Alternative integrable equations of nonlinear optics

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We present a list of nonlinear optics equations [the reduced Maxwell-Bloch system with pumping (or with a damping of a special type), nonlinear Schrödinger-Bloch equations with pumping, etc.] that can be solved by the inverse-scattering transform with a variable spectral parameter. We obtain the corresponding Lax pairs and equations for the spectral parameter.

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I. INTRODUCTION

Exactly solvable problems occupy a special place in mathematical and theoretical physics. Crystal clear in formulation and quite universal in application, they constitute the foundation for progress in theoretical physics. As a vivid example we can mention the soliton models applied to nonlinear optics, plasma physics, hydrodynamics, etc. Moreover, on the basis of an exactly solvable problem one can construct perturbation theory and evaluate the effect of small additional physical factors.

The development of the inverse-scattering transform (IST) and the theory of nonlinear optics equations has been proceeding hand in hand for the past 25 years, each fruitfully affecting the other. In fact, one of the first "solitonic" papers [1], published in 1967, was devoted to the integration of the sine-Gordon equation, applied to the modeling of an electromagnetic-pulse propagation through (and interaction with) an optical medium containing resonant two-level atoms. Since then, a number of physically important equations—the Maxwell-Bloch system (a complex generalization of the sine-Gordon equation) [2–5], the three-wave resonant interaction system [6,7], etc.—were incorporated into the IST, with a nonlinear Schrödinger equation [8] being so far the most promising from the view of potential telecommunication technology.

II. INVERSE SCATTERING WITH A VARIABLE SPECTRAL PARAMETER

The linear overdetermined system (Lax pair) associated with a soliton equation contains a spectral parameter λ, which must be constant according to conventional soliton methods. It was suggested in [9] that the spectral parameter be regarded as a function (of time, coordinate, and an additional complex constant, called the "hidden" spectral parameter) which satisfies an additional, overdetermined system of nonlinear differential equations. The proposed method was named IST with a variable spectral parameter (or the method of nonisospectral deformations). This technique makes it possible to construct many new integrable equations. For each soliton equation which is integrable within the framework of conventional IST one can produce a whole class of new equations, integrable by IST with a variable spectral parameter. The elements of this class were named deformations of the initial soliton equation [9] (i.e., deformed equations). To derive soliton solutions of the deformations, one can use the "dressing" technique developed initially for the case of a constant λ [10]. A procedure for generating the finite-gap solutions was developed in [26].

It should be noted the method suggested in [9] is a development of ideas presented in [12–14]. Furthermore, the series of soliton equations with variable spectral parameter were constructed in [15–17]. The symmetry approach was applied to this problem in [18]. The relationship between nonisospectral and isomonodromic deformations, and deformations of Painlevé equations, were studied in [19].

The application of the IST with a variable spectral parameter is not restricted to the area of classical nonlinear physics. It was also successfully (and independently) applied to solve a quantum nonlinear Schrödinger equation [31].

The most interesting are deformations of the following Maxwell-Bloch system:

$$
\begin{aligned}
&E_\eta=\rho\ ,\\
&N_\xi+(\rho\bar{E}+\bar{\rho}E)/2=0\ ,\\
&\rho_\xi+i\varepsilon\rho=NE\ ,\\
&\varepsilon_\eta=0\ ,\\
&x=\eta\ ,\quad t=\eta+\xi\ .
\end{aligned}
\tag{1}
$$

The additional real field $\varepsilon=\varepsilon(\xi)$ (which models the frequency shift from the resonance) is trivial because it can

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be eliminated from (1) by rescaling:

$$E \to E \exp(-i \int \varepsilon(\xi) d\xi) ,$$
$$\rho \to \rho \exp(-i \int \varepsilon(\xi) d\xi) .$$

The function ε will become nontrivial in certain deformations of (1), so we retain it. The Lax pair for (1) is

$$\Phi_\xi + U\Phi = 0 ,\quad \Phi_\eta + V\Phi = 0 , \tag{2}$$

where

$$U = i\lambda u_1 + u_0 ,\quad V = R/4i\lambda ,$$
$$u_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} ,\quad u_0 = \tfrac{1}{2}\begin{bmatrix} i\varepsilon & E \\ -\bar{E} & -i\varepsilon \end{bmatrix} ,\quad R = \begin{bmatrix} N & \rho \\ \bar{\rho} & -N \end{bmatrix} . \tag{3}$$

If we insert (3) into the compatibility condition for Eqs. (2), namely,

$$U_\eta - V_\xi + [U,V] = 0 , \tag{4}$$

and require that (4) be fulfilled identically in λ, then we get the Maxwell-Bloch system (1). According to [9], the main idea of the IST with a variable spectral parameter is as follows [in the limits of example (1)].

Let λ be a function of ξ, η and a "hidden" spectral parameter z (due to z dependence, functions λ^s; $s = 0, \pm 1, \ldots$, are linearly independent). This function is not completely arbitrary. It must obey a system of equations that is uniquely fixed by the requirement that Eq. (4) should be fulfilled identically in z and the corresponding nonlinear system is a determined one. The resulting equations for λ are [9]

$$\frac{\partial \lambda}{\partial \eta} = b + \frac{c}{\lambda} , \tag{5a}$$

$$\frac{\partial}{\partial \xi}\frac{1}{\lambda} = \tilde{b} + \frac{\tilde{c}}{\lambda} . \tag{5b}$$

Substituting (3) into (4) and taking into account (5a) and (5b), we get a class of deformations of the Maxwell-Bloch system:

$$\begin{aligned} &E_\eta - \rho = -i\tilde{b}\rho/2 , \\ &N_\xi + (\rho\bar{E} + \bar{\rho}E)/2 = -\tilde{c}N - 4c , \\ &\rho_\xi + i\varepsilon\rho - NE = -\tilde{c}\rho , \\ &\varepsilon_\eta = -\tilde{b}N/2 - 2b . \end{aligned} \tag{6}$$

First of all, we should analyze the overdetermined system of equations [(5a) and (5b)] for the spectral parameter λ. Its compatibility condition generates a nonlinear system [in addition to (6) and independent of it] for the unknown functions b, $\tilde{b}$, c, and $\tilde{c}$:

$$\begin{aligned} &\tilde{b}_\eta = 0 ,\quad c_\xi + 2\tilde{c}c = 0 , \\ &b_\xi + \tilde{c}b + 3c\tilde{b} = 0 ,\quad \tilde{c}_\eta + 2b\tilde{b} = 0 . \end{aligned} \tag{7}$$

If $\tilde{b} = \tilde{b}_0(\xi)$ is nonzero, then this function can be transformed to be a constant, say, 1, by a change of variables. Thus, the system (7) will become simpler, taking the form

$$\tilde{c}_\eta + 2b = 0 ,\quad c_\xi + 2\tilde{c}c = 0 ,\quad b_\xi + \tilde{c}b + 3c = 0 . \tag{8}$$

It was rigorously proved in [19] that the self-similar reduction of (8) is without the Painlevé property. The construction of a general solution of (7) and (8) is a problem which is still not solved. However, there are particular cases of (8) which can be fully solved.

(a) $c = 0$. If $b \neq 0$, then the substitution $b = e^{\varphi}$; $\tilde{c} = -\varphi_\xi$ in (8) leads to the Liouville equation

$$\varphi_{\xi\eta} = 2e^{\varphi} .$$

(b) If $b = 0 = c = \tilde{c}$, then we obtain

$$\begin{aligned} &E_\eta = (1 - i\tilde{b}_0/2)\rho , \\ &\varepsilon_\eta = -\tilde{b}_0 N/2 , \\ &N_\xi + (\rho\bar{E} + \bar{\rho}E)/2 = 0 , \\ &\rho_\xi + i\varepsilon\rho = NE . \end{aligned} \tag{9}$$

Here $\tilde{b}_0$ is a real constant. At the moment we cannot say anything definite about possible physical application of (9).

The case $\tilde{b} = 0 = b$ is the most interesting from the physical point of view. If, further, $\tilde{c} = 0$; $c \neq 0$, then we get the Maxwell-Bloch system with pumping:

$$\begin{aligned} &E_\eta = \rho , \\ &N_\xi + (\rho\bar{E} + \bar{\rho}E)/2 = -4c , \\ &\rho_\xi = NE . \end{aligned} \tag{10}$$

Here the parameter c defines the pumping of an optical medium. The Lax double pair for (10) consists of the linear system (2)–(4) for the function Φ plus a pair of nonlinear equations for the spectral parameter λ:

$$\lambda_\eta = c/\lambda ,\quad \lambda_\xi = 0 ,\quad \lambda(\eta, z) = \sqrt{2c\eta + z} . \tag{11}$$

We want to point out that calculation of two-point equal-time temperature correlators for the quantum Schrödinger equation was reduced in [31] to the solution of some classical nonlinear equation. The remarkable fact is that the equation almost coincides with the Maxwell-Bloch system with pumping: both systems demand the same linear system, but use different reductions.

The opposite case—$\tilde{c} \neq 0$, $c = 0$—leads to the Maxwell-Bloch system with damping:

$$\begin{aligned} &E_\eta = \rho , \\ &N_\xi + \tilde{c}N + (\rho\bar{E} + \bar{\rho}E)/2 = 0 , \\ &\rho_\xi + \tilde{c}\rho = NE , \end{aligned} \tag{12}$$

with another set of equations for the λ:

$$\lambda_\eta=0\ ,\quad \lambda_\xi=\tilde{c}\lambda\ ,\quad \lambda(\xi,z)=z\exp(\tilde{c}\xi)\ . \tag{13}$$

Formulas (10)–(13) were derived in [9], but only in [11] was the physical meaning of the constant c in (10) clearly interpreted, and the general theory of (1) derived.

To describe correctly a physical model associated with (10), we should first introduce the Maxwell-Bloch system written in a more convenient form:

$$\begin{aligned}&\partial E/\partial\eta=\rho\ ,\\&\partial N_1/\partial\xi-(\rho\bar{E}+\bar{\rho}E)/2=0\ ,\\&\partial N_2/\partial\xi+(\rho\bar{E}+\bar{\rho}E)/2=0\ ,\\&\partial\rho/\partial\xi=(N_2-N_1)E/2\ .\end{aligned} \tag{14}$$

Here the function $E=E(\xi,\eta)$ is a complex electromagnetic field envelope; N_1 and N_2 are populations of a lower and an upper level, respectively; ρ is a polarization of the medium.

It is important to emphasize that when formulating the theory of radiation in such a system, one must take into account the difference between "passive" media (i.e., self-induced transparency) and "active" media (i.e., pulse amplification and superfluorescence). For active media the applicability conditions are much more rigid. The two-level approach may be a good approximation, but in reality atoms are multilevel. In the course of the process of population inversion (by pumping), atoms are excited from the ground state (population N_1) not only to the upper "working" level (population N_2), but also (in fact, predominantly) to higher-energy nonworking levels. Then, as a result of atomic transitions (fast nonradiative ones in particular), atoms fall from higher-energy levels down to working levels. Finally, one cannot sometimes ignore various dissipative processes in both passive and in active media. The simplest way to incorporate these additional physical factors into a mathematical model is to replace the system (14) with a more general one:

$$\begin{aligned}&\frac{\partial E}{\partial\eta}+\sigma E=\rho\ ,\\&\frac{\partial N_1}{\partial\xi}+N_1/T_1-(\rho\bar{E}+\bar{\rho}E)/2=a_1\ ,\\&\frac{\partial N_2}{\partial\xi}+N_2/T_2+(\rho\bar{E}+\bar{\rho}E)/2=a_2\ ,\\&\frac{\partial\rho}{\partial\xi}+\rho/T=(N_2-N_1)E/2\ .\end{aligned} \tag{15}$$

Parameters a_1 and a_2 model constant pumping into lower and upper working levels, respectively. We assume that the pumping does not change the microscopic polarization ρ (corresponding to the working atomic transition $1\rightarrow2$). Define T as the relaxation time of polarization, which is determined, e.g., in gas lasers, by atomic collisions. We note by T_1 (T_2) the time constant for the decay from state 1 (2) to all other states. We neglect the decay from state 2 to state 1. Finally, σ is the conductivity (at the frequency of the atomic transition $1\rightarrow2$). The results (10)–(13) are the following.

With the exception of two particular cases, system (15) cannot be incorporated into the IST with a variable spectral parameter [9], namely, when the following occurs.

(1) Pumping is present, $a_{1,2}\neq0$, and there is no damping:

$$\sigma=0=T_1^{-1}=T_2^{-1}=T^{-1}\ .$$

A change of variables, $N=(N_2-N_1)/2$, $-4c=(a_2-a_1)/2$, transforms Eqs. (15) into (10).

(2) Pumping is absent ($a_{1,2}=0$), and the damping must be of a special type ($\sigma=0$, $T_1=T_2=T$).

System (12), unlike (10), is a trivial deformation of the Maxwell-Bloch system—it may be converted to the ordinary Maxwell-Bloch system (1) ($\varepsilon=0$) via a change of variables [9]. But it is nontrivial from a physical point of view. The constraint $T_1=T_2=T$ restricts the applicability of the solution (commonly $T_{1,2}\gg T$), but nevertheless the system is applicable to metallic vapors.

To construct special solutions for deformations, one can use the standard "dressing" technique [10]. This was clearly demonstrated in [12,14,20]. The author of [21] applied a Darboux transformation to construct solitons of the Maxwell-Bloch system with pumping. In this section we present, as an example, the formulas (omitting calculations) for the self-induced transparency soliton under the effect of damping (not small) (12):

$$\begin{aligned}&E(x,t)=4\beta e^{\tilde{c}(x-t)+iq}/\cosh(r)\ ,\\&\rho(x,t)=4\beta e^{\tilde{c}(x-t)}[\partial_t+\partial_x]e^{iq}/\cosh(r)\ ,\\&N(x,t)=N_0e^{-\tilde{c}t}+4\beta e^{\tilde{c}(x-t)}[\partial_t+\partial_x]e^{-r}/\cosh(r)\ .\end{aligned} \tag{16}$$

Here,

$$q=-2\alpha(1-e^{\tilde{c}(x-t)})/\tilde{c}+\frac{\alpha N_0}{2\tilde{c}(\alpha^2+\beta^2)}(1-e^{-\tilde{c}x})\ ,$$

$$r=2\beta(1-e^{\tilde{c}(x-t)})/\tilde{c}+\frac{\beta N_0}{2\tilde{c}(\alpha^2+\beta^2)}(1-e^{-\tilde{c}x})\ .$$

Formulas (16) are written in the laboratory coordinate system and are parametrized by three constants: N_0, α, and β. In the limit $\tilde{c}\rightarrow0$, (16) will transform into the well-known expression describing undamped soliton. Soliton (16) does not look like any conventional soliton. In the course of propagation its shape, velocity, and amplitude change due to damping. Moreover, generally, this is the case for any soliton in any equation with a variable spectral parameter.

Finally, to analyze the structure of (16) we fix the coordinate system moving with the velocity of light: $x-t=\underline{x}$. In the limit $t\rightarrow\infty$, we get

$$E=4\beta e^{\tilde{c}\underline{x}+i\underline{q}}/\cosh(\underline{r})\ ,\quad N=0=\rho \tag{17}$$

where

$$\underline{q}=-2\alpha(1-e^{\tilde{c}\underline{x}})/\tilde{c}+\frac{\alpha N_0}{2\tilde{c}(\alpha^2+\beta^2)}\ ,$$

$$\underline{r}=2\beta(1-e^{\tilde{c}\underline{x}})/\tilde{c}+\frac{\beta N_0}{2\tilde{c}(\alpha^2+\beta^2)}\ .$$

Formulas (17) describe a soliton of an asymmetric shape, which moves with the velocity of light in transparent media. The soliton is alive, despite the damping.

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APPENDIX

We present in this appendix a list of nonlinear optics equations [(A1), (A3), (A5), (A9), (A10), (A11), (A16)–(A18); in addition the systems (10) and (12)] for which we managed to find the Lax double pair (with a variable spectral parameter). We hope the list may be expanded further.

In reality, unless special measures are taken, atomic transitions are usually degenerate. The authors of [22] put forward the Maxwell-Bloch system that incorporates this additional effect and also presented the Lax pair (with a constant spectral parameter). The system, like the original Maxwell-Bloch system, can be pumped:

$$\partial e_\pm/\partial\eta = v_\pm ,\qquad \partial R/\partial\xi + [u_0, R] = -4cu_1 , \tag{A1}$$

where

$$u_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \alpha I ,\qquad u_0 = \frac{1}{2}\begin{bmatrix} 0 & e_- & e_+ \\ -\bar{e}_- & 0 & 0 \\ -\bar{e}_+ & 0 & 0 \end{bmatrix} ,\qquad R = \begin{bmatrix} N & v_- & v_+ \\ \bar{v}_- & n_- & \mu \\ \bar{v}_+ & \bar{\mu} & n_+ \end{bmatrix} , \tag{A2}$$

where α=const, and I is an identity 3×3 matrix. The quantity cu_1 in (A1) describes the process of constant pumping (regarding the physical meaning of all other variables, see [22]). The Lax double pair for (A1) is defined by formulas (2), (3), and (11) with matrices $u_{0,1}, R$ fixed by (A2). Note that in this appendix we assume that, for each Maxwell-Bloch systems, $\varepsilon = 0$.

Propagation of ultrafast electromagnetic pulses in optical media with multilevel resonant atoms is analyzed in [23]. The authors of [23] presented the Lax pair for the system, generalizing the Maxwell-Bloch system (1) for the N-level case (provided certain restrictions on physical parameters hold true). Here we produce the appropriate system with pumping in the simplest, $N=3$ case:

$$\partial E_1/\partial\eta = \rho_{12} ,\qquad \partial E_2/\partial\eta = \kappa^2\rho_{23} ,\qquad \partial E_3/\partial\eta = (1+\kappa^2)\rho_{13} ,\qquad \partial R/\partial\xi + [u_0, R] = -4cu_1 . \tag{A3}$$

Here, κ=const:

$$u_1 = \begin{bmatrix} 1+\kappa^2 & 0 & 0 \\ 0 & -1+\kappa^2 & 0 \\ 0 & 0 & -1-\kappa^2 \end{bmatrix} + \alpha I ,\qquad u_0 = \frac{1}{2}\begin{bmatrix} 0 & E_1 & E_3 \\ -\bar{E}_1 & 0 & E_2 \\ -\bar{E}_3 & -\bar{E}_2 & 0 \end{bmatrix} ,\qquad R = \begin{bmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ \bar{\rho}_{12} & \rho_{22} & \rho_{23} \\ \bar{\rho}_{13} & \bar{\rho}_{23} & \rho_{33} \end{bmatrix} . \tag{A4}$$

Here, as before, α=const and I is a 3×3 identity matrix. The Lax double pair for (A3) is fixed by (2), (3), (11), and (A4).

In addition to Eqs. (A1) and (A3), one can generate systems with damping of the type (12) and also deformations like (9).

In [24] a nonlinear Schrödinger-Bloch system was put forward (with the Lax pair) which took into account quadratic nonlinearity of the medium (Kerr type), in addition to the resonant interaction. We present the nonlinear Schrödinger-Bloch equations with pumping:

$$iE_\eta + E_{\xi\xi}/2 + |E|^2E/4 = i\rho ,\qquad N_\xi + (\rho\bar{E} + \bar{\rho}E)/2 = -4c ,\qquad \rho_\xi = NE , \tag{A5}$$

which require the Lax double pair (2) and (11) with the following matrix functions U and V:

$$U = i\lambda u_1 + u_0 ,\qquad V = R/4i\lambda + Q + \lambda u_0 + i\lambda^2 u_1 ,$$
$$u_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} ,\quad u_0 = \frac{1}{2}\begin{bmatrix} 0 & E \\ -\bar{E} & 0 \end{bmatrix} ,\quad R = \begin{bmatrix} N & \rho \\ \bar{\rho} & -N \end{bmatrix} ,\quad Q = 1/4i \begin{bmatrix} |E|^2/2 & -E_\xi \\ -\bar{E}_\xi & -|E|^2/2 \end{bmatrix} . \tag{A6}$$

In [25] Zabolotskii generalized the nonlinear Schrödinger-Bloch system for the case of multilevel optical media. We construct the Zabolotskii system with pumping, and the Lax double pair:

$$i\Phi_t + U\Phi = 0 ,\qquad i\Phi_x + V\Phi = 0 , \tag{A7}$$

$$\lambda_x = c/\lambda ,\quad \lambda_t = 0 ,\qquad U = \lambda u_1 + u_0 ,\qquad V = \lambda v_1 + v_0 - R/\lambda . \tag{A8}$$

Here u_1 and v_1 are real diagonal $N\times N$ matrices:

$$u_1 = \mathrm{diag} b_i ,$$
$$v_1 = \mathrm{diag} a_i ,$$

$a_1 > a_2 > \ldots > a_N$, $u_0 = [u_1, Q]$, $v_0 = [v_1, Q]$. The matrix function $Q = w_{ij}/\sqrt{a_i - a_j}$ is anti-Hermitian; the density matrix $R = \rho_{ij}$. The compatibility condition for (A7) [taking into account (A8)] reads as follows:

$$(\partial_t + v_{ij}\partial_x) w_{ij} = -i\kappa\alpha_{ijk} w_{ik} w_{kj} + i g_{ij}\rho_{ij} \ , \quad i<j \tag{A9}$$

$$R_t - i[u_0, R] = -4cu_1 \ . \tag{A10}$$

Here

$$g_{ij} = -(b_i - b_j)/\sqrt{a_i - a_j} \ ,$$
$$v_{ij} = -(b_i - b_j)/(a_i - a_j) \ ,$$

and α_{ijk} is an antisymmetric tensor,

$$\kappa = \frac{a_1 b_3 - a_3 b_1 + a_3 b_2 - a_2 b_3 + a_2 b_1 - a_1 b_2}{\sqrt{(a_1 - a_2)(a_1 - a_3)(a_2 - a_3)}} \ .$$

Finally, we rewrite system (A9) for the simplest nontrivial case $N=3$:

$$(\partial_t + v_1\partial_x)\psi_1 = i\kappa\bar{\psi}_2\psi_3 + i g_{12}\rho_{12} \ ,$$
$$(\partial_t + v_2\partial_x)\psi_2 = i\kappa\bar{\psi}_1\psi_3 + i g_{23}\rho_{23} \ ,$$
$$(\partial_t + v_3\partial_x)\psi_3 = i\kappa\psi_1\psi_2 + i g_{13}\rho_{13} \ ,$$

where $\psi_1 = w_{12}$, $\psi_2 = w_{23}$, $\psi_3 = w_{13}$ and $v_1 = v_{12}$, $v_2 = v_{23}$, $v_3 = v_{13}$.

Note that one can "pump" (but not "damp") the Maxwell-Bloch system in the presence of an inhomogeneous broadening effect:

$$\begin{aligned} &E_\eta = \langle \rho \rangle \ , \\ &N_\xi + (\rho\bar{E} + \bar{\rho}E)/2 = -4c \ , \\ &\rho_\xi + 2is\rho = NE \ , \\ &\langle \rho \rangle = \int_{-\infty}^{+\infty} g(s)\rho(\xi,\eta,s)\,ds \ , \\ &\int_{-\infty}^{+\infty} g(s)\,ds = 1 \ . \end{aligned} \tag{A11}$$

Here the function $g(s)$ models the shape of the spectral line and s is a frequency shift from resonance. The Lax double pair for Eqs. (A11) is just like (2) and (3) with the matrix function V:

$$V = \int_{-\infty}^{+\infty} ds\, g(s) R / 4i(s - \lambda) \tag{A12}$$

and the spectral parameter obeying the equations

$$\lambda_\eta = \int_{-\infty}^{+\infty} ds\, g(s) c / (-s + \lambda) \ , \quad \lambda_\xi = 0 \ . \tag{A13}$$

Here, c may be an arbitrary function of the two variables η and s (but not of time ξ).

In all the above-mentioned nonlinear equations, we work with a complex electromagnetic-pulse envelope E. It was discovered in [27] that we can remove this restriction when dealing with a resonant interaction. We can work with the original electric field $\mathcal{E}$ and still retain the integrability of the model equations:

$$\begin{aligned} &\mathcal{E}_\eta = \alpha \langle\langle \omega v \rangle\rangle \ , \\ &n_\xi + v\mathcal{E} = 0 \ , \\ &u_\xi + \omega v = 0 \ , \\ &v_\xi - \omega u = n\mathcal{E} \ , \\ &\langle\langle \omega v \rangle\rangle = \int_0^{+\infty} f(\omega)\omega v(\eta,\xi,\omega)\,d\omega \ . \end{aligned} \tag{A14}$$

The system (A14) is called the "reduced" Maxwell-Bloch system.

Between "ordinary" (1) and reduced (A14) Maxwell-Bloch systems, there are not one, but two links. The first link is apparent: if one introduces a complex field envelope approximation into the reduced system, then one gets the ordinary system. The second link [27] is not as apparent, but is much more straightforward. The change of variables

$$\begin{aligned} &2s = \omega \ , \quad E = -i\mathcal{E} \ , \quad \rho = u - iv \ , \quad N = n \ , \\ &g = \alpha\omega[f(\omega) - f(-\omega)] \end{aligned} \tag{A15}$$

transforms the ordinary system into the reduced one. The reduced system can also be pumped:

$$\begin{aligned} &\mathcal{E}_\eta = \alpha \langle\langle \omega v \rangle\rangle \ , \\ &n_\xi + v\mathcal{E} = -4c \ , \\ &u_\xi + \omega v = 0 \ , \\ &v_\xi - \omega u = n\mathcal{E} \ , \end{aligned} \tag{A16}$$

and damped [for the resonant case, $f = \delta(\omega - \omega_0)$]. Via the change of variables (A15) the double pair (2–3, A12, A13) yields the Lax representation for the reduced Maxwell-Bloch system with pumping.

Finally, we derive deformations of equations [28–30] that describe both stimulated Raman scattering and resonant two-photon absorption and emission. There are two different nontrivial deformations: type I,

$$\begin{aligned} &\partial_\xi r_3 = i(\bar{s}r - s\bar{r})/2 - \tilde{c}r_3 \ , \\ &\partial_\xi r = isr_3 + igs_3 r - \tilde{c}r \ , \\ &\partial_\eta s_3 = i(\bar{r}s - r\bar{s})/2 + \epsilon\tilde{a}r_3/4 \ , \\ &\partial_\eta s = igr_3 s + i\epsilon r s_3 + \tilde{a}(if - g/2)\epsilon r/2 \ , \\ &\tilde{c} = \tilde{a}/4 \ , \end{aligned} \tag{A17}$$

and type II,

$$\begin{aligned} &\partial_\xi r_3 = i(\bar{s}r - s\bar{r})/2 + 4cs_3 \ , \\ &\partial_\xi r = isr_3 + igs_3 r - 8c(if + g/2)s \ , \\ &\partial_\eta s_3 = i(\bar{r}s - r\bar{s})/2 - as_3 \ , \\ &\partial_\eta s = igr_3 s + i\epsilon r s_3 - as \ , \\ &c = \frac{if}{\epsilon(if + g/2) - g/2}\,\frac{a}{4} \ . \end{aligned} \tag{A18}$$

For both (A17) and (A18), $f^2=(\epsilon-g^2)/4$. We should assume $\epsilon=1$ for stimulated Raman scattering and $\epsilon=-1$ for two-photon processes; a and $\tilde{a}$ are arbitrary real constants.

Equations (A17) and (A18) demand the following matrix functions U, V in the Lax pair (2):

$$U=i\lambda u_1+u_0 ,$$

$$V=v_0+\frac{v_1}{\lambda+g/2} ,$$

$$u_1=\begin{bmatrix} is_3 & -s \\ \epsilon\bar{s} & -is_3 \end{bmatrix} , \quad u_0=-if\begin{bmatrix} 0 & s \\ \epsilon\bar{s} & 0 \end{bmatrix} ,$$

$$v_1=-\tfrac{1}{2}\begin{bmatrix} -i\epsilon r_3/2 & (if-g/2)\epsilon r \\ (if+g/2)\bar{r} & i\epsilon r_3/2 \end{bmatrix} ,$$

$$v_0=-\tfrac{1}{2}\begin{bmatrix} igr_3 & \epsilon r \\ -\bar{r} & -igr_3 \end{bmatrix} .$$

The spectral parameter λ is a solution of the system

$$\lambda_\eta=a\lambda-2gc+\frac{c}{\lambda+g/2} , \quad \text{(A19)}$$

$$\lambda_\xi=0$$

in the case of (A18) and of the system

$$\lambda_\eta=0 , \quad \text{(A20)}$$

$$\partial_\xi\frac{1}{\lambda+g/2}=\tilde{a}\lambda-2\epsilon g\tilde{c}+\frac{\tilde{c}}{\lambda+g/2}$$

for the (A17) case.

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