-\omega') . \tag{4'''}$$

In second-harmonic generation, the light field is strong at the fundamental frequency ω_0. A harmonic field at the frequency $2\omega_0$ is generated when the phases of the two fields are matched. Higher harmonic components are usually very weak because they do not satisfy the phase-matching conditions and require much higher power to be significantly strong. Thus we can assume that there only exist two bands of frequencies for the field which are centered at ω_0 and $2\omega_0$, respectively. For the harmonic band, $\omega \sim 2\omega_0$, and we find from Eqs. (4) that only the term of Eq. (4''') contributes to the polarization $P^{(\mathrm{nl})}(\omega)$ that is responsible for the generation of harmonic frequency $\omega \sim 2\omega_0$. Thus we have

$$P_2^{(\mathrm{nl})}(\omega) = \int_{[\omega_0]} d\omega' \chi^{(2)}(\omega', \omega-\omega') A_1(z,\omega') A_1(z,\omega-\omega') \quad (\omega \sim 2\omega_0) , \tag{6}$$

where the subscripts 1,2 denote the fundamental and harmonic bands, respectively, and $[\omega_0]$ represents the integration range around ω_0 for ω'. Similarly for the fundamental band, we have

$$\begin{aligned} P_1^{(\mathrm{nl})}(\omega) &= \int_{[\omega_0]} d\omega' \chi^{(2)}(-\omega', \omega+\omega') A_1^*(z,\omega') A_2(z,\omega'+\omega) \\ &\quad + \int_{[2\omega_0]} d\omega'' \chi^{(2)}(\omega'', \omega-\omega'') A_2(z,\omega'') A_1^*(z,\omega''-\omega) \\ &= 2\int_{[\omega_0]} d\omega' \chi^{(2)}(-\omega', \omega+\omega') A_1^*(z,\omega') A_2(z,\omega'+\omega) \quad (\omega \sim \omega_0) , \end{aligned} \tag{7}$$

where after we change the integration variable from ω'' to $\omega'+\omega$ in the second term, we used the symmetry property of $\chi^{(2)}(\tau_1,\tau_2) = \chi^{(2)}(\tau_2,\tau_1)$. Usually, within the bandwidth of interest, $\chi^{(2)}(\omega',\omega'')$ does not change very much with frequency. Furthermore, under conditions of low-frequency excitation (frequencies ω_0 and $2\omega_0$ are much smaller than the lowest resonant frequency of the medium), the nonlinear susceptibility $\chi^{(2)}$ is essentially independent of frequency for the bandwidth of interest [3]. Thus we can replace it with a constant $\chi^{(2)}$ in Eqs. (6) and (7) and obtain

$$P_1^{(\mathrm{nl})}(\omega) = 2\chi^{(2)} \int_{[\omega_0]} d\omega' A_1^*(z,\omega') A_2(z,\omega'+\omega) \quad (\omega \sim \omega_0) , \tag{8a}$$

$$P_2^{(\mathrm{nl})}(\omega) = \chi^{(2)} \int_{[\omega_0]} d\omega' A_1(z,\omega') A_1(z,\omega-\omega') \quad (\omega \sim 2\omega_0) . \tag{8b}$$

Next we consider the wave equation with the nonlinear polarization $P^{(\mathrm{nl})}$ given in Eqs. (6) and (7), in order to derive equations of propagation of the field. Because the spatial mode of the field is plane wave, we only need to consider z direction for propagation. Therefore we have, for the one-dimensional wave equation [3],

$$-\frac{\partial^2}{\partial z^2} E(z,t) + \frac{1}{c^2}\frac{\partial^2}{\partial t^2} E(z,t) = -\frac{4\pi}{c^2}\frac{\partial^2}{\partial t^2} P(z,t) , \tag{9}$$

where $P(z,t) = P^{(L)}(z,t) + P^{(\mathrm{nl})}(z,t)$ is the polarization of the medium. Substituting Eq. (1b), Eqs. (2), and the reverse transformation of Eq. (3) into Eq. (9), we obtain for the frequency component ω

$$\frac{d^2}{dz^2} A(z,\omega) + \frac{\omega^2 n^2(\omega)}{c^2} A(z,\omega) = -\frac{4\pi\omega^2}{c^2} P^{(\mathrm{nl})}(\omega) , \tag{10}$$

where $n(\omega) \equiv \sqrt{1+4\pi\chi^{(1)}(\omega)}$ is the index of refraction of the medium. By substituting Eq. (5) into Eq. (10) and making the well-known slowly-varying-amplitude approximation

$$\left| \frac{d^2 a(z,\omega)}{dz^2} \right| \ll \left| k \frac{da(z,\omega)}{dz} \right| ,$$

we end up with

$$\left[\frac{\hbar\omega}{cn(\omega)} \right]^{1/2} \frac{da(z,\omega)}{dz} = \frac{2\pi\omega^2}{kc^2} P^{(\mathrm{nl})}(\omega) e^{-ikz} . \tag{11}$$

Substituting $P^{(\mathrm{nl})}(\omega)$ in Eqs. (6) and (7) into Eq. (11), we obtain the equations for the propagation of the fundamental and harmonic frequency components along the z direction in the nonlinear medium,

$$\frac{da_1(z,\omega)}{dz} = 2K \int_{[\omega_0]} d\omega' a_1^*(z,\omega') a_2(z,\omega+\omega') e^{i\Delta k_1 z} \quad (\omega \sim \omega_0) , \tag{12a}$$

$$\frac{da_2(z,\omega)}{dz} = -K \int_{[\omega_0]} d\omega' a_1(z,\omega') a_1(z,\omega-\omega') e^{i\Delta k_2 z} \quad (\omega \sim 2\omega_0) , \tag{12b}$$

where $K \equiv (2\hbar\omega_0/n_2 c)^{1/2} 2\pi\omega_0\chi^{(2)}/n_1 c$ with $n_1 \equiv n(\omega_0)$, $n_2 \equiv n(2\omega_0)$. $\Delta k_1(\omega') \equiv k_2(\omega'+\omega) - k_1(\omega') - k_1(\omega)$ and $\Delta k_2(\omega') \equiv k_1(\omega') + k_1(\omega-\omega') - k_2(\omega)$ are the phase mismatches with subscripts 1 and 2 representing the fundamental and harmonic fields, respectively. We used the narrow bandwidth approximation that the integration range $[\omega_0] \ll \omega_0$ on the constants for the value of K. Here only the slowly-varying-amplitude approximation and narrow bandwidth approximation are used for the derivation of Eqs. (12). Equations (12) are a set of nonlinear differential-integral equations that couple the harmonic field with the fundamental field and vice versa for the propagation of the fields in the nonlinear medium (single-pass case). Given the initial conditions on the fundamental and harmonic field, we should be able to find their values along the nonlinear medium for any given length z. Although they are derived from classical wave equation, they are also valid for the quantized fields as long as we treat the amplitude $a(z,\omega)$ as the annihilation operator.

III. LINEARIZED INPUT-OUTPUT RELATIONS FOR THE QUANTUM FLUCTUATIONS

Because of the nonlinear nature of the coupling equations in (12), it is impossible to solve them analytically without making any approximation. For the problem of second-harmonic generation, it usually starts with a strong fundamental field. Let us assume that the fundamental field has a strong coherent component that is monochromatic at frequency ω_0 while the quantum fluctuations are weak, that is,

$$a_1(z,\omega)=c_1(z)\delta(\omega-\omega_0)+\Delta\hat{a}_1(z,\omega)\ , \tag{13}$$

where c_1 is a c number whereas $\Delta\hat{a}_1$ is an operator characterizing the quantum fluctuation of the field with $\langle|\Delta\hat{a}_1|^2\rangle \ll |c_1|^2$. After substituting Eq. (13) into Eq. (12b), we find that the harmonic field has similar form:

$$a_2(z,\omega)=c_2(z)\delta(\omega-2\omega_0)+\Delta\hat{a}_2(z,\omega)\ , \tag{14}$$

with center frequency at $\omega=2\omega_0$. Substituting Eqs. (13) and (14) into Eqs. (12) and keeping only the terms up to the first order in $\Delta\hat{a}_i$ $(i=1,2)$, we have

$$\begin{aligned}\frac{dc_1}{z}\delta(\omega-\omega_0)+\frac{d}{dz}\Delta\hat{a}_1(\omega) &= 2K[c_1^*c_2\delta(\omega-\omega_0)e^{iz\Delta k_1^0}+c_1^*\Delta\hat{a}_2(\omega+\omega_0)e^{iz\Delta k_1(\omega_0)} \\ &\quad +c_2\Delta\hat{a}_1^\dagger(2\omega_0-\omega)e^{iz\Delta k_1(2\omega_0-\omega)}]\ ,\end{aligned} \tag{15a}$$

$$\begin{aligned}\frac{dc_2}{dz}\delta(\omega-2\omega_0)+\frac{d}{dz}\Delta\hat{a}_2(\omega) &= -K[c_1^2\delta(\omega-2\omega_0)e^{iz\Delta k_2^0} \\ &\quad +2c_1\Delta\hat{a}_1(\omega-\omega_0)e^{iz\Delta k_2(\omega_0)}]\ .\end{aligned} \tag{15b}$$

Here $\Delta k_1^0=-\Delta k_2^0\equiv 2\omega_0(n_2-n_1)/c$ with $n_1\equiv n_1(\omega_0)$ and $n_2\equiv n_2(2\omega_0)$ being the indices of refraction for the fundamental and harmonic fields. When the phase-matching condition is satisfied by the fundamental and harmonic fields, we have $n_1=n_2$, thus $\Delta k_i^0=0$ $(i=1,2)$. By equating the zeroth-order terms in $\Delta\hat{a}$ on both sides of Eqs. (15), we have for the coherent parts of the fields

$$\begin{aligned}\frac{dc_1}{dz}&=2Kc_1^*c_2\ ,\\ \frac{dc_2}{dz}&=-Kc_1^2\ .\end{aligned} \tag{16}$$

Similarly, for the first-order terms in Δa, we obtain the equations for the quantum fluctuations of the fields,

$$\begin{aligned}\frac{d}{dz}\Delta\hat{a}_1(\omega)&=2K[c_1^*\Delta\hat{a}_2(\omega+\omega_0)e^{iz\Delta k_1(\omega_0)} \\ &\quad +c_2\Delta\hat{a}_1^\dagger(2\omega_0-\omega)e^{iz\Delta k_1(2\omega_0-\omega)}] \\ &\qquad (\omega\sim\omega_0)\ ,\end{aligned} \tag{17a}$$

$$\frac{d}{dz}\Delta\hat{a}_2(\omega)=-2Kc_1\Delta\hat{a}_1(\omega-\omega_0)e^{iz\Delta k_2(\omega_0)}\quad(\omega\sim 2\omega_0)\ . \tag{17b}$$

Because of dispersion of the medium, $\Delta k_1,\Delta k_2\neq 0$ for $\omega\neq\omega_0, 2\omega_0$. Let us expand $n_1(\omega)$ around ω_0 and $n_2(\omega)$ around $2\omega_0$. We then obtain the phase mismatches

$$\begin{aligned}\Delta k_1(\omega_0)&=(\omega-\omega_0)[(\omega+\omega_0)\Delta n_2-\omega\Delta n_1]/c \\ &\simeq(\omega-\omega_0)\omega_0(2\Delta n_2-\Delta n_1)/c\equiv(\omega-\omega_0)\Delta n/c \\ &\qquad \text{for } \omega\sim\omega_0\ ,\end{aligned} \tag{18a}$$

$$\Delta k_1(2\omega_0-\omega)=-2(\omega-\omega_0)^2\Delta n_1/c\quad\text{for } \omega\sim\omega_0\ , \tag{18b}$$

$$\begin{aligned}\Delta k_2(\omega_0)&=(\omega-2\omega_0)[(\omega-\omega_0)\Delta n_1-\omega\Delta n_2]/c \\ &\simeq-(\omega-2\omega_0)\omega_0(2\Delta n_2-\Delta n_1)/c \\ &\equiv-(\omega-2\omega_0)\Delta n/c\quad\text{for } \omega\sim 2\omega_0\ .\end{aligned} \tag{18c}$$

Here $\Delta n_i\equiv\partial n_i/\partial\omega$ $(i=1,2)$, $\Delta n\equiv(2\Delta n_2-\Delta n_1)\omega_0$, and we keep only the lowest-order nonzero terms in $(\omega-\omega_0)$ and $(\omega-2\omega_0)$ in the approximation. Hence Eqs. (17) change to

$$\begin{aligned}\frac{d}{dz}\Delta\hat{a}_1(\omega)&=2K[c_1^*\Delta\hat{a}_2(\omega+\omega_0)e^{i(\omega-\omega_0)z\Delta n/c} \\ &\quad +c_2\Delta\hat{a}_1^\dagger(2\omega_0-\omega)e^{-2i(\omega-\omega_0)^2z\Delta n_1/c}] \\ &\qquad (\omega\sim\omega_0)\ ,\end{aligned} \tag{19a}$$

$$\frac{d}{dz}\Delta\hat{a}_2(\omega)=-2Kc_1\Delta\hat{a}_1(\omega-\omega_0)e^{-i(\omega-2\omega_0)z\Delta n/c}\quad(\omega\sim 2\omega_0)\ . \tag{19b}$$

Thus the quantum fluctuation of the fundamental frequency at ω is coupled to the harmonic frequency component at $\omega+\omega_0$ around $2\omega_0$ and to the conjugate of the fundamental frequency component at $2\omega_0-\omega$ around ω_0 whereas that of the harmonic field is only directly coupled to the fundamental field.

Let us first solve Eqs. (16) for the coherent components. Actually, the solution is already given in the standard nonlinear optics textbook [3]. For harmonic generation with initial fundamental input of $c_1(0)=\sqrt{N_0}e^{i\varphi_0}$ and no harmonic input $[c_2(0)=0]$, the solutions to Eqs. (16) are given as

$$\begin{aligned}c_1(z)&=e^{i\varphi_0}\sqrt{N_0}\,\text{sech}\zeta\ ,\\ c_2(z)&=-e^{i2\varphi_0}\sqrt{N_0/2}\tanh\zeta\ ,\end{aligned} \tag{20}$$

where $\zeta\equiv z/z_0$ is the normalized distance with $z_0\equiv 1/\sqrt{2K^2N_0}$ being the characteristic distance. When $z=z_0$, $\zeta=1$, and a significant amount of fundamental field is converted into harmonic field with conversion efficiency $\tanh^2(1)=58\%$. Notice the relation

$$|c_1(z)|^2+2|c_2(z)|^2=N_0 \tag{21}$$

and recall that $\hat{a}(\omega)$ is the photon annihilation operator, then Eq. (21) simply means energy conservation: to generate one harmonic photon, two fundamental photons are annihilated.

With solutions in Eqs. (20), we can substitute in Eqs. (19) to find the equations for the propagation of the quantum fluctuations. Equations (19) change to

$$\frac{d}{d\zeta}\Delta\hat{a}_1(\Omega)=e^{-i\varphi_0}\sqrt{2}\Delta\hat{a}_2(\Omega)e^{i\zeta\delta}\,\text{sech}\zeta - e^{i2\varphi_0}\Delta\hat{a}_1^{\dagger}(-\Omega)e^{-i\zeta\sigma^2}\tanh\zeta\ , \qquad (22a)$$

$$\frac{d}{d\zeta}\Delta\hat{a}_2(\Omega)=-e^{i\varphi_0}\sqrt{2}\Delta\hat{a}_1(\Omega)e^{-i\zeta\delta}\,\text{sech}\zeta\ , \qquad (22b)$$

where we changed ω to $\omega_0+\Omega$ in Eq. (18a) and ω to $2\omega_0+\Omega$ in Eq. (18b) and dropped the center frequencies $\omega_0, 2\omega_0$ in the notation. The dimensionless frequency offsets $\delta\equiv\Omega/\Omega_1$ and $\sigma\equiv\Omega/\Omega_2$ with $\Omega_1\equiv c/z_0\Delta n$ and $\Omega_2\equiv\sqrt{c/2z_0\Delta n_1}$. As we will see in Sec. IV C, usually $\Omega_2\gg\Omega_1$ so that we can divide the frequency Ω into three ranges: (i) $\Omega\ll\Omega_1/\zeta, \Omega_2/\zeta^{1/2}$ or $\zeta\delta, \zeta\sigma^2\ll 1$; (ii) $\Omega\sim\Omega_1/\zeta\ll\Omega_2/\zeta^{1/2}$ or $\zeta\delta\sim 1$, but $\zeta\sigma^2\ll 1$; (iii) $\Omega\sim\Omega_2/\zeta^{1/2}$ or $\zeta\sigma^2\sim 1$. At this moment, for the simplicity of discussion, let us assume Ω is in ranges (i) and (ii) so that $\zeta\sigma^2\ll 1$ and we can set $e^{-i\zeta\sigma^2}\simeq 1$ (we will come back to the effect of this term in Sec. IV C). With this approximation, we now rewrite Eqs. (22) in terms of the quadrature-phase amplitudes of the fields defined as

$$\begin{aligned}
\hat{x}_1(\Omega)&\equiv[\Delta\hat{a}_1(\Omega)e^{-i\varphi_0}+\Delta\hat{a}_1^{\dagger}(-\Omega)e^{i\varphi_0}]/\sqrt{2}\ ,\\
\hat{y}_i(\Omega)&\equiv[\Delta\hat{a}_1(\Omega)e^{-i\varphi_0}-\Delta\hat{a}_1^{\dagger}(-\Omega)e^{i\varphi_0}]/i\sqrt{2}\ ,\\
\hat{x}_2(\Omega)&\equiv[\Delta\hat{a}_2(\Omega)e^{-i2\varphi_0}+\Delta\hat{a}_2^{\dagger}(-\Omega)e^{i2\varphi_0}]/\sqrt{2}\ ,\\
\hat{y}_2(\Omega)&\equiv[\Delta\hat{a}_2(\Omega)e^{-i2\varphi_0}-\Delta\hat{a}_2^{\dagger}(-\Omega)e^{i2\varphi_0}]/i\sqrt{2}\ ,
\end{aligned} \qquad (23)$$

which should satisfy the canonical commutation relations $[\hat{x}_j(\Omega),\hat{y}_j(\Omega')]=i\delta(\Omega+\Omega')$ $(j=1,2)$. Hence Eqs. (22) change to

$$\frac{d\hat{x}_1}{d\zeta}=\sqrt{2}\hat{x}_2e^{i\zeta\delta}\,\text{sech}\zeta-\hat{x}_1\tanh\zeta\ , \qquad (24a)$$

$$\frac{d\hat{x}_2}{d\zeta}=-\sqrt{2}\hat{x}_1e^{-i\zeta\delta}\,\text{sech}\zeta\ , \qquad (24b)$$

$$\frac{d\hat{y}_1}{d\zeta}=\sqrt{2}\hat{y}_2e^{i\zeta\delta}\text{sech}\zeta+\hat{y}_1\tanh\zeta\ , \qquad (25a)$$

$$\frac{d\hat{y}_2}{d\zeta}=-\sqrt{2}\hat{y}_1e^{-i\zeta\delta}\,\text{sech}\zeta\ . \qquad (25b)$$

Because of the phases in the coherent components of the fields [Eqs. (20)] and in the definitions of $\hat{x}_{1,2}$, $\hat{y}_{1,2}$ [Eqs. (23)], it can be proved that $\hat{x}_i(\Omega)$, $\hat{x}_2(\Omega)$ are associated to the intensity fluctuations of the fundamental and the harmonic fields, respectively, and $\hat{y}_1(\Omega)$, $\hat{y}_2(\Omega)$ to the phase fluctuations. It can be easily seen that Eqs. (24a) and (24b) are coupled and so are Eqs. (25a) and (25b) but Eqs. (24) are independent of Eqs. (25). Thus we can solve them separately. Notice that Eqs. (24) and (25) are linear sets of equations. From the theory of linear differential equations, we know that their solutions can be expressed linearly in terms of the initial conditions, that is,

$$\begin{aligned}
\hat{x}_1(\zeta)&=f_1^x(\zeta)\hat{x}_1(0)+f_2^x(\zeta)\hat{x}_2(0)\ ,\\
\hat{x}_2(\zeta)&=h_1^x(\zeta)\hat{x}_1(0)+h_2^x(\zeta)\hat{x}_2(0)\ ,\\
\hat{y}_1(\zeta)&=f_1^y(\zeta)\hat{y}_1(0)+f_2^y(\zeta)\hat{y}_2(0)\ ,\\
\hat{y}_2(\zeta)&=h_1^y(\zeta)\hat{y}_1(0)+h_2^y(\zeta)\hat{y}_2(0)\ ,
\end{aligned} \qquad (26)$$

where the functions f,h are c numbers and satisfy the initial conditions expressed as

$$\begin{aligned}
&f_1^x(0)=1,\quad h_2^x(0)=1,\quad f_1^y(0)=1,\quad h_2^y(0)=1\ ,\\
&f_2^x(0)=0,\quad h_1^x(0)=0,\quad f_2^y(0)=0,\quad h_1^y(0)=0\ .
\end{aligned} \qquad (27)$$

Substituting Eqs. (26) into Eqs. (24) and (25), we find the functions f,h satisfying the following sets of equations:

$$\begin{aligned}
\frac{df_i^x}{d\zeta}&=\sqrt{2}h_i^xe^{i\zeta\delta}\,\text{sech}\zeta-f_i^x\tanh\zeta\ ,\\
\frac{dh_i^x}{d\zeta}&=-\sqrt{2}f_i^xe^{-i\zeta\delta}\,\text{sech}\zeta
\end{aligned} \qquad (28a)$$

$(i=1,2)$, and

$$\begin{aligned}
\frac{df_i^y}{d\zeta}&=\sqrt{2}h_i^ye^{i\zeta\delta}\text{sech}\zeta+f_i^y\tanh\zeta\ ,\\
\frac{dh_i^y}{d\zeta}&=-\sqrt{2}f_i^ye^{-i\zeta\delta}\,\text{sech}\zeta
\end{aligned} \qquad (28b)$$

$(i=1,2)$, with initial conditions given in Eqs. (27). From the above equations, we can prove the identities

$$\begin{aligned}
f_i^{x,y}(\zeta,-\delta)&=[f_i^{x,y}(\zeta,\delta)]^*\ ,\\
h_i^{x,y}(\zeta,-\delta)&=[h_i^{x,y}(\zeta,\delta)]^*
\end{aligned} \qquad (29)$$

$(i=1,2)$, which will be useful for later calculations of spectra of squeezing and correlation functions. To solve the sets of linear differential equations in Eqs. (28), we first make the transformation of

$$f_i^x(\zeta)=\bar{f}_i^x(\zeta)/\cosh\zeta,\quad f_i^y(\zeta)=\bar{f}_i^y(\zeta)\cosh\zeta \qquad (30)$$

$(i=1,2)$. Then we find that $\{\bar{f}_2^x, h_2^x\}$ satisfy a same set of transformed equations and initial conditions as $\{-(h_1^y)^*, (\bar{f}_1^y)^*\}$. Therefore they should be equivalent to each other:

$$\begin{aligned}
\bar{f}_2^x(\zeta)&=-[h_1^y(\zeta)]^*\ ,\\
h_2^x(\zeta)&=[\bar{f}_1^y(\zeta)]^*\ .
\end{aligned} \qquad (31a)$$

Similarly,

$$\begin{aligned}
\bar{f}_2^y(\zeta)&=-[h_1^x(\zeta)]^*\ ,\\
h_2^y(\zeta)&=[\bar{f}_1^x(\zeta)]^*\ .
\end{aligned} \qquad (31b)$$

Or from Eqs. (30),

$$\begin{aligned}
f_1^x(\zeta)&=[h_2^y(\zeta)]^*\,\text{sech}\zeta\ ,\\
f_2^x(\zeta)&=-[h_1^y(\zeta)]^*\,\text{sech}\zeta\ ,\\
f_1^y(\zeta)&=[h_2^x(\zeta)]^*\cosh\zeta\ ,\\
f_2^y(\zeta)&=-[h_1^x(\zeta)]^*\cosh\zeta\ .
\end{aligned} \qquad (32)$$

Hence we only need to find $h^{x,y}_{1,2}(\zeta)$ which satisfy the sets of equations

$$\frac{d(h^y_2)^*}{d\zeta}=\sqrt{2}h^x_1 e^{i\zeta\delta},\quad \frac{dh^x_1}{d\zeta}=-\sqrt{2}(h^y_2)^* e^{-i\zeta\delta}\operatorname{sech}^2\zeta\ ,$$
$$\text{with } h^y_2(0)=1,\quad h^x_1(0)=0\ ; \qquad (33a)$$

and

$$\frac{d(h^y_1)^*}{d\zeta}=-\sqrt{2}h^x_2 e^{i\zeta\delta},\quad \frac{dh^x_2}{d\zeta}=\sqrt{2}(h^y_1)^* e^{-i\zeta\delta}\operatorname{sech}^2\zeta\ ,$$
$$\text{with } h^y_1(0)=0\quad h^x_2(0)=1\ . \qquad (33b)$$

It is easy to check from the above equations that

$$\frac{d}{d\zeta}\{h^x_1(\zeta,\Omega)[h^y_1(\zeta,\Omega)]^*+h^x_2(\zeta,\Omega)[h^y_2(\zeta,\Omega)]^*\}=0\ .$$

Hence

$$\begin{aligned} h^x_1(\zeta,\Omega)[h^y_1(\zeta,\Omega)]^*+h^x_2(\zeta,\Omega)[h^y_2(\zeta,\Omega)]^* &= h^x_1(0,\Omega)[h^y_1(0,\Omega)]^*+h^x_2(0,\Omega)[h^y_2(0,\Omega)]^* \\ &=1\ . \end{aligned} \qquad (34)$$

This identity together with Eqs. (29) and (32) can be used to prove the canonical commutation relations

$$[\hat{x}_j(\zeta,\Omega),\hat{y}_j(\zeta,\Omega')]=i\delta(\Omega+\Omega')\quad (j=1,2) \qquad (35)$$

for any ζ and Ω just as the inputs $\hat{x}_j(0,\Omega)$, $\hat{y}_j(0,\Omega)$ $(j=1,2)$.

For arbitrary value of δ, we are not able to solve Eqs. (33) analytically. However, for the bandwidth in which we are interested, Ω is usually small so that we can make the approximation that $\zeta\delta\ll 1$ and set $e^{i\zeta\delta}\simeq 1$, which is equivalent to perfect phase match with $\Delta k_1=\Delta k_2=0$ for all frequencies. Under this approximation, Eqs. (33) can be solved with another transformation of $h^y_i(\zeta)=\bar{h}^y_i(\zeta)\tanh\zeta$. The solutions of Eqs. (33) for $\delta=0$ are given as

$$\begin{aligned} h^x_1(\zeta)&=-(\tanh\zeta+\zeta\operatorname{sech}^2\zeta)/\sqrt{2}\ ,\\ h^x_2(\zeta)&=\operatorname{sech}^2\zeta\ ,\\ h^y_1(\zeta)&=-\sqrt{2}\tanh\zeta\ ,\\ h^y_2(\zeta)&=1-\zeta\tanh\zeta\ , \end{aligned} \qquad (36a)$$

and from Eqs. (32), we have

$$\begin{aligned} f^x_1(\zeta)&=(1-\zeta\tanh\zeta)\operatorname{sech}\zeta\ ,\\ f^x_2(\zeta)&=\sqrt{2}\tan\zeta\operatorname{sech}\zeta\ ,\\ f^y_1(\zeta)&=\operatorname{sech}\zeta\ ,\\ f^y_2(\zeta)&=(\sinh\zeta+\zeta\operatorname{sech}\zeta)/\sqrt{2}\ . \end{aligned} \qquad (36b)$$

Substituting the functions f,h in Eqs. (36) into Eqs. (26), we obtain the input-output relations for the quantum fluctuations of fundamental and harmonic fields for arbitrary interaction length ζ. They have the following form:

$$\begin{aligned} \hat{x}_1(\zeta)&=\hat{x}_1(0)(1-\zeta\tanh\zeta)\operatorname{sech}\zeta\\ &\quad+\hat{x}_2(0)(\sqrt{2}\tanh\zeta\operatorname{sech}\zeta)\ ,\\ \hat{x}_2(\zeta)&=-\hat{x}_1(0)(\tanh\zeta+\zeta\operatorname{sech}^2\zeta)/\sqrt{2}+\hat{x}_2(0)\operatorname{sech}^2\zeta\ ,\\ \hat{y}_1(\zeta)&=\hat{y}_1(0)\operatorname{sech}\zeta+\hat{y}_2(0)(\sinh\zeta+\zeta\operatorname{sech}\zeta)/\sqrt{2}\ ,\\ \hat{y}_2(\zeta)&=-\hat{y}_1(0)(\sqrt{2}\tanh\zeta)+\hat{y}_2(0)(1-\zeta\tanh\zeta)\ . \end{aligned} \qquad (37)$$

As for $\delta\neq 0$, the solutions to Eqs. (24) and (25) will have the general form in Eqs. (26). We can solve Eqs. (33) numerically for the coefficients f,h. We will postpone the discussion on the case of $\delta\neq 0$ to Sec. IV C.

IV. QUANTUM FLUCTUATIONS IN HARMONIC GENERATION

A. Quadrature-phase squeezing

With the input-output relations in Eqs. (26) and solutions in Eqs. (37) for the coefficients f,h at $\zeta\delta\ll 1$, we find the propagation of spectrum of squeezing in the nonlinear medium as

$$\begin{aligned} S^x_1(\zeta,\Omega)&=|f^x_1(\zeta)|^2S^x_1(0,\Omega)+|f^x_2(\zeta)|^2S^x_2(0,\Omega)\ ,\\ S^x_2(\zeta,\Omega)&=|h^x_1(\zeta)|^2S^x_1(0,\Omega)+|h^x_2(\zeta)|^2u^x_2(0,\Omega)\ ,\\ S^y_1(\zeta,\Omega)&=|f^y_1(\zeta)|^2S^y_1(0,\Omega)+|f^y_2(\zeta)|^2S^y_2(0,\Omega)\ ,\\ S^y_2(\zeta,\Omega)&=|h^y_1(\zeta)|^2S^y_1(0,\Omega)+|h^y_2(\zeta)|^2S^2_y(0,\Omega)\ , \end{aligned} \qquad (38)$$

where $S^{x,y}_i(0,\Omega)$, $S^{x,y}_i(\zeta,\Omega)$ $(i=1,2)$ are the input and output spectra of squeezing for the quadrature-phase amplitudes of the fundamental and harmonic fields, respectively, and are defined from the relations

$$\langle x_i(\Omega)x_i(\Omega')\rangle=\delta(\Omega+\Omega')S^x_i(\Omega)\ ,$$
$$\langle y_i(\Omega)y_i(\Omega')\rangle=\delta(\Omega+\Omega')S^y_i(\Omega)$$

$(i=1,2)$. To obtain Eqs. (38) from Eqs. (26), we assumed that the inputs of the fundamental and the harmonic field are independent of each other and used the relations in Eqs. (29). It can be proved from Eqs. (2), (13), (14), (20), and (23) that $S^x_i(\Omega)$ $(i=1,2)$ is the spectrum for the intensity fluctuations of the fundamental and the harmonic fields and $S^y_i(\Omega)$ $(i=1,2)$ is for the phase fluctuations.

Because there are analytical solutions for the functions f,h at $\zeta\delta\ll 1$, we will mainly concentrate on the case of $\zeta\delta\ll 1$ in this and following subsections. For coherent-state input of the fundamental field and vacuum-state input of the harmonic field, $S^{x,y}_i(0)=1$ $(i=1,2)$. Therefore, from the solutions of f,h in Eqs. (37), we can find that

$$\begin{aligned} S^x_1(\zeta,\Omega)&=(1-\zeta\tanh\zeta)^2\operatorname{sech}^2\zeta+2\tanh^2\zeta\operatorname{sech}^2\zeta\ ,\\ S^x_2(\zeta,\Omega)&=(\tanh\zeta+\zeta\operatorname{sech}^2\zeta)^2/2+\operatorname{sech}^4\zeta\ ,\\ S^y_1(\zeta,\Omega)&=\operatorname{sech}^2\zeta+(\sinh\zeta+\zeta\operatorname{sech}\zeta)^2/2\ ,\\ S^y_2(\zeta,\Omega)&=2\tanh^2\zeta+(1-\zeta\tanh\zeta)^2\ , \end{aligned} \qquad (39)$$

which are plotted in Fig. 1. It is seen that the intensity fluctuations (x quadrature) of both fundamental and harmonic fields are squeezed below the vacuum fluctuation of 1. For large enough ζ, the intensity fluctuation of the fundamental field can be suppressed and an arbitrary amount of squeezing is achieved whereas only half of the input vacuum fluctuation is suppressed for the intensity fluctuation of the harmonic field. On the other hand, the phase fluctuations (y quadrature) of both fields, especially of the fundamental field, increase quickly as the fields propagate along the nonlinear medium. By calculating the products $S_1^x(\zeta)S_1^y(\zeta)$ and $S_2^x(\zeta)S_2^y(\zeta)$, we find that $S_i^x(\zeta)S_i^y(\zeta)>1$ $(i=1,2)$ and therefore the output fundamental and harmonic fields are not in a minimum uncertainty state.

The presence of squeezing in the fundamental field is attributed to the sechζ factor in the transformation of (32). This factor comes from the tanhζ-dependent intensity of the harmonic field [Eqs. (20)]. Therefore squeezing of the fundamental field in the second-harmonic generation process results from parametric amplification (deamplification) pumped by the generated harmonic field. This point can be better understood in type-II harmonic generation where two orthogonally polarized fundamental fields are coupled to each other as well as to the harmonic field. After a rotation of 45° of the polarizations, the fundamental fields are transformed to two other orthogonally polarized fields denoted as a and b which are decoupled from each other but each field is still coupled to the harmonic field in the same way as type-I second-harmonic generation process [17]. Thus the type-II process is decomposed into two type-I processes. The two processes are coupled indirectly through the harmonic field. If we only allow the coherent beam input to one of the fundamental fields, say field a, then the equations for the propagation of this field and the harmonic field are exactly the same as Eqs. (12) and Eqs. (16) and (17) under the linearization approximation. On the other hand, for the other orthogonally polarized fundamental field b, it behaves like the subharmonic field in parametric amplification process with the harmonic field as the pump field. Actually, for the quantum fluctuations $\Delta\hat{b}$ of field b, the equations are the same as Eqs. (17) except that the coherent component c_1 of the fundamental field is zero for field b and c_2 is given in Eq. (20). The quadrature-phase amplitudes $\bar{x}_1, \bar{y}_1$ of the field b can be proved to satisfy the following equations for $\zeta\delta \ll 1$:

$$\frac{d\bar{x}_1}{d\zeta}=\bar{x}_1 \tanh\zeta \; , \qquad \frac{d\bar{y}_1}{d\zeta}=-\bar{y}_1 \tanh\zeta \; , \tag{40}$$

where the tanhζ factor comes from the harmonic field which acts as the pump. The solutions to Eqs. (40) are given as

$$\bar{x}_1(\zeta)=\bar{x}_1(0)\cosh\zeta \; , \qquad \bar{y}_1(\zeta)=\bar{y}_1(0)\,\mathrm{sech}\zeta \; . \tag{41}$$

Thus $\bar{x}_1$ is amplified and $\bar{y}_1$ is deamplified just like in parametric amplifier. If the input to $\bar{x}_1, \bar{y}_1$ is in vacuum state, then the output is in a minimum uncertainty state with $\bar{y}_1$ squeezed. It is interesting to note that in this case the amount of squeezing $S_1^{\bar{y}}(\zeta)=\mathrm{sech}^2\zeta=1-\eta$ with $\eta=\tanh^2\zeta$ being the efficiency of mean field energy conversion from the fundamental field to the harmonic field [Eqs. (20)]. The situation here is different from a parametric amplifier with a constant undepleted pump because the strength of the pump field is a function of ζ $[|c_2|^2=(N_0/2)\tanh^2\zeta]$, which results in the coshζ- and sechζ-dependent gains for field b.

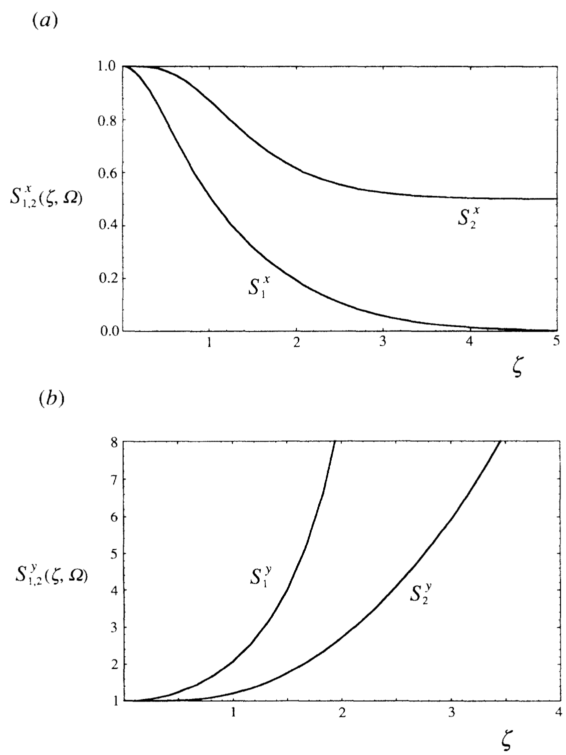

FIG. 1. (a) Intensity noise and (b) phase noise of the harmonic and fundamental fields for $\Omega\simeq 0$ as they propagate along the nonlinear medium. $\zeta\equiv z/z_0$ is the normalized distance.

B. Quantum correlations in harmonic generation

From Eqs. (36), we find that f_1^y, f_1^x, f_2^x, and h_2^x go to zero for large ζ with $f_1^y, f_2^x \sim e^{-\zeta}$, $f_1^x \sim \zeta e^{-\zeta}$, and $h_2^x \sim e^{-2\zeta}$, whereas f_2^y and h_2^y go to infinity for large ζ with $f_2^y \sim e^{\zeta}$ and $h_2^y \sim \zeta$. Therefore, for large ζ, Eqs. (34) can be approximated as

$$\begin{aligned}
\hat{x}_1(\zeta)&=f_1^x(\zeta)\hat{x}_1(0)+f_2^x(\zeta)\hat{x}_2(0) \; , \\
\hat{x}_2(\zeta)&\simeq h_1^x(\zeta)\hat{x}_1(0) \; , \\
\hat{y}_1(\zeta)&\simeq f_2^y(\zeta)\hat{y}_2(0) \; , \\
\hat{y}_2(\zeta)&=h_1^y(\zeta)\hat{y}_1(0)+h_2^y(\zeta)\hat{y}_2(0)
\end{aligned} \tag{42}$$

for $e^{-\zeta} \ll 1$. So $\hat{x}_2(\zeta)$ of the output fundamental field is perfectly correlated with $\hat{x}_1(0)$ of the input harmonic field while $\hat{y}_1(\zeta)$ of the output harmonic field is perfectly correlated with $\hat{y}_2(0)$ of the input harmonic field while

$\hat{y}_1(\zeta)$ of the output harmonic field is perfectly correlated with $\hat{y}_2(0)$ of the input fundamental fields. This can be seen in the correlation functions C defined as

$$
\begin{aligned}
C_{x_2^{\rm out}x_1^{\rm in}}(\zeta) &\equiv \frac{|\langle x_2(\zeta)x_1(0)\rangle|}{\sqrt{\langle[x_2(\zeta)]^2\rangle\langle[x_1(0)]^2\rangle}} \\
&= \frac{|h_1^x(\zeta)|}{\sqrt{|h_1^x|^2+|h_2^x|^2}}\ , \\
C_{y_1^{\rm out}y_2^{\rm in}}(\zeta) &\equiv \frac{|\langle y_1(\zeta)y_2(0)\rangle|}{\sqrt{\langle[y_2(0)]^2\rangle\langle[y_1(\zeta)]^2\rangle}} \\
&= \frac{|f_2^y(\zeta)|}{\sqrt{|f_1^y|^2+|f_2^y|^2}}\ .
\end{aligned}
\qquad (43)
$$

Because of the relations in Eqs. (32), we have $C_{x_2^{\rm out}x_1^{\rm in}}(\zeta)=C_{y_1^{\rm out}y_2^{\rm in}}(\zeta)$. In Fig. 2(a), we plot $C_{x_2^{\rm out}x_1^{\rm in}}(\zeta)$, $C_{y_1^{\rm out}y_2^{\rm in}}(\zeta)$ as a function of ζ and find that $C\simeq 1$ for $\zeta>2$. On the other hand, because $f_1^x/f_2^x=h_2^y/h_1^y\sim\zeta\rightarrow\infty$ for large ζ, there are also good correlations between the input and output of the amplitudes $\hat{x}_1,\hat{y}_2$. In Fig. 2(b), we plot the correlation function $C_{x_1^{\rm out}x_1^{\rm in}}(\zeta)=C_{y_2^{\rm out}y_2^{\rm in}}(\zeta)$ as a function of ζ and find that they approach 1 for a larger ζ. Because of the correlations discussed above, there also exist good correlations in the output between the amplitudes $\hat{x}_1(\zeta),\hat{x}_2(\zeta)$ as well as between $\hat{y}_1(\zeta),\hat{y}_2(\zeta)$. In Fig. 2(c), we plot the correlation functions $C_{x_1^{\rm out}x_2^{\rm out}}(\zeta)=C_{y_1^{\rm out}y_2^{\rm out}}(\zeta)$ as a function of ζ and find that they approach the value of 1 for perfect correlation for large ζ.

The correlations between the input and output as well as between the outputs of the fundamental and harmonic fields suggest the possible implementation of this process for QND measurement under the criteria given by Holland *et al.* [18] (HCWL criteria). Consider the amplitudes $\hat{x}_1$ and $\hat{y}_2$. The probes for the two quantities are $\hat{x}_2$ and $\hat{y}_1$, respectively. The first two HCWL criteria for QND measurement, i.e., the criteria on the measurement and the degradation of the signal field are satisfied by both $\hat{x}_1$ and $\hat{y}_2$ because the correlation functions $\{C_{x_1^{\rm out}x_1^{\rm in}}(\zeta),C_{x_2^{\rm out}x_1^{\rm in}}(\zeta)\}$ and $\{C_{y_2^{\rm out}y_2^{\rm in}}(\zeta),C_{y_1^{\rm out}y_2^{\rm in}}(\zeta)\}\sim 1$ as $\zeta\rightarrow\infty$. As for the third criterion on the state preparation, we find the conditional variances

$$V(x_1^{\rm out}|x_2^{\rm out})=V[x_1(\zeta)]\{1-C^2_{x_1^{\rm out}x_2^{\rm out}}\}\rightarrow 0 \quad \text{as } \zeta\rightarrow\infty\ , \qquad (44a)$$

$$V(y_2^{\rm out}|y_1^{\rm out})=V[y_2(\zeta)]\{1-C^2_{y_1^{\rm out}y_2^{\rm out}}\}\rightarrow 2 \quad \text{as } \zeta\rightarrow\infty\ . \qquad (44b)$$

Therefore x_1 is a QND quantity by the HCWL criteria although it is attenuated heavily ($\sim\zeta e^{-\zeta}$) in the process. As for the quantity y_2, if we attenuate $y_2(\zeta)$ by the amount of $1/h_2^y(\zeta)$ in such a way that is noise-free, i.e., $\hat{y}'_2=\hat{y}_2(\zeta)/h_2^y(\zeta)$, then $V(y_2^{\rm out}|y_1^{\rm out})\rightarrow 0$ for large ζ and therefore y'_2 is also a QND quantity. Even without the noise-free attenuator, because of the good correlation as expressed in the correlation functions $\{C_{y_2^{\rm out}y_2^{\rm in}}(\zeta),C_{y_1^{\rm out}y_2^{\rm in}}(\zeta)\}$ (~ 1 as $\zeta\rightarrow\infty$), the process is simply a realization of optical tap or quantum beam splitter for $\hat{y}_2$ [19,20], where input quantum signal is split without change of signal-to-noise ratio. Actually, the input signal is greatly amplified because $f_2^y\sim e^{\zeta}$ and $h_2^y\sim\zeta$ for large ζ.

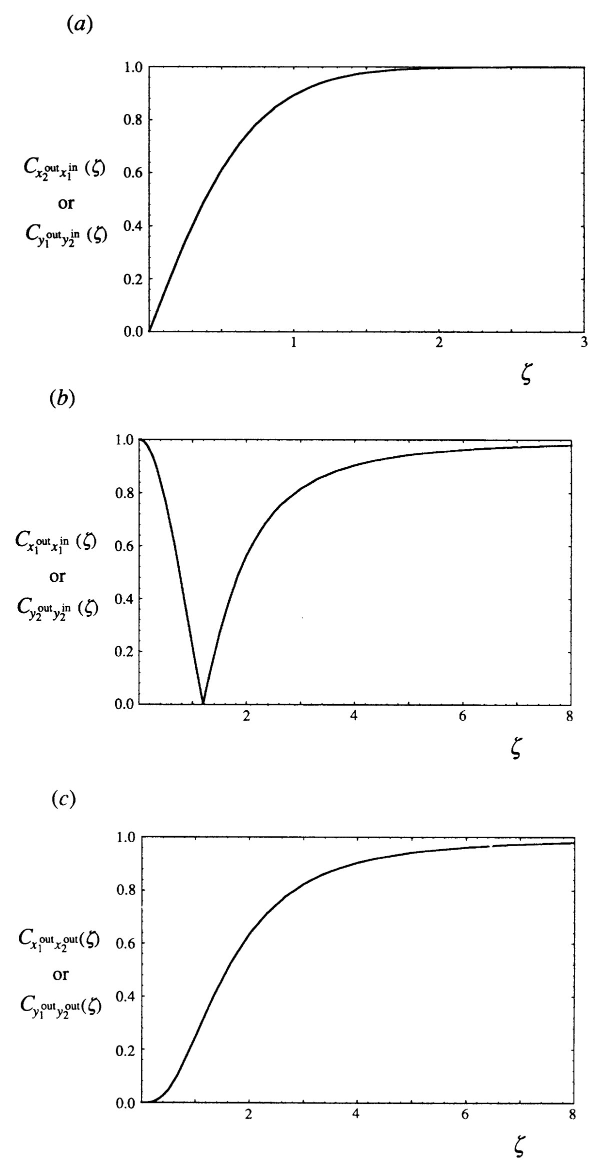

FIG. 2. (a) Correlations between intensity fluctuations of the output of the harmonic field $[\hat{x}_2(\zeta)]$ and the input of the fundamental field $[\hat{x}_1(0)]$ or between phase fluctuations of the output of the fundamental field $[\hat{y}_1(\zeta)]$ and the input of the harmonic field $[\hat{y}_2(0)]$. (b) Correlations between intensity fluctuations of the output and the input of the fundamental field $[\hat{x}_1(\zeta)$ and $\hat{x}_1(0)]$ or between phase fluctuations of output and the input of the harmonic field $[\hat{y}_2(\zeta)$ and $\hat{y}_2(0)]$. (c) Correlations between intensity fluctuations of the outputs of the fundamental and harmonic fields $[\hat{x}_1(\zeta)$ and $\hat{x}_2(\zeta)]$ or between phase fluctuations of the outputs of the fundamental and harmonic fields $[\hat{y}_1(\zeta)$ and $\hat{y}_2(\zeta)]$. $\zeta\equiv z/z_0$.

C. Effect of large frequency offset ($\delta \neq 0$)

When $\delta \neq 0$, there are no analytical solutions available for Eqs. (33). However, we can integrate them by numerical method. In Figs. 3(a) and 3(b), we plot the spectrum $S_1^x(\zeta,\Omega)$ of squeezing calculated from Eqs. (38) for the fundamental field as a function of $\delta(\propto \Omega)$ with the distance ζ fixed at two values of (0.5,2). Both the spectra decrease as δ increases and eventually reach the value of $\mathrm{sech}^2\zeta$ (dashed lines in Fig. 3) for large δ. This result is somewhat counter intuitive because in most schemes for the generation of squeezed states, the amount of squeezing decreases for large offset frequency Ω ($=\omega-\omega_0$ or $\omega-2\omega_0$). Usually, squeezing is best for either $\Omega=0$ or nearby and will diminish as Ω gets large. This is because the coupling between the amplitudes $\hat{a}(\Omega)$ and $\hat{a}^\dagger(-\Omega)$ is responsible for the squeezing effect and decreases for large Ω. To understand the counterintuitive effect for the case here, let us go back to Eqs. (24) and (25). Notice the fact that the terms with the $e^{i\zeta\delta}$ factor in Eqs. (24) and (25) will have a $1/\delta$ dependence for large δ after the integration and can be neglected for large δ. Without the $e^{i\zeta\delta}$ terms, Eqs. (24a), (24b), (25a), and (25b) are independent of each other, which means that there is no coupling between the fundamental and harmonic fields as is the case for small δ. It is not surprising for $S_1^x(\zeta,\Omega)$ to approach $\mathrm{sech}^2\zeta$ in large δ if we notice that Eqs. (24a) and (25b) without the $e^{i\zeta\delta}$ terms are similar to Eqs. (40) with solution in Eqs. (41) which are for a parametric amplifier pumped by the generated harmonic field. This indicates that as δ increases, the coupling between the quantum fluctuations of the harmonic and fundamental fields becomes weak and the fundamental field will act as in a parametric amplifier. Therefore the coupling between the harmonic and fundamental fields introduces extra noise to the fundamental output from the harmonic input so that the noise reduction in fundamental output field is better when the two fields are decoupled.

On the other hand, as shown in Fig. 4 where we plot the spectrum $S_2^x(\zeta,\Omega)$ as a function of δ for two values of ζ ($=2,5$), the effect of squeezing for the harmonic field diminishes as δ becomes large, and eventually $S_2^x(\zeta,\Omega)$ reaches the vacuum noise level of 1 for large δ. Therefore the squeezing in the harmonic field can be understood as coupling between the harmonic and fundamental field and will decrease as the coupling between the fundamental and harmonic fields becomes weak for large δ.

Recall that the coupling between the quantum fluctuations of the harmonic and fundamental fields is responsible for the correlation shown in Fig. 2. Therefore, as δ becomes large, we should expect a degradation in these correlations. Indeed, as shown in Fig. 5, where we plot $C_{x_2^{\mathrm{out}}x_1^{\mathrm{in}}}, C_{x_1^{\mathrm{out}}x_2^{\mathrm{out}}}$ as a function of δ for fixed $\zeta=2$, the correlation functions decrease as δ increases. On the other hand, $C_{x_1^{\mathrm{out}}x_1^{\mathrm{in}}}$ increases with increasing δ and becomes 1 for some values of δ as shown in Fig. 6. This is because the fundamental field acts as in a degenerate parametric amplifier for large δ and there exists perfect correlation between input and output in degenerate parametric amplifier as seen in Eqs. (41).

We may estimate the bandwidth for the squeezing of the harmonic field and the correlations between the two fields from the definition of δ ($\equiv \Delta n \Omega z_0/c$). Usually in harmonic generation, the power is strong enough to obtain significant conversion efficiency for a crystal of 10 mm length, so we can set the characteristic $z_0 \equiv 1/\sqrt{2K^2N_0}=10$ mm. By definition,

$$\Delta n \equiv \omega_0 \left| 2\frac{\partial n_2}{\partial \omega} - \frac{\partial n_1}{\partial \omega} \right| = -\frac{2\pi c}{\omega_0} \left| \frac{1}{2}\frac{\partial n_2}{\partial \lambda} - \frac{\partial n_1}{\partial \lambda} \right|$$

and for the harmonic generation at 860 nm with $KNbO_3$

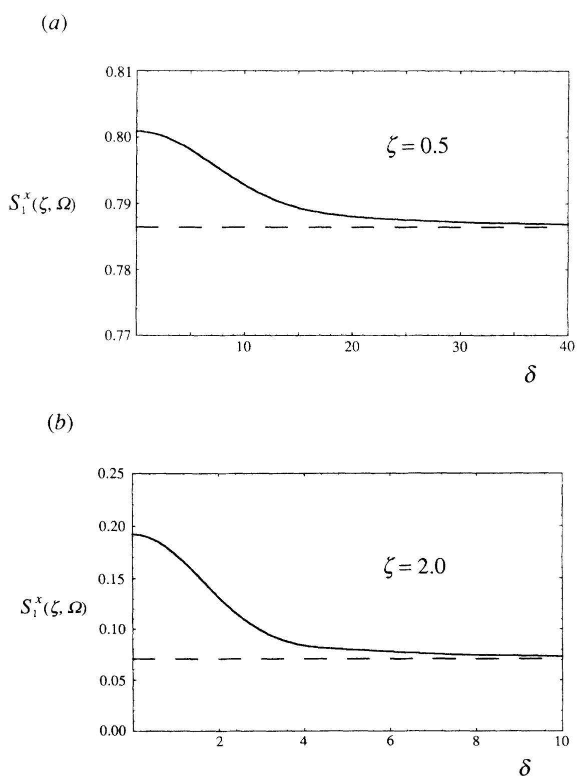

FIG. 3. Spectra of squeezing for intensity fluctuations of the fundamental field for Ω around $\Omega_1 \equiv c/z_0\Delta n$ but $\ll \Omega_2 \equiv (c/z_0\Delta n_1)^{1/2}$ at propagation distances (a) $\zeta=0.5$, (b) $\zeta=2.0$. $\delta \equiv \Omega/\Omega_1$ is the dimensionless frequency normalized to Ω_1.

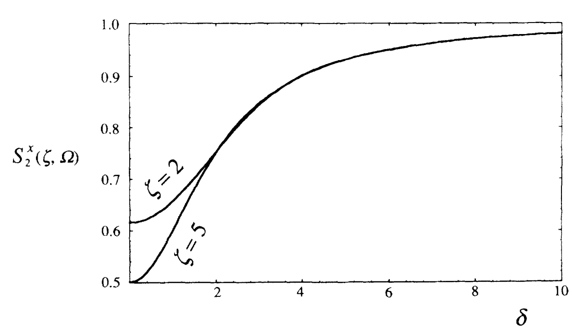

FIG. 4. Spectra of squeezing for intensity fluctuations of the harmonic field at propagation distances $\zeta=2$ and 5. $\delta \equiv \Omega/\Omega_1$ is the dimensionless frequency normalized to Ω_1.

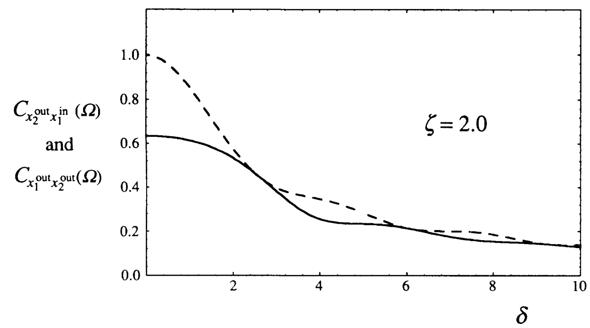

FIG. 5. Correlation functions between intensity fluctuations of the output of the harmonic field and the input of the fundamental fields (dashed curve) and between the intensity fluctuations of the outputs of the fundamental and harmonic fields (solid curve) as a function of normalized frequency $\delta \equiv \Omega/\Omega_1$ at the distance $\zeta=2$.

crystal, we have $\partial n_2/\partial\lambda = -1.4\ \mu m^{-1}$ and $\partial n_1/\partial\lambda = -0.14\ \mu m^{-1}$. Thus

$$\frac{\Omega}{\omega_0} = \frac{\delta\sqrt{2K^2N_0}}{2\pi}\left[-\frac{1}{2}\frac{\partial n_2}{\partial\lambda}+\frac{\partial n_1}{\partial\lambda}\right] = 3.7\times10^{-5}\delta \ .$$

With $\delta=1$ as the characteristic frequency, $\Omega=\Omega_1 = 3.7\times10^{-5}\omega_0 = 81$ GHz at 860 nm. This is the bandwidth for the correlations between the harmonic and fundamental fields as well as the bandwidth for the intensity squeezing in harmonic field.

For the squeezing in the fundamental field, we find that from numerical solutions of Eqs. (24), the amount of squeezing approaches $\text{sech}^2\zeta$ for large δ [Figs. 3(a) and 3(b)] and no frequency limit on squeezing. This is because we made the approximation $e^{-i\zeta\sigma^2}\simeq 1$ in obtaining Eqs. (24). To estimate the bandwidth of squeezing for the fundamental field, we need to keep the $e^{-i\zeta\sigma^2}$ in Eq. (22a). However, when the $e^{-i\zeta\sigma^2}$ term becomes significant, δ will be very large because δ comes from first order in the expansion. Thus we can neglect the first term on the right-hand side of Eq. (22a) because it has $1/\delta$ dependence for large δ after integration. Therefore, for $\Omega \sim \Omega_2/\zeta^{1/2}$, Eq. (22a) changes to

$$\frac{d}{d\zeta}\Delta\hat{a}_1(\Omega) = -e^{i2\varphi_0}\Delta\hat{a}_1^{\dagger}(-\Omega)e^{-i\zeta\sigma^2}\tanh\zeta \ , \tag{45}$$

where $\sigma\equiv\Omega/\Omega_2$ is the normalized frequency to another characteristic frequency $\Omega_2\equiv(c/2\Delta n_1 z_0)^{1/2} = \omega_0[4\pi(-\partial n_1/\partial\lambda)z_0]^{-1/2}$. Usually, $\Omega_2 \gg \Omega_1$. For the quadrature-phase amplitudes defined in Eqs. (23), we have

$$\frac{d\hat{x}_1}{d\zeta} = -(\hat{x}_1\cos\zeta\sigma^2 - \hat{y}_1\sin\zeta\sigma^2)\tanh\zeta \ ,$$
$$\frac{d\hat{y}_1}{d\zeta} = (\hat{y}_1\cos\zeta\sigma^2 + \hat{x}_1\sin\zeta\sigma^2)\tanh\zeta \ . \tag{46}$$

It is easy to check that $(d/d\zeta)[\hat{x}_1(\zeta,\Omega),\hat{y}_1(\zeta,\Omega')] = 0$, thus $[\hat{x}_1(\zeta,\Omega),\hat{y}_1(\zeta,\Omega')] = [\hat{x}_1(0,\Omega),\hat{y}_1(0,\Omega')] = i\delta(\Omega+\Omega')$ for any ζ. As in Eqs. (24) and (25), the solutions to Eq. (46) have the form

$$\hat{x}_1(\zeta) = g_x(\zeta,\sigma)\hat{x}_1(0) + g_y(\zeta,\sigma)\hat{y}_1(0) \ ,$$
$$\hat{y}_1(\zeta) = k_x(\zeta,\sigma)\hat{x}_1(0) + k_y(\zeta,\sigma)\hat{y}_1(0) \ , \tag{47}$$

with $g_x(0)=1$, $g_y(0)=0$ $k_x(0)=0$, $k_y(0)=1$. By solving the coefficients g,k numerically, we can calculate the spectrum of squeezing $S_1^x(\zeta,\Omega)$ $(=[g_x(\zeta,\sigma)]^2+[g_y(\zeta,\sigma)]^2)$. In Fig. 7, we plot $S_1^x(\zeta,\Omega)$ as a function of σ for fixed $\zeta=2$. As σ increases, $S_1^x(\zeta,\Omega)$ arises from nearly zero ($\text{sech}^2\zeta$) to above 1 and oscillates around 1, and eventually converges for large σ to the level of 1 for the vacuum fluctuation. Therefore from Fig. 7 we find that the bandwidth for squeezing for the fundamental field is such that $\sigma\sim0.5$, which corresponds to $\Omega=0.5\Omega_2=16$ THz and $\delta\simeq200$ with the ratio $\Omega_2/\Omega_1=400\gg1$ for the data given earlier on $KNbO_3$ crystal. This bandwidth is much wider than the one calculated earlier for the correlations and the harmonic field. The spectrum of squeezing $S_1^x(\zeta,\Omega)$ for the fundamental field at $\zeta=2.0$ is a combination of Fig. 3(b) for $\Omega\sim\Omega_1\ll\Omega_2$ and Fig. 7 for $\Omega\sim\Omega_2$.

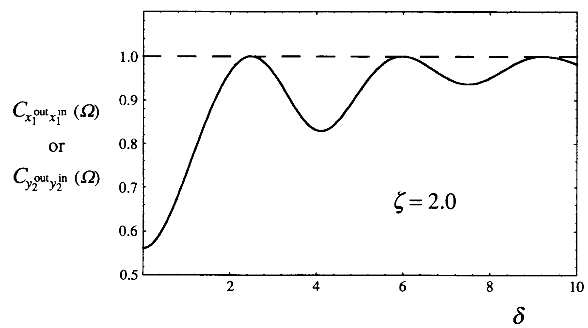

FIG. 6. Correlation functions between intensity fluctuations of the output and the input of the fundamental field or phase fluctuations of the output and the input of the harmonic field as a function of normalized frequency $\delta\equiv\Omega/\Omega_1$ at the distance $\zeta=2$.

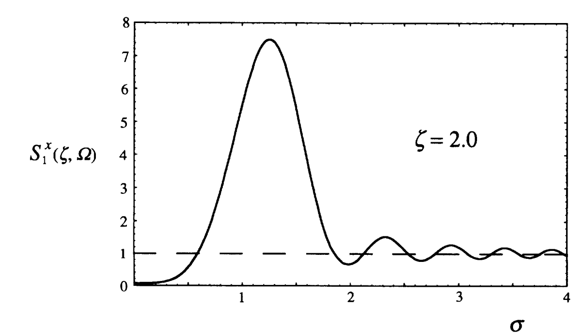

FIG. 7. Spectrum of squeezing $S_1^x(\zeta,\Omega)$ of the intensity fluctuation of the fundamental field for frequency Ω around $\Omega_2\equiv(c/z_0\Delta n_1)^{1/2}\gg\Omega_1\equiv c/z_0\Delta n$ at $\zeta=2$. $\sigma\equiv\Omega/\Omega_2$ is the dimensionless frequency normalized to Ω_2.

V. DISCUSSION AND SUMMARY

In this paper, we derived the input-output relations for the propagation of the quantum fluctuations of the fields in second-harmonic generation. From these relations, we obtained the propagation of the spectra of squeezing and found correlations in the quantum fluctuations of the amplitudes between the fundamental and harmonic fields. The bandwidths of the spectra of squeezing and the correlations are determined by the phase mismatch in the frequencies other than the center frequencies ω_0 and $2\omega_0$ though the first-order phase mismatch that is linear in frequency offset Ω will enhance the effect of squeezing for the fundamental field due to decoupling of the harmonic and fundamental fields. The bandwidth of squeezing for the fundamental field is limited by the phase mismatch in second order of the frequency offset Ω and is much larger than the bandwidth for correlations and squeezing in harmonic field.

In the linearization approximation, we assumed strong fundamental field at frequency ω_0 as compared to the quantum fluctuations and the fields at other frequencies. Therefore the propagation relations in Eqs. (26) with solutions of Eqs. (37) are valid for any kind of input field, quantum or classical. Of course, when input at fundamental frequency is small, linearization fails. Thus we should not expect to see the oscillation effect [7] found in harmonic intensity for small intensity of fundamental input where quantum fluctuations are large enough to influence the dynamics of the process. On the other hand, the existence of such effect requires large nonlinearity of the material as expressed by the coupling constant K. For second-harmonic generation in the nonlinear materials available so far, we are safe in making the linearization approximation.

Although we assumed that the mean input fundamental field is monochromatic [the δ function in Eqs. (13) and (14)], the treatment should be valid for quasimonochromatic input fields whose bandwidth is much narrower than Ω_1, Ω_2. On the other hand, it is easy to extend the treatment to include wide band input fundamental field. Finally, we should point out that the treatment given in this paper does not include any dissipative loss which will generally degrade the effect of squeezing as well as the quantum correlations.

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