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Inequivalence between the Schrödinger equation and the Madelung hydrodynamic equations

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By differentiating the Schrödinger equation and separating the real and imaginary parts, one obtains the Madelung hydrodynamic equations, which have inspired numerous classical interpretations of quantum mechanics. Such interpretations frequently assume that these equations are equivalent to the Schrödinger equation, and thus provide an alternative basis for quantum mechanics. This paper proves that this is incorrect: to recover the Schrödinger equation, one must add by hand a quantization condition, as in the old quantum theory. The implications for various alternative interpretations of quantum mechanics are discussed.

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I. INTRODUCTION

Classical interpretations of quantum mechanics are as old as quantum mechanics itself. In 1926, Erwin Madelung showed that if one writes the wave function in the form e^{R+iS}, the Schrödinger equation implies that S is governed by a classical Hamilton-Jacobi-like equation, or alternatively that $v = \nabla S$ is governed by a Newton-like equation [1]. The only formal difference between these equations and their purely classical counterparts is the existence of an additional "quantum" potential. One also easily derives a continuity equation for v and ρ.

Since that time these equations have provided the basis for numerous classical interpretations of quantum mechanics, including the hydrodynamic interpretation first proposed by Madelung [1–10], the theory of stochastic mechanics due to Nelson and others [11–34], the hidden-variable and double-solution theories of Bohm and de Broglie, respectively [35–38], and quite possibly other interpretations as well [39,40].

In some of these theories, such as the hydrodynamic interpretation and stochastic mechanics, the Madelung equations are taken as fundamental, and the Schrödinger equation is viewed as a mathematical consequence. This is based on the belief that the two equations are mathematically equivalent, a claim which goes back to Madelung himself [1].

The purpose of this paper is to demonstrate the following: the Madelung equations are not equivalent to the Schrödinger equation unless a quantization condition is imposed. This condition is that the wave function be single valued; translated into the quantities of the Madelung equation, this means that

$$\oint_L v \cdot dl = 2\pi j, \tag{1.1}$$

where j is an integer and L is any closed loop. To the best of my knowledge, this condition has not yet found any convincing explanation outside the context of the Schrödinger equation. And yet, without it, alternative interpretations are incapable of explaining perhaps the central mystery of quantum mechanics, the emergence of the quantum.

This inequivalence has undoubtedly been noticed many times. In the stochastic mechanics literature, however, the necessity of a quantization condition appears to have been completely unknown until it was recently observed by the present author [41]. It turns out, however, that in 1952, the very year that Fényes introduced stochastic mechanics [18], this same observation had been made by Takabayashi, working in the hydrodynamic interpretation. Takabayashi, moreover, understood its physical implications perfectly: "We are led to a new postulate...which is so to speak the 'quantum condition' for fluidal motion and of *ad hoc* and compromising character for our formulation, just as the quantum condition for old quantum theory" [7, p. 155].

This observation has not previously been discussed in any detail, and in my experience many of those accustomed to regarding the Madelung and Schrödinger equations as equivalent tend to suspect, initially, that something crucial must have been overlooked. It seems worthwhile, therefore, to provide a thorough discussion. After developing the basic result in the context of the Newton-Madelung equation, I show how explicit solutions to the Madelung equations not satisfying the Schrödinger equation may be constructed (Sec. III), and discuss the nature of the problem for the Hamilton-Jacobi-Madelung equation (Sec. IV) and the gauge invariance of the results (Sec. V). I conclude with a discussion (Sec. VI).

II. THE MADELUNG EQUATIONS AND THE SCHRÖDINGER EQUATION

We begin by showing how the Madelung equations are obtained from the Schrödinger equation. Then we review the usual argument for recovering the Schrödinger equation, and indicate where it breaks down.

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A. The Madelung equations

We write the Schrödinger equation in units where $\hbar = m = 1$:

$$i\frac{\partial\psi}{\partial t} = \left(-\tfrac{1}{2}\Delta + V\right)\psi, \tag{2.1}$$

Now write the wave function in so-called Madelung form (the "de Broglie ansatz"), $\psi = \exp(R + iS)$, insert into the Schrödinger equation, divide by ψ, and separate into real and imaginary parts. This yields two coupled nonlinear partial differential equations, valid wherever $\psi \neq 0$. The first is a differentiated form of the continuity equation. The usual form,

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot(v\rho), \qquad (v = \nabla S), \tag{2.2}$$

can be derived directly from the Schrödinger equation without differentiation, and we use that instead. The more interesting equation is the second,

$$\frac{\partial S}{\partial t} = -V - U - \tfrac{1}{2}(\nabla S)^2, \tag{2.3}$$

where U is the so-called "quantum-mechanical potential"

$$U(\rho) = -\frac{\Delta(\rho^{1/2})}{2\rho^{1/2}}\ . \tag{2.4}$$

Equation (2.3) is sometimes called the Hamilton-Jacobi-Madelung (HJM) equation. Often, another differentiation is performed; this leads to the so-called Newton-Madelung (NM) or stochastic Newton equation, expressed in terms of $v = \nabla S$:

$$\frac{\partial v}{\partial t} = -\nabla V - (v\cdot\nabla)v - \nabla U(\rho). \tag{2.5}$$

If we write this in terms of the hydrodynamic derivative, $D = \partial/\partial t + v\cdot\nabla$, the NM equation reads $Dv = -\nabla(V + U)$. The similarity to Newton's law is striking. The question is whether either the NM or HJM equation, coupled to the continuity equation, is equivalent to the Schrödinger equation. When considering the coupled system, we speak of the NM or HJM equations (plural).

B. Recovering the Schrödinger equation from the Madelung equations

To recover the Schrödinger equation from Eqs. (2.5,2.2), it is necessary first of all to assume that v is equal to a gradient, since this is the case for all v derived from a wave function. Substitute $v = \nabla S$, and integrate (2.5). The integration constant can be set equal to zero, because it will only contribute a global phase to the wave function. We thus obtain (2.3). Now add (2.3) to i times the gradient of (2.2), multiply by $\psi = e^{R+iS}$, and formally identify $\nabla\psi$ with $(\nabla R + i\nabla S)\psi$, and similarly for other terms. The result is the Schrödinger equation.

The difficulty with this derivation has to do with our assumption that v is the gradient of a function S, which was stated somewhat imprecisely. In general, we must allow S to be a many-valued function, so v is only locally a gradient. S is many valued, for example, in wave functions with angular momentum, which typically contain a factor like $e^{im\varphi}$, where m is an integer, φ is an (azimuthal) angle, and $S = m\varphi$. [We can make S single valued only at the price of making it discontinuous at some point. But then $\nabla\psi = (\nabla R + i\nabla S)\psi$ would develop a singularity.] Once we allow S to be many valued, however, there is nothing in the Madelung equations to constrain $\psi = e^{R+iS}$ to be single valued. For a generic solution to these equations, it will not be, and the connection to the Schrödinger equation breaks down.

From a more abstract perspective, note that S is undefined wherever $\psi = 0$. If, once the nodal surface is removed, the space on which ψ is defined is no longer simply connected, the fact that v is locally a gradient no longer implies that v is globally a gradient, i.e., that v can be expressed as the gradient of a globally defined single-valued function. S will therefore be many valued in general, and so will $\psi = e^{R+iS}$.

In order for the wave function to be single valued, the different values of S must differ by integral multiples of 2π. In terms of v, this condition is $\oint_L v\cdot dl = 2\pi j$, where j is an integer and L is any closed loop. It appears that this condition re-establishes the formal equivalence of the Madelung equations with the Schrödinger equation.

III. A WORKED EXAMPLE

The question now arises as to whether the nonquantized solutions of the Madelung equations can be excluded in any natural way. One possibility is that nonquantized solutions are simply pathological, from a mathematical point of view.

This turns out not to be the case. Consider the solution of the Schrödinger equation for a particle in a well-behaved two-dimensional central potential $V(r)$. $V(r)$ could, for example, be the harmonic oscillator potential $\frac{1}{2}kr^2$. This problem is solved by separation of variables; we substitute for $\psi_a(r,\varphi)$ the ansatz $R_a(r)e^{ia\varphi}$, where (r,φ) are polar coordinates. The Schrödinger equation then implies that $R_a(r)$ is the solution of the radial equation:

$$\left[-\frac{1}{2}\frac{d^2}{dr^2} + \frac{a^2}{2r^2} + V(r)\right]R_a(r) = E_a R_a(r). \tag{3.1}$$

If we insist that ψ be single valued, we have that $\psi_a(r,\phi) = \psi_a(r,\phi + 2\pi j)$ (j an integer), and this implies that a must be an integer.

Note, however, that in the context of the Madelung equations, there is no requirement that ψ be single valued. And indeed, all of the solutions $\rho_a = |R_a(r)|^2$ and $v_a = (a/r)\hat{\varphi}$ satisfy the Madelung equations for the potential V, regardless of whether a is an integer. They are certainly local solutions of the Schrödinger equation with potential V, and the derivation of the Madelung equations from the Schrödinger equation is local.

The solutions to the NM equations contain the solutions to the Schrödinger equation as a proper subset, and smoothly interpolate between them. Clearly (ρ_a, v_a) only corresponds to a single-valued solution of the Schrödinger equation when a is an integer. The angular momentum is a, which takes on a continuum of values, and the energy can also be shown to assume a continuum of values.

IV. THE HAMILTON-JACOBI-MADELUNG EQUATION

We now discuss formalisms which are based on the Hamilton-Jacobi-Madelung equations. In general, any solution to the NM equations will also satisfy the HJM equations, because there is nothing in the HJM equations to enforce a single-valued wave function. Occasionally, however, such a formalism is developed under the assumption that S is single valued, and the formalism winds up actually excluding the possibility of a many-valued phase. Then ψ is indeed single valued, but all wave functions which require many-valued phases, such as those possessing angular momentum, are excluded [22,25]. Again, the elegant quantization properties of the Schrödinger equation have broken down.

For example, in the Guerra-Morato variational approach to stochastic mechanics [22], a quantity J is defined as a conditional expectation on the manifold of the stochastic process. A variational principle is then proposed, with the intention that the physical states will be critical states of the variational problem. It is then proven that a diffusion is critical if and only if ρe^{iJ} satisfies the Schrödinger equation. Since J is single valued, however, this also proves that wave functions with many-valued phases S are not critical; if they were, we would have $S = J$, where J is single valued. For an in-depth analysis of the Guerra-Morato theory, including worked examples where diffusions with angular momentum are proven to be not critical, see [41]. This problem is worse than having too many solutions, because there is no way to recover the correspondence with the Schrödinger equation, even with *ad hoc* assumptions.

V. ISSUES RELATED TO GAUGE INVARIANCE

We re-examine our results when a vector potential is added to the Schrödinger equation [32]. We then have

$$i\frac{\partial\psi}{\partial t} = \left[-\frac{1}{2}\left(-i\nabla - A\right)^2 + V\right]\psi, \tag{5.1}$$

where we have absorbed e and c into the potential terms. Again taking $\psi = e^{R+iS}$, we find that the continuity equation has the usual form if we take $v = \nabla S - A$, which is gauge invariant. Adopting this definition, we can derive the Madelung equation for v in the presence of a magnetic field, by taking derivatives as before:

$$\frac{\partial v}{\partial t} = E + v \times B - v\cdot\nabla v - \nabla U, \tag{5.2}$$

where U is again the quantum potential, $B = \nabla\times A$ is the magnetic field, and $E = -\partial A/\partial t - \nabla V$ as usual. Note that the force $-\nabla V$ has been replaced by the Lorentz force, and that the vector potential does not occur directly in the expression.

The inequivalence between the Schrödinger equation and the Madelung equations persists in this setting. In any well-defined physical problem, $\alpha = \oint_L A\cdot dl$ is fixed, since it is equal to the magnetic flux through L. Therefore, if v corresponds to a single-valued solution of the Schrödinger equation, then

$$\oint_L v\cdot dl = -\alpha + 2\pi j, \tag{5.3}$$

where we have replaced $\oint_L \nabla S$ by $2\pi j$ (j an integer). Even though $\oint_L v\cdot dl$ can now achieve nonintegral values, it is still quantized, and this still cannot be explained by the Madelung equation. There will be solutions to the Madelung equation which lead to nonintegral j. The value of α is given by the physics. It cannot be adjusted for each v, to make j an integer or even zero [21,42], even in the context of a multiply connected space.

We have assumed throughout that ψ is single valued. It is possible, in the context of multiply connected spaces, to use many-valued wave functions in a carefully controlled way. It is not necessary, however, so we lose nothing in generality by assuming ψ is single valued [43].

VI. DISCUSSION

The problems described in this paper only arise in dimensions two or greater, since it is only in 2 or more dimensions that the removal of the nodal set can lead to a nontrivial topology. (They can arise in one dimension if the topology of the space is nontrivial.) Nevertheless, space is three-dimensional (and configuration space is $3n$ dimensional), so we expect these problems to arise quite generally.

It is sometimes observed that if $\oint v\cdot dl$ is quantized initially, this condition will be maintained by the time evolution. This is analogous to Kelvin's law in hydrodynamics, and is easily proven in the present context as well. Is it really reasonable to suppose, however, that quantization is entirely the result of initial conditions? Why do these conditions just happen to correspond to a single-valued "wave function"? Would these conditions be preserved in interactions? Would they be stable?

The Madelung equations seem classical, but they are inherently nonlocal. The "quantum-mechanical potential" U exists in configuration space, which gives it a very different character than a typical physical potential. It is this feature that gives rise to nonlocality, so that the change in the density of one particle can affect U, and hence the motion of another correlated particle, no matter how distant.

Many theorists who study hidden variables, however, accept nonlocality as a fact of nature that is no more an indictment of hidden variables than it is of conventional quantum mechanics. The present work shows that, even if one is willing to accept nonlocality, theories based on the Madelung equations simply do not reproduce the Schrödinger equation; they fail on technical grounds, regardless of how one feels about the esthetics.

The significance of this result for the formalisms considered in the introduction depends on the formalism, as discused below.

A. Stochastic mechanics

The theory of stochastic mechanics was first discovered by Fényes in 1952 [18], and was rediscovered in a somewhat different form by Nelson in 1966 [30]. (We should note, incidentally, that stochastic mechanics is distinct from stochastic quantization [44].) In stochastic mechanics, it is assumed that the quantum particle possesses an actual trajectory, and that this trajectory is described by a stochastic differential equation, consisting of a drift term and a diffusion term:

$$d\xi(t) = b(\xi(t), t)dt + dw(t), \tag{6.1}$$

where $\xi(t)$ is the random variable describing the particle's trajectory, $b(x,t)$ is the drift, and $w(t)$ is a Wiener process with covariance given by $dw(t)dw(t) = \hbar/m\, dt$. Nelson postulates a plausible definition for the acceleration of this diffusion process, and invokes Newton's law. This yields the Newton-Madelung equation, where the quantum potential U arises naturally from the definition of the acceleration. (For a more thorough discussion, see [32].) Many similar derivations of either the NM or HJM equations have been given [11–17,22,23,25–27,33,34]. Similar derivations corresponding to other equations, such as the Klein-Gordon [45] and Pauli [46,47] equations, have also been provided; these have the same difficulties.

In the context of stochastic mechanics, it is very difficult to see how the circulation of the current velocity might be quantized in a natural way. The wave equation, for example, is understood as a technique for linearizing the more fundamental equations of the stochastic theory, and the wave function is seen as a mathematical artifact. Because of this, however, the constraint that $\oint v\cdot dl$ be 2π times an integer looks totally *ad hoc*. There seems to be nothing within the particle-oriented world of stochastic mechanics which can lead to what is effectively a condition on the "wave function". For other critiques of stochastic mechanics, see [8,41,48–52].

B. The hydrodynamic interpretation

The hydrodynamic interpretation was pioneered by Madelung. Madelung does state that his hydrodynamical equations are equivalent to the Schrödinger equation: "Die hydrodynamischen Gleichungen sind also gleichwertig mit denen von Schrödinger" [1]. He concludes that the *quantization* problem has thus found its solution in hydrodynamics, although as we have just seen, the Madelung equations do not lead to quantization.

Interest in the hydrodynamic interpretation waned for many years, but was reawakened by Takabayashi in 1952 [7], who was in turn stimulated by Bohm's then recent work on hidden variables. It has continued to be studied since that time [2–6,9,10,53,54].

As noted in the introduction, already in 1952 Takabayashi realized the necessity for the quantization condition, and understood clearly its implications for the hydrodynamical interpretation. This problem appears to have been largely or completely forgotten in the hydrodynamic literature, however. While the quantization condition is usually mentioned as an auxiliary condition, its significance is not emphasized, and I have been unable to find any real attempts to justify it in terms of a classical model. The uses of the hydrodynamical model range from the foundational to the very practical. The issues raised in this paper are of importance primarily to those seeking to provide a hydrodynamical foundation for quantum mechanics.

C. Other theories

Finally, there is the hidden-variable theory of Bohm [35,36] and the pilot wave theory of de Broglie [37,38]. Both of these theories postulate the existence of an underlying particle motion obeying the Newton-Madelung equation. Furthermore, the probability density of the particles corresponds to the density of the wave function. These theories are different from those considered above, however, in that they also assume the existence of a "ψ field" which evolves in accordance with the Schrödinger equation. Significantly, the ψ field is assumed to be what determines the value of the quantum potential. The evolution of the particle density does not influence the dynamics of the problem. Indeed, the dynamical evolution of the system is given by the Schrödinger equation, not the Madelung equations, and the Madelung equations are used only to provide an interpretation of the density evolution in terms of particle motion. As a result, the results of this paper do not seem to apply. We note, however, that these theories assume the Schrödinger equation as a physical principle, rather than seeking to derive it or explain it in classical terms.

These results do not apply to the current algebra formalism [55,56]. In this setting, one does arrive at what are essentially hydrodynamic equations governing ρ and j, where j is the current density. ρ and j are operators, however, and act on the Hilbert space of single-valued wave functions. Here, the introduction of a single-valued wave function enforces quantization, and what we really have is a rewriting of conventional quantum mechanics.

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[1] E. Madelung, Z. Phys. **40**, 322 (1926).
[2] S. K. Ghosh and B. M. Deb, Phys. Rep. **92**, 1 (1982).
[3] L. Jánossy, Found. Phys. **3**, 185 (1973).
[4] L. Jánossy and M. Ziegler, Acta Phys. Hung. **16**, 37 (1963).
[5] K.-K. Kan and J. J. Griffin, Phys. Rev. C **15**, 1126 (1977).
[6] S. Sonego, Found. Phys. **21**, 1135 (1991).
[7] T. Takabayasi, Prog. Theor. Phys. **8**, 143 (1952).
[8] T. Takabayasi, Prog. Theor. Phys. **9**, 187 (1953).
[9] H. E. Wilhelm, Phys. Rev. D **1**, 2278 (1970).
[10] C. Y. Wong, J. Math. Phys. **17**, 1008 (1976).
[11] E. Aldrovandi, D. Dohrn, and F. Guerra, Acta. Appl. Math. **26**, 219 (1992).
[12] M. Baublitz, Jr., Prog. Theor. Phys. **80**, 232 (1988).
[13] P. Blanchard, P. Combes, and W. Zheng, in *Mathematical and Physical Aspects of Stochastic Mechanics*, Lecture Notes in Physics Vol. 281 (Springer-Verlag, Berlin, 1987).
[14] M. Davidson, Lett. Math. Phys. **3**, 271 (1979).
[15] I. M. Davies, J. Phys. A **22**, 3199 (1989).
[16] L. de la Peña Auerbach, Phys. Lett. **24A**, 603 (1967).
[17] L. de la Peña Auerbach and A. M. Cetto, Rev. Mex. Fis. **17**, 327 (1968).
[18] I. Fényes, Z. Phys. **132**, 81 (1952).
[19] P. Garbaczewski, Phys. Lett. A **162**, 129 (1992).
[20] P. Garbaczewski and J.-P. Vigier, Phys. Rev. A **46**, 4634 (1992).
[21] S. Goldstein, J. Stat. Phys. **47**, 645 (1987); see especially the discussion of the Aharanov-Bohm effect, §10.1.
[22] F. Guerra and L. M. Morato, Phys. Rev. D **27**, 1774 (1983).
[23] K. Hajra, J. Math. Phys. **32**, 1505 (1991).
[24] D. Kershaw, Phys. Rev. **136**, B1850 (1964).
[25] J. D. Lafferty, Trans. Am. Math. Soc. **633**, 699 (1988).
[26] R. Marra, Phys. Rev. D **36**, 1724 (1987).
[27] L. M. Morato, Phys. Rev. D **32**, 1982 (1985).
[28] K. Namsrai, *Nonlocal Quantum Field Theory and Stochastic Quantum Mechanics*, *Fundamental Theories of Physics* (D. Reidel Publishing Company, Dordrecht, 1986).
[29] K. Namsrai, Y. Hulree, and N. Njamtseren, Int. J. Mod. Phys. **7**, 2661 (1992).
[30] E. Nelson, Phys. Rev. **150**, 1079 (1966).
[31] E. Nelson, *Dynamical Theories of Brownian Motion* (Princeton University Press, Princeton, 1967).
[32] E. Nelson, *Quantum Fluctuations* (Princeton University Press, Princeton, 1985).
[33] M. S. Wang, Phys. Lett. A **137**, 437 (1989).
[34] K. Yasue, J. Funct. Anal. **41**, 327 (1981).
[35] D. Bohm, Phys. Rev. **85**, 166 (1952).
[36] D. Bohm, Phys. Rev. **85**, 180 (1952).
[37] L. de Broglie, *Une Tentative d'Interpretation Causale et Non Linéaire de la Mécanique Ondulatoire (la Théorie de la Double Solution)* (Gauthier-Villars, Paris, 1956).
[38] L. de Broglie, *The Current Interpretation of Wave Mechanics. A Critical Study* (Elsevier, Amsterdam, 1964).
[39] M. Jammer, *The Philosophy of Quantum Mechanics* (Wiley, New York, 1974).
[40] J. A. Wheeler and W. H. Zurek, *Quantum Theory and Measurement*, *Princeton Series in Physics* (Princeton University Press, Princeton, 1983).
[41] T. C. Wallstrom, Found. Phys. Lett. **2**, 113 (1989).
[42] E. A. Carlen and M. I. Loffredo, Phys. Lett. A **141**, 9 (1989).
[43] E. Merzbacher, Am. J. Phys. **30**, 237 (1962).
[44] P. H. Damgaard and H. Hüffel, Phys. Rep. **152**, 227 (1987).
[45] L. de la Peña Auerbach, J. Math. Phys. **10**, 1620 (1969).
[46] T. G. Dankel, Jr., Arch. Ration. Mech. Anal. **37**, 192 (1970).
[47] T. C. Wallstrom, Trans. Am. Math. Soc. **318**, 749 (1990).
[48] T. C. Wallstrom, Phys. Lett. A **184**, 229 (1994).
[49] E. Nelson, in *Stochastic Processes in Classical and Quantum Systems*, edited by S. Albeverrio, G. Casati, and D. Merlini, Lecture Notes in Physics Vol. 262 (Springer-Verlag, New York, 1986), pp. 438–469.
[50] D. Bohm and B. J. Hiley, Phys. Rep. **172**, 93 (1989).
[51] J. G. Gilson, Proc. Cambridge Philos. Soc. **64**, 1061 (1968).
[52] A. F. Nicholson, Aust. J. Phys. **7**, 14 (1954).
[53] G. Casati and I. Guarneri, Phys. Rev. Lett. **42**, 1579 (1979).
[54] I. Bialynicki-Birula and Z. Bialynicka-Birula, Phys. Rev. D **3**, 2410 (1971).
[55] G. A. Goldin and D. H. Sharp, in *Symmetries in Science III*, edited by B. Gruber and F. Iachello (Plenum, New York, 1989), pp. 181–205.
[56] G. A. Goldin and D. H. Sharp, Int. J. Mod. Phys. **5**, 2625 (1991).