Quantum-noise matrix for multimode systems: U(n) invariance, squeezing, and normal forms

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We present a complete analysis of variance matrices and quadrature squeezing for arbitrary states of quantum systems with any finite number of degrees of freedom. Basic to our analysis is the recognition of the crucial role played by the real symplectic group $\mathrm{Sp}(2n,\mathbb{R})$ of linear canonical transformations on n pairs of canonical variables. We exploit the transformation properties of variance (noise) matrices under symplectic transformations to express the uncertainty-principle restrictions on a general variance matrix in several equivalent forms, each of which is manifestly symplectic invariant. These restrictions go beyond the classically adequate reality, symmetry, and positivity conditions. Towards developing a squeezing criterion for n-mode systems, we distinguish between photon-number-conserving passive linear optical systems and active ones. The former correspond to elements in the maximal compact $\mathrm{U}(n)$ subgroup of $\mathrm{Sp}(2n,\mathbb{R})$, the latter to *noncompact* elements outside $\mathrm{U}(n)$. Based on this distinction, we motivate and state a $\mathrm{U}(n)$-invariant squeezing criterion applicable to any state of an n-mode system, and explore alternative ways of expressing it. The set of all possible quantum-mechanical variance matrices is shown to contain several interesting subsets or subfamilies, whose definitions are related to the fact that a general variance matrix is not diagonalizable within $\mathrm{U}(n)$. Definitions, characterizations, and canonical forms for variance matrices in these subfamilies, as well as general ones, and their squeezing nature, are established. It is shown that all conceivable variance matrices can be generated through squeezed thermal states of the n-mode system and their symplectic transforms. Our formulas are developed in both the real and the complex forms for variance matrices, and ways to pass between them are given.

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I. INTRODUCTION

Squeezed states of the radiation field are distinctly nonclassical in nature [1]. Over the past decade their study has developed into a major area of quantum optics [2]. Their experimental realization by several groups [3–5] has certainly contributed to the enormous interest and activity in this area. In addition to quadrature squeezing [6–10], attention has also been focused on amplitude [11] as well as higher-order [12] squeezing. While many of these studies have centered around states of single- and two-mode systems [9,10] of immediate relevance to current experimental activity, there nevertheless has been interest in the multimode case as well [13,14].

All the information regarding the quadrature squeezing properties of any state of a multimode (quantum) system is contained in the noise or variance matrix of that state. In the single-mode case this is a 2×2 real symmetric positive-definite matrix V: the diagonal entries are the expectation values $\langle(\hat{q}-\langle\hat{q}\rangle)^2\rangle$ and $\langle(\hat{p}-\langle\hat{p}\rangle)^2\rangle$ while the off-diagonal one is $\langle\frac{1}{2}(\hat{q}\hat{p}+\hat{p}\hat{q})\rangle-\langle\hat{q}\rangle\langle\hat{p}\rangle$, where $\hat{q}$ and $\hat{p}$ are the quadrature operator components of the mode annihilation operator $\hat{a}=(\hat{q}+i\hat{p})/\sqrt{2}$.

For a classical probability distribution over a classical two-dimensional phase space with variables q and p, any real symmetric positive-definite matrix is a valid, that is, physically realizable, variance matrix. In the quantum case, however, the variance matrix has to satisfy the additional condition $\det V\geq\frac{1}{4}$. This is a precise and complete statement of Heisenberg's uncertainty principle for one pair of operator canonical variables.

The group of real linear canonical transformations $\mathrm{Sp}(2,\mathbb{R})\equiv\mathrm{Sl}(2,\mathbb{R})\equiv\mathrm{SU}(1,1)$ plays a basic role [1,2] in the study of squeezing in a single-mode system. (This is so whether or not one wishes to make explicit use of the language and machinery of Lie groups.) This group has as its maximal compact subgroup a one-parameter group U(1) corresponding to phase-space rotations; its generator is the harmonic-oscillator Hamiltonian $(\hat{a}^\dagger\hat{a}+\hat{a}\hat{a}^\dagger)/2$. An important aspect of the definition of squeezing in the single-mode case is that it has a basic built-in invariance under this U(1) subgroup, which is physically reasonable and justifiable.

Our interest in this paper is in n-mode systems. We present a comprehensive analysis of the properties of variance matrices for states of such systems. Such matrices are real, symmetric, $2n$ dimensional, and positive

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definite. In addition, they obey certain specifically quantum-mechanical inequalities in the Heisenberg sense. We make effective use of elementary concepts and results related to the group $Sp(2n,\mathbb{R})$ of real linear canonical transformations in $2n$-dimensional phase space, which is naturally available, to analyze both the Heisenberg inequalities and the notion of squeezing for n-mode states. While, as noted above, $Sp(2,\mathbb{R})$ is isomorphic to the pseudounitary group SU(1,1) in the single-mode case, $Sp(2n,\mathbb{R})$ is not isomorphic to the pseudounitary group $SU(n,n)$ but rather to a proper subgroup of it when $n \geq 2$.

The material of this paper is organized as follows. In Sec. II we address the following basic question: Given a $2n \times 2n$ real symmetric positive-definite matrix, how do we test whether it qualifies to be the variance matrix of some physically realizable state of the (quantum) n-mode system? Clearly this is the same as asking for a complete statement of the Heisenberg uncertainty relations for such systems. We solve this problem by making use of a classic theorem due to Williamson [15] on the normal forms of real symmetric matrices under symmetric symplectic transformations. The nontrivial aspect of this theorem hinges on the facts that only some phase-space rotations are canonical transformations, and that a symmetric symplectic transformation in general is not a similarity transformation. It thus turns out that the normal form is diagonal only for some special subsets of symmetric matrices, and there are several distinct normal forms. What is relevant for our problem is the fact that for symmetric positive-definite matrices the Williamson normal form is a diagonal one, and variance matrices are positive definite.

As can be seen from the work of Caves and Schumaker [6], it is important to be able to describe the variance matrix and squeezing in terms of the real (Hermitian) quadrature components $\hat{q}_j,\hat{p}_j$ as well as in terms of the complex (non-Hermitian) operators $\hat{a}_j,\hat{a}_j^\dagger$, and to switch easily between these two descriptions. While a given canonical transformation is specified in terms of a real matrix in the former description, it is specified by a complex matrix in the latter. These two matrices are related through a similarity transformation by a fixed numerical matrix. Similarly the real variance matrix becomes a complex Hermitian one when transcribed to the $\hat{a}$, $\hat{a}^\dagger$ description. We formulate our complete characterization of the variance matrix in Sec. II in the form of necessary and sufficient conditions, and these are expressed in both descriptions. Transformation formulas for passing easily between them are presented. Our notations are somewhat different from those of Caves and Schumaker [6] for the single-mode and two-mode cases, but we believe they are more convenient, typographically and otherwise.

The role of U(1) in the single-mode squeezing group $Sp(2,\mathbb{R})$ is played by the $U(n) \equiv K(n)$ subgroup of $Sp(2n,\mathbb{R})$ in the n-mode case. This subgroup consists of all those phase-space *rotations* which are also canonical transformations. It should be appreciated that for $n \geq 2$ *most* phase-space rotations are not canonical. Motivated by the U(1)-invariant squeezing criterion in the single-mode case and by the familiar division of quantum optical systems into passive and active types, we formulate, in Sec. III, a $U(n)$-invariant squeezing criterion for n-mode systems and explain why it is reasonable. Since the $K(n)$ subgroup of $Sp(2n,\mathbb{R})$ is too small to diagonalize a general variance matrix, it would appear at first sight that our squeezing criterion is not expressible in terms of the eigenvalue spectrum of the variance matrix. However, the identity of the two coset spaces $SO(2n)/SO(2n-1)=U(n)/U(n-1)=S^{2n-1}$, where S^{2n-1} is the unit sphere in the $2n$-dimensional Euclidean space $\mathbb{R}^{2n}$, enables us to establish that a state is squeezed if and only if the smallest eigenvalue of the corresponding variance matrix is less than $\frac{1}{2}$.

We take up in Sec. IV the question of $K(n)$ canonical forms for variance matrices. Though it is true that a generic variance matrix cannot be diagonalized by $K(n)$ transformations, there are two special families $\mathcal{S}_G$ and $\mathcal{S}_H$ whose $K(n)$ canonical form is diagonal. The family $\mathcal{S}_G$ consists of variance matrices which, apart from a factor of $\frac{1}{2}$, are also elements of $Sp(2n,\mathbb{R})$. Members of the family $\mathcal{S}_H$ are built up from $n \times n$ Hermitian matrices H obeying a positivity and spectrum condition. Both families $\mathcal{S}_G$ and $\mathcal{S}_H$ are contained in the larger family $\mathcal{S}_K$ of all those variance matrices whose $K(n)$ canonical form is diagonal. We state and prove a simple matrix algebraic necessary and sufficient condition characterizing the elements of $\mathcal{S}_K$. Finally, we also develop canonical (nondiagonal) forms for variance matrices which are not contained in $\mathcal{S}_K$.

In Sec. V we construct examples of n-mode states and their variance matrices to render transparent the physical meanings of the $K(n)$ canonical forms and the families $\mathcal{S}_G$, $\mathcal{S}_H$, and $\mathcal{S}_K$. It turns out that any acceptable variance matrix can be realized through a suitable squeezed thermal state (zero-mean Gaussian state). Finally, in Sec. VI we present some concluding remarks.

At several places throughout the paper we make use of properties of the symplectic groups $Sp(2n,\mathbb{R})$. We take care to state and explain them clearly at first encounter.

II. CHARACTERIZATION OF VARIANCE MATRICES BY UNCERTAINTY PRINCIPLES

Consider an n-mode quantum system with annihilation and creation operators $\hat{a}_j,\hat{a}_j^\dagger$, $j=1,2,\ldots,n$, obeying the standard boson commutation relations

$$[\hat{a}_j,\hat{a}_k^\dagger]=\delta_{jk} \, , \quad [\hat{a}_j,\hat{a}_k]=[\hat{a}_j^\dagger,\hat{a}_k^\dagger]=0 \, . \tag{2.1}$$

In terms of the Hermitian operators $\hat{q}_j,\hat{p}_j$ defined in the usual manner,

$$\hat{a}_j=\frac{\hat{q}_j+i\hat{p}_j}{\sqrt{2}} \, , \quad \hat{a}_j^\dagger=\frac{\hat{q}_j-i\hat{p}_j}{\sqrt{2}} \, , \tag{2.2}$$

we have the equivalent commutation relations

$$[\hat{q}_j,\hat{p}_k]=i\delta_{jk} \, , \quad [\hat{q}_j,\hat{q}_k]=[\hat{p}_j,\hat{p}_k]=0 \, . \tag{2.3}$$

It should be noted that these $\hat{q}_j$ and $\hat{p}_j$ differ from the familiar quadrature components of $\hat{a}_j$ by a factor of $\sqrt{2}$.

It will prove convenient to arrange the Hermitian $\hat{q}_j,\hat{p}_j$ and the non-Hermitian $\hat{a}_j,\hat{a}_j^\dagger$ into $2n$-component column vectors as follows:

$$\hat{\xi}^{(r)}=\begin{bmatrix}\hat{q}_1\\ \vdots\\ \hat{q}_n\\ \hat{p}_1\\ \vdots\\ \hat{p}_n\end{bmatrix},\quad \hat{\xi}^{(c)}=\begin{bmatrix}\hat{a}_1\\ \vdots\\ \hat{a}_n\\ \hat{a}_1^\dagger\\ \vdots\\ \hat{a}_n^\dagger\end{bmatrix}. \tag{2.4}$$

The associated row vectors are

$$\hat{\xi}^{(r)T}=\hat{\xi}^{(r)\dagger}=(\hat{q}_1\cdots\hat{q}_n\hat{p}_1\cdots\hat{p}_n),\qquad \hat{\xi}^{(c)\dagger}=(\hat{a}_1^\dagger\cdots\hat{a}_n^\dagger\hat{a}_1\cdots\hat{a}_n). \tag{2.5}$$

Therefore $\hat{\xi}^{(r)}\hat{\xi}^{(r)\dagger}$ and $\hat{\xi}^{(c)}\hat{\xi}^{(c)\dagger}$ are $2n\times 2n$ Hermitian matrices with operator-valued entries. The columns $\hat{\xi}^{(r)},\hat{\xi}^{(c)}$ (and hence the rows $\hat{\xi}^{(r)\dagger},\hat{\xi}^{(c)\dagger}$) are linearly related by a fixed numerical matrix Ω determined by Eq. (2.2):

$$\hat{\xi}^{(c)}=\Omega\hat{\xi}^{(r)},\quad \hat{\xi}^{(c)\dagger}=\hat{\xi}^{(r)\dagger}\Omega^\dagger,\qquad \Omega=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & i1\\ 1 & -i1\end{bmatrix}. \tag{2.6}$$

Since Ω is unitary,

$$\Omega^{-1}=\Omega^\dagger=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ -i1 & i1\end{bmatrix}, \tag{2.7}$$

we have the reverse relationships

$$\hat{\xi}^{(r)}=\Omega^\dagger\hat{\xi}^{(c)},\quad \hat{\xi}^{(r)\dagger}=\hat{\xi}^{(c)\dagger}\Omega. \tag{2.8}$$

This matrix Ω will play an important role in the sequel.

The fundamental commutation relations (2.1) and (2.3) can now be compactly written as

$$[\hat{\xi}_\mu^{(r)},\ \hat{\xi}_\nu^{(r)}]=i\beta_{\mu\nu}, \tag{2.9a}$$

$$[\hat{\xi}_\mu^{(c)},\ \hat{\xi}_\nu^{(c)\dagger}]=i(\Sigma_3)_{\mu\nu},\quad \mu,\nu=1,2,\ldots,2n, \tag{2.9b}$$

where the $2n\times 2n$ matrices β and Σ_3 are given in block form by

$$\beta=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix},\quad \Sigma_3=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}. \tag{2.10}$$

Actually, since $\hat{\xi}^{(c)}$ consists just of the $\hat{a}_j$'s and their adjoints, Eq. (2.9b) could be equally well expressed by

$$[\hat{\xi}_\mu^{(c)},\ \hat{\xi}_\nu^{(c)}]=\beta_{\mu\nu}. \tag{2.11}$$

Consider now a real linear transformation on the variables $\hat{q}_j,\hat{p}_j$ specified by a $2n\times 2n$ real matrix $S^{(r)}$:

$$\hat{\xi}^{(r)}\to\hat{\xi}^{r\prime}=S^{(r)}\hat{\xi}^{(r)}. \tag{2.12}$$

The condition that this be canonical means that $\hat{\xi}^{(r)\prime}$ must obey the same commutation relations (2.9a) as do $\hat{\xi}^{(r)}$. This places the following condition on the matrix $S^{(r)}$:

$$S^{(r)}\beta S^{(r)T}=\beta. \tag{2.13}$$

This is the well-known defining property for the elements of the real symplectic group $Sp(2n,\mathbb{R})$: the set of all matrices $S^{(r)}$ obeying Eq. (2.13) gives us in fact the defining representation of this group [16,17]. Thus real canonical linear transformations in $2n$ dimensions and $Sp(2n,\mathbb{R})$ matrices are in one-to-one correspondence.

The importance of such transformations for squeezing problems arises from the well-known fact that unitary evolutions generated by Hermitian Hamiltonians which are quadratic in $\hat{q}_j,\hat{p}_j$ (equivalently, in $\hat{a}_j,\hat{a}_j^\dagger$) produce these very transformations on the canonical variables:

$$\begin{aligned}&\hat{H}=\hat{\xi}^{(r)\dagger}h^{(r)}\hat{\xi}^{(r)}=\sum_{\mu,\nu}h^{(r)}_{\mu\nu}\hat{\xi}^{(r)}_\mu\hat{\xi}^{(r)}_\nu,\\ &h^{(r)}_{\mu\nu}=h^{(r)}_{\nu\mu}=h^{(r)*}_{\mu\nu},\\ &\hat{U}=\exp(-i\hat{H})\\ &\quad\Longrightarrow \hat{U}^\dagger\hat{\xi}^{(r)}\hat{U}=S^{(r)}(h^{(r)})\hat{\xi}^{(r)},\\ &S^{(r)}(h^{(r)})\in Sp(2n,\mathbb{R}).\end{aligned} \tag{2.14}$$

Expressions for $S^{(r)}(h^{(r)})$ in terms of $h^{(r)}$ can be found in Ref. [14]. The converse is also true: Given any $S^{(r)}\in Sp(2n,\mathbb{R})$, there exists a unitary evolution of the above type, fixed up to a phase which can be narrowed down to a sign ambiguity, which transforms $\hat{\xi}^{(r)}$ by $S^{(r)}$. The relevance to squeezing problems is now clear, since squeeze operators belong to this class of evolutions.

It is clear that when $\hat{\xi}^{(r)}$ undergoes the linear canonical transformation (2.12), $\hat{\xi}^{(c)}$ behaves as follows:

$$\begin{aligned}&\hat{\xi}^{(c)}\to\hat{\xi}^{(c)\prime}=S^{(c)}\hat{\xi}^{(c)}\\ &S^{(c)}=\Omega S^{(r)}\Omega^\dagger,\\ &S^{(r)}\in Sp(2n,\mathbb{R}).\end{aligned} \tag{2.15}$$

This is determined by the relationships (2.6) and (2.8). It is important to realize that $S^{(c)}$, though complex, represents the same *real* linear canonical transformation $S^{(r)}$ that appears in Eq. (2.12): it is a complex representation (in the complex $\hat{a}_j,\hat{a}_j^\dagger$ basis) for the real transformation.

A. Real form for the noise matrix

Squeezing deals with second-order noise moments. We want to be able to deal collectively with the set of all second-order moments for any state of a multimode system. Earlier studies of such systems have largely concentrated on the squeezed coherent states, or two-photon coherent states (TCS). We shall, however, build up a formalism capable of studying noise and squeezing in an arbitrary (pure or mixed) state specified by a density operator $\hat{\rho}$, with the expectation value of any observable $\hat{O}$ being given by $\langle\hat{O}\rangle=\mathrm{Tr}(\hat{\rho}\hat{O})$.

Let us assume without loss of generality (see below) that the state $\hat{\rho}$ has zero-mean values for the basic variables: $\langle\hat{\xi}^{(r)}\rangle=\langle\hat{\xi}^{(c)}\rangle=0$. Consider the operator matrix $\hat{\xi}^{(r)}\hat{\xi}^{(r)T}$. A general element can be written in terms of the anticommutator and commutator of its factors thus:

$$(\hat{\xi}^{(r)}\hat{\xi}^{(r)T})_{\mu\nu}=\hat{\xi}^{(r)}_{\mu}\hat{\xi}^{(r)}_{\nu}$$
$$=\tfrac{1}{2}\{\hat{\xi}^{(r)}_{\mu},\hat{\xi}^{(r)}_{\nu}\}+\frac{i}{2}\beta_{\mu\nu} . \quad (2.16)$$

We now define the $2n\times 2n$ real variance (noise) matrix $V^{(r)}$ for the state $\hat{\rho}$ by

$$\langle \hat{\xi}^{(r)}\hat{\xi}^{(r)T}\rangle=\mathrm{Tr}(\hat{\rho}\hat{\xi}^{(r)}\hat{\xi}^{(r)T})$$
$$=V^{(r)}+\frac{i}{2}\beta ,$$
$$V^{(r)}_{\mu\nu}=\tfrac{1}{2}\mathrm{Tr}(\hat{\rho}\{\hat{\xi}^{(r)}_{\mu},\hat{\xi}^{(r)}_{\nu}\}) . \quad (2.17)$$

We can decompose $V^{(r)}$ usefully into $n\times n$ blocks in this way:

$$V^{(r)}=\begin{bmatrix} V_1 & V_2 \\ V_2^T & V_3 \end{bmatrix} ,$$
$$(V_1)_{jk}=\langle \hat{q}_j\hat{q}_k\rangle ,$$
$$(V_2)_{jk}=\tfrac{1}{2}\langle\{\hat{q}_j,\hat{p}_k\}\rangle , \quad (2.18)$$
$$(V_3)_{jk}=\langle \hat{p}_j\hat{p}_k\rangle , \quad j,k=1,2,\ldots,n .$$

Thus V_1 gives the noise and correlations among the $\hat{q}$ variables, V_3 among the $\hat{p}$ variables, and V_2 comprises the correlations between $\hat{q}$'s and $\hat{p}$'s. The submatrices V_1 and V_3 are individually symmetric, and so is $V^{(r)}$ as a whole.

The restriction that the state $\hat{\rho}$ have zero means is easily removed. For, if $\hat{\rho}$ is such that $\langle\hat{\xi}^{(r)}\rangle\neq 0$, we simply replace $\hat{\xi}^{(r)}$ by $\Delta\hat{\xi}^{(r)}=\hat{\xi}^{(r)}-\langle\hat{\xi}^{(r)}\rangle$ in Eq. (2.17) in defining the variance matrix $V^{(r)}$. This corresponds to a *rigid* translation of the state in the (quantum) phase space by an amount $-\langle\hat{\xi}^{(r)}\rangle$, implemented by the displacement operator $\hat{D}(-\langle\hat{\xi}^{(c)}\rangle)$ familiar from the context of multimode coherent states. Also, such a rigid translation does not affect variances.

As we have mentioned in the Introduction, if we had a classical probability distribution over a classical $2n$-dimensional phase space, the only restriction on a $2n\times 2n$ variance matrix, for it to be realizable, is that it be positive definite (apart from being real and symmetric). In the present quantum case, however, $V^{(r)}$ has to satisfy additional *uncertainty inequalities.* We wish to now derive them, keeping in evidence always their $\mathrm{Sp}(2n,\mathbb{R})$ invariance.

For a single-mode system, the 2×2 variance matrix is

$$V^{(r)}=\begin{bmatrix} \langle\hat{q}^2\rangle & \tfrac{1}{2}\langle\{\hat{q},\hat{p}\}\rangle \\ \tfrac{1}{2}\langle\{\hat{q},\hat{p}\}\rangle & \langle\hat{p}^2\rangle \end{bmatrix} , \quad (2.19)$$

the means being assumed to vanish. The naive statement of the uncertainty principle,

$$\langle\hat{q}^2\rangle\langle\hat{p}^2\rangle\geq\tfrac{1}{4} , \quad (2.20)$$

is, as is well known, a necessary consequence of the commutation relation between $\hat{q}$ and $\hat{p}$. It is, however, not sufficient to characterize the variance matrix completely. The correct statement for this purpose is

$$\det V^{(r)}\equiv\langle\hat{q}^2\rangle\langle\hat{p}^2\rangle-[\tfrac{1}{2}\langle\{\hat{q},\hat{p}\}\rangle]^2\geq\tfrac{1}{4} . \quad (2.21)$$

Any $V^{(r)}$ obeying this condition is physically realizable as the variance matrix of some state $\hat{\rho}$. Moreover, the inequality (2.21) is explicitly $\mathrm{Sp}(2,\mathbb{R})$ invariant (see below).

A special canonical case, which may appear trivial, will turn out to be the most important case in the multimode situation, since any multimode variance matrix will be seen to be reducible essentially to this case. This occurs when the 2×2 matrix $V^{(r)}$ in Eq. (2.19) has the diagonal form

$$V^{(r)}=\begin{bmatrix} \kappa & 0 \\ 0 & \kappa \end{bmatrix} . \quad (2.22)$$

Clearly, the statement (2.20) is adequate in this case, and the necessary and sufficient condition for this $V^{(r)}$ to be acceptable is

$$\kappa\geq\tfrac{1}{2} . \quad (2.23)$$

Indeed, for any such κ, we can exhibit a thermal state with variance matrix (2.22) (see Sec. VI).

Now we turn to n-mode systems. Given a state $\hat{\rho}$ with variance matrix $V^{(r)}$, consider the state $\hat{\rho}(S^{(r)})=\hat{U}(S^{(r)})\hat{\rho}\hat{U}(S^{(r)})^{\dagger}$, where $\hat{U}(S^{(r)})$ is a unitary operator implementing the symplectic transformation $S^{(r)}\in\mathrm{Sp}(2n,\mathbb{R})$ in the sense of Eq. (2.14). Let $V^{(r)'}$ be the variance matrix of this latter state:

$$\mathrm{Tr}[\hat{\rho}(S^{(r)})\hat{\xi}^{(r)}\hat{\xi}^{(r)T}]=V^{(r)'}+\tfrac{1}{2}i\beta . \quad (2.24)$$

Making use of the cyclic invariance of traces, and Eqs. (2.14) and (2.17) in that order, we have

$$V^{(r)'}+\frac{i}{2}\beta=\mathrm{Tr}\left[\hat{\rho}\hat{U}(S^{(r)})^{\dagger}\hat{\xi}^{(r)}\hat{\xi}^{(r)T}\hat{U}(S^{(r)})\right]$$
$$=\mathrm{Tr}(\hat{\rho}S^{(r)}\hat{\xi}^{(r)}\hat{\xi}^{(r)T}S^{(r)T})$$
$$=S^{(r)}(V^{(r)}+\tfrac{1}{2}i\beta)S^{(r)T} . \quad (2.25)$$

Finally making use of the defining property (2.13) for symplectic matrices, we obtain the important result

$$V^{(r)'}=S^{(r)}V^{(r)}S^{(r)T} . \quad (2.26)$$

We term $V^{(r)'}$ the symmetric symplectic transform of $V^{(r)}$ under $S^{(r)}$. This then is the connection between the variance matrices of two states $\hat{\rho}$ and $\hat{\rho}(S^{(r)})$ related unitarily through the canonical evolution $S^{(r)}$. Incidentally we can now appreciate the $\mathrm{Sp}(2,\mathbb{R})$ invariance of the uncertainty principle (2.21): indeed, for any n, every symplectic matrix is known to be unimodular, so from Eq. (2.26) the determinant of $V^{(r)}$ is a symplectic invariant.

We now tackle the question: Given a real symmetric positive-definite $2n\times 2n$ matrix $V^{(r)}$, what are the necessary and sufficient conditions that ensure that it is the variance matrix of some state of the quantum n-mode system? A similar question was raised by Littlejohn [18] in the limited context of *Gaussian* Wigner distributions, and solved subsequently by some of the present authors [19]. The basic principles behind that solution apply to the present problem as well.

From the arguments leading to Eq. (2.26) it is clear that if a given $V^{(r)}$ is physically realizable, then so is its symplectic transform $V^{(r)'}$ given by (2.26), for every $S^{(r)}\in \mathrm{Sp}(2n,\mathbb{R})$. Conversely, the invertibility of $S^{(r)}$ implies the opposite statement as well. If $V^{(r)}$ does not qualify to be a variance matrix, neither can $V^{(r)'}$ for any $S^{(r)}\in \mathrm{Sp}(2n,\mathbb{R})$.

This suggests the following approach to solve the problem: Given $V^{(r)}$, we look for a symplectic transform $V^{(r)'}$ which has a particularly simple (canonical or normal) form allowing us to *test by inspection* whether it is physically realizable or not. Then the given $V^{(r)}$ is a variance matrix if and only if $V^{(r)'}$ is a bona fide variance matrix.

The existence of such a canonical form is guaranteed by *Williamson's theorem* [15]: For any real symmetric positive-definite $2n\times 2n$ matrix $V^{(r)}$, there exists an $S^{(r)}\in \mathrm{Sp}(2n,\mathbb{R})$ such that the symplectic transform of $V^{(r)}$ by $S^{(r)}$ has the canonical scaled diagonal form, unique up to the ordering of the κ_j,

$$\begin{aligned}\mathring{V}^{(r)}&=S^{(r)}V^{(r)}(S^{(r)})^T\\&=\mathrm{diag}\,(\kappa_1,\kappa_2,\ldots,\kappa_n,\kappa_1,\kappa_2,\ldots,\kappa_n)\ .\end{aligned} \tag{2.27}$$

Clearly, in the canonical form of $\mathring{V}^{(r)}$ the $2n$ phase-space variables $\hat{q}_j$, $\hat{p}_j$ are not correlated with one another, and what we have here are simply n *copies* of the two-dimensional example (2.22). It follows that $\mathring{V}^{(r)}$, and hence $V^{(r)}$, is a bona fide variance matrix if and only if

$$\kappa_j\ge \tfrac{1}{2}\ ,\quad j=1,2,\ldots,n\ . \tag{2.28}$$

It now remains to write these conditions in a more convenient $\mathrm{Sp}(2n,\mathbb{R})$ invariant form, so that in implementing or checking them we may deal directly with $V^{(r)}$.

We must note that the transformation (2.26) is in general not a similarity transformation, so the κ_j appearing in Eq. (2.27) are in general not the eigenvalues of $V^{(r)}$. However, Eq. (2.26) combined with Eq. (2.13) implies that the product matrix $V^{(r)}\beta$ does undergo a similarity transformation:

$$\begin{aligned}&V^{(r)}\to S^{(r)}V^{(r)}S^{(r)T}\\&\Longrightarrow V^{(r)}\beta\to S^{(r)}V^{(r)}\beta S^{(r)-1}\ .\end{aligned} \tag{2.29}$$

Therefore also the square of $V^{(r)}\beta$ transforms by

$$(V^{(r)}\beta)^2\to S^{(r)}(V^{(r)}\beta)^2(S^{(r)})^{-1}\ . \tag{2.30}$$

Now when $V^{(r)}$ takes the canonical diagonal form $\mathring{V}^{(r)}$ of Eq. (2.27), the product $\mathring{V}^{(r)}\beta$ takes an off-diagonal form, while $(\mathring{V}^{(r)}\beta)^2$ is diagonal:

$$(\mathring{V}^{(r)}\beta)^2=-\mathrm{diag}(\kappa_1^2,\kappa_2^2,\ldots,\kappa_n^2,\kappa_1^2,\kappa_2^2,\ldots,\kappa_n^2)\ . \tag{2.31}$$

We deduce that $\kappa_1^2,\kappa_2^2,\ldots,\kappa_n^2$ are the (at least doubly degenerate) eigenvalues of $-(V^{(r)}\beta)^2=V^{(r)}\beta V^{(r)}\beta^T$. Now this last matrix is in general neither symmetric nor manifestly positive definite, and it would clearly be convenient to have such a matrix for which the eigenvalues are the κ_j^2 for $j=1,2,\ldots,n$. This is easily achieved. We let $(V^{(r)})^{1/2}$ denote the unique real symmetric positive-definite square root of $V^{(r)}$, and subject $V^{(r)}\beta V^{(r)}\beta^T$ to a similarity transformation applying $(V^{(r)})^{-1/2}$ on the left and $(V^{(r)})^{1/2}$ on the right. This will not alter the eigenvalue spectrum at all, so we can say that $\kappa_1^2,\kappa_2^2,\ldots,\kappa_n^2$ are the (at least doubly degenerate) eigenvalues of the real symmetric positive-definite matrix $(V^{(r)})^{1/2}\beta V^{(r)}\beta^T(V^{(r)})^{1/2}$. We can now express condition (2.28) in the following invariant manner.

Theorem 1. The necessary and sufficient condition for a real symmetric positive-definite $2n\times 2n$ matrix $V^{(r)}$ to be a bona fide (quantum) variance matrix is that

$$4(V^{(r)})^{1/2}\beta V^{(r)}\beta^T(V^{(r)})^{1/2}\ge 1\ . \tag{2.32}$$

In passing we note that the conditions (2.28), namely, $\kappa_j^2\ge\frac{1}{4}$ for $j=1,2,\ldots,n$, can be written in an $\mathrm{Sp}(2n,\mathbb{R})$ invariant form in terms of the $\mathrm{Sp}(2n,\mathbb{R})$ invariant traces $\mathrm{Tr}(V^{(r)}\beta V^{(r)}\beta^T)^l$, $l=1,2,\ldots,n$. These traces are evidently the sums of even powers of the κ_j, aside from a factor of 2: for $l=1$, we have the sum of κ_j^2; for $l=2$, the sum of κ_j^4; and so on. The procedure for expressing the inequalities (2.28) in terms of these traces is the same as that developed earlier for a related problem. The details are given in Ref. [19], so we do not pursue them here.

Let us instead show that our necessary and sufficient conditions (2.28) and (2.32) can be cast in yet another instructive form. For this purpose, consider the Hermitian matrix

$$\mathring{V}^{(r)}+\frac{i}{2}\beta=\begin{bmatrix}\kappa_1 & & & \frac{i}{2} & & \\ & \ddots & & & \ddots & \\ & & \kappa_n & & & \frac{i}{2}\\ -\frac{i}{2} & & & \kappa_1 & & \\ & \ddots & & & \ddots & \\ & & -\frac{i}{2} & & & \kappa_n\end{bmatrix}\ . \tag{2.33}$$

The eigenvalues are easily determined because each $\hat{q}_j$ is coupled only to its own $\hat{p}_j$, and not to any $\hat{q}_k$ or $\hat{p}_k$ for $k\ne j$. Thus this matrix has the spectrum of eigenvalues $\kappa_j\pm\frac{1}{2}$ for $j=1,2,\ldots,n$. It follows that our condition (2.28) amounts to demanding that $\mathring{V}^{(r)}+(i/2)\beta$ be positive semidefinite. Now the transformation

$$\mathring{V}^{(r)}+\frac{i}{2}\beta=S^{(r)}\left[V^{(r)}+\frac{i}{2}\beta\right]S^{(r)T} \tag{2.34}$$

connecting the two Hermitian matrices is not a similarity transformation but a real symmetric one by a nonsingular matrix. Thus,, while it may not preserve the spectrum of eigenvalues, it certainly preserves positive semidefiniteness (in fact, it preserves the signature of the eigenvalues and the rank of the concerned matrix). We have thus established the following result as fully equivalent to (2.28) and (2.32):

Theorem 2. A real symmetric positive-definite $2n\times 2n$ matrix $V^{(r)}$ is a bona fide (quantum) variance matrix if and only if the Hermitian matrix $V^{(r)}+(i/2)\beta$ is positive semidefinite:

$$V^{(r)}+\frac{i}{2}\beta\geq 0\ . \tag{2.35}$$

We may remark that the necessity of this condition (2.35) is actually quite obvious when we look at the structure of Eq. (2.17) defining $V^{(r)}$, and also remember that each entry in $\hat{\xi}^{(r)}$ is Hermitian. What is quite nontrivial is therefore the sufficiency of this condition.

B. Complex form for the noise matrix

To conclude this section we derive some results concerning the complex representation $V^{(c)}$ of the real variance matrix $V^{(r)}$. Analogous to Eq. (2.17), and in view of (2.9), we define $V^{(c)}$ by

$$\langle \hat{\xi}^{(c)}\hat{\xi}^{(c)\dagger}\rangle = V^{(c)}+\tfrac{1}{2}\Sigma_3\ . \tag{2.36}$$

Written out in detail in terms of $n\times n$ blocks this reads:

$$V^{(c)}=\begin{bmatrix} A & B\\ B^* & A^*\end{bmatrix}\ ,$$
$$A_{jk}=A^*_{kj}=\tfrac{1}{2}\langle\{\hat{a}_j,\hat{a}_k^\dagger\}\rangle\ , \tag{2.37}$$
$$B_{jk}=B_{kj}=\langle \hat{a}_j\hat{a}_k\rangle\ .$$

(Remember again that the means of $\hat{a}_j$ and $\hat{a}_j^\dagger$ are assumed, without loss of generality, to be zero.) Thus A is Hermitian, so $A^*=A^T$; and B is symmetric, so $B^*=B^\dagger$. These imply the hermiticity of $V^{(c)}$. Conversely, $(V^{(c)})^\dagger=V^{(c)}$ implies $A^\dagger=A$, and $B^T=B$.

We relate $V^{(c)}$ to $V^{(r)}$ easily by using Eq. (2.6):

$$V^{(c)}=\Omega V^{(r)}\Omega^\dagger\ ,\quad V^{(r)}=\Omega^\dagger V^{(c)}\Omega\ . \tag{2.38}$$

Here we have also used the connections

$$\Omega\beta\Omega^\dagger=-i\Sigma_3\ ,$$
$$\Omega^\dagger\Sigma_3\Omega=i\beta\ . \tag{2.39}$$

Passing between the real and complex forms of the variance matrix via Eq. (2.38) is consistent with the fact that reality of $V^{(r)}$ implies the special form (2.37) for $V^{(c)}$. Further, the real symmetric positive-definite nature of $V^{(r)}$ implies that $V^{(c)}$ is complex Hermitian positive definite. Written out in terms of the blocks of $V^{(r)}$ and $V^{(c)}$, we have

$$A=\tfrac{1}{2}[V_1+V_3+i(V_2^T-V_2)]\ ,$$
$$B=\tfrac{1}{2}[V_1-V_3+i(V_2^T+V_2)]\ ; \tag{2.40a}$$

$$V_1=\tfrac{1}{2}(A+A^*+B+B^*)\ ,$$
$$V_2=\frac{i}{2}(A-A^*-B+B^*)\ , \tag{2.40b}$$
$$V_3=\tfrac{1}{2}(A+A^*-B-B^*)\ .$$

Now consider a linear transformation $\hat{\xi}^{(c)}\to\hat{\xi}^{(c)\prime}=S^{(c)}\hat{\xi}^{(c)}$ where $S^{(c)}$ is a complex $2n\times 2n$ matrix. The requirement that $\xi^{(c)\prime}$ satisfy the same commutation relations as $\hat{\xi}^{(c)}$ can be expressed as a condition on $S^{(c)}$ in two ways, corresponding to Eqs. (2.9b) and (2.11), respectively:

$$S^{(c)}\Sigma_3(S^{(c)})^\dagger=\Sigma_3\ , \tag{2.41a}$$

$$S^{(c)}\beta(S^{(c)})^T=\beta\ . \tag{2.41b}$$

These are actually equivalent requirements since the reality of $S^{(r)}$ implies

$$(S^{(c)})^*=\Sigma_1 S^{(c)}\Sigma_1\ ,$$
$$\Sigma_1=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\ . \tag{2.42}$$

One can verify that, by virtue of Eq. (2.15) connecting $S^{(r)}$ and $S^{(c)}$, and the properties of β, Σ_1, and Σ_3, including (2.39), the real symplectic nature of $S^{(r)}$, namely, Eq. (2.13), guarantees that $S^{(c)}$ obeys Eqs. (2.41).

Since $S^{(r)}$ and $S^{(c)}$ thus represent one and the same element of Sp($2n,\mathbb{R}$), they are generated by the same quadratic Hamiltonian and associated unitary operator. Indeed we have from Eqs. (2.14) and (2.8) (and some abuse of notation),

$$\hat{U}(S^{(r)})=\hat{U}(S^{(c)})=\exp(-i\hat{H})\ ,$$
$$\hat{H}=\xi^{(r)^T}h^{(r)}\xi^{(r)}$$
$$=\xi^{(c)\dagger}h^{(c)}\xi^{(c)}\ , \tag{2.43}$$
$$h^{(c)}=\Omega h^{(r)}\Omega^\dagger\ ,$$
$$h^{(r)^T}=h^{(r)}\ ,\quad h^{(c)\dagger}=h^{(c)}\ .$$

Thus the complex variance matrices $V^{(c)}$ and $V^{(c)\prime}$ for two states $\hat{\rho}$ and $\hat{\rho}(S^{(r)})=\hat{\rho}(S^{(c)})$ are related by

$$V^{(c)\prime}=S^{(c)}V^{(c)}(S^{(c)})^\dagger\ . \tag{2.44}$$

Once again, one may verify that this law is consistent with Eqs. (2.15), (2.26), and (2.39).

Finally, we deal with the necessary and sufficient conditions for a given complex matrix $V^{(c)}$ to qualify as a (quantum) variance matrix. We have already seen that $V^{(c)}$ must be Hermitian, positive definite, and take the special form (2.37). Beyond this, first note from Eq. (2.27) that the canonical form $\mathring{V}^{(r)}$ of $V^{(r)}$ commutes with Ω, so by Eq. (2.38) the corresponding canonical form $\mathring{V}^{(c)}$ of $V^{(c)}$ coincides with $\mathring{V}^{(r)}$:

$$\mathring{V}^{(c)}=\mathring{V}^{(r)}\ . \tag{2.45}$$

Further the matrices on the left-hand sides of Eqs. (2.32) and (2.35) can be expressed thus in terms of $V^{(c)}$:

$$4(V^{(r)})^{1/2}\beta V^{(r)}\beta^T(V^{(r)})^{1/2}=4\Omega^\dagger(V^{(c)})^{1/2}\Sigma_3 V^{(c)}\times\Sigma_3(V^{(c)})^{1/2}\Omega\ ,$$
$$V^{(r)}+\frac{i}{2}\beta=\Omega^\dagger(V^{(c)}+\tfrac{1}{2}\Sigma_3)\Omega\ . \tag{2.46}$$

Thus, going back to Eqs. (2.32) and (2.35), we arrive at a complete characterization of $V^{(c)}$:

Theorem 3. The necessary and sufficient conditions for a $2n\times 2n$ complex matrix $V^{(c)}$ to be a bona fide (quantum) variance matrix (in the complex representation) is that it be Hermitian positive definite with the special form (2.37) and in addition satisfy (one of) the following equivalent conditions:

$$4(V^{(c)})^{1/2}\Sigma_3 V^{(c)}\Sigma_3(V^{(c)})^{1/2}\geq 1 \ ,$$
$$V^{(c)}+\tfrac{1}{2}\Sigma_3\geq 0 \ . \tag{2.47}$$

These are a complete statement of the uncertainty principles in complex representation; and as before with $V^{(r)}$, the necessity of the second inequality above is obvious from Eq. (2.36).

III. U(n) INVARIANCE AND THE n-MODE SQUEEZING CRITERION

Having completely characterized the variance matrices of n-mode systems from the point of view of the quantum-mechanical uncertainty principles, it is now natural to pose the following question. Given an acceptable variance matrix, how shall we decide whether it describes a squeezed state or not? The aim of this section is to motivate and develop an answer to this question.

It is clear that we need a squeezing criterion possessing certain desirable properties. Given $V^{(r)}$, suppose one of its diagonal elements is already less than $\frac{1}{2}$. Then we would like to conclude that the state is manifestly squeezed. If $V^{(r)}_{\mu\mu}<\frac{1}{2}$ for some μ, then the squeezed quadrature component is $\hat{q}_\mu$ if $\mu\leq n$, and $\hat{p}_{\mu-n}$ if $\mu>n$. The question becomes nontrivial only if $V^{(r)}_{\mu\mu}\geq\frac{1}{2}$ for all $\mu=1,2,\dots,2n$.

For guidance let us turn again to a single-mode system. To be specific, consider the variance matrix

$$V^{(r)}=\frac{1}{2}\begin{bmatrix}\cosh\eta & \sinh\eta\\ \sinh\eta & \cosh\eta\end{bmatrix} , \tag{3.1}$$

for some real $\eta>0$. Both diagonal elements exceed $\frac{1}{2}$, so there is no manifest evidence of squeezing. Yet we know that squeezing is buried in this variance matrix, since the elements here are the variances of the squeezed coherent state

$$|\alpha;\eta\rangle=\exp\left\{i\frac{\eta}{4}(\hat{a}^2+\hat{a}^{\dagger 2})\right\}|\alpha\rangle \ , \tag{3.2}$$

where $|\alpha\rangle$ is any coherent state.

To understand the situation *in a form that will help us to generalize to the n-mode case,* we note that Hermitian quadratic Hamiltonians in the single-mode case are of two types. The first type, labeled by one real parameter θ, is

$$\hat{H}(\theta)=\frac{\theta}{4}(\hat{a}^\dagger\hat{a}+\hat{a}\hat{a}^\dagger) \ . \tag{3.3}$$

It generates the linear canonical transformations

$$e^{i\hat{H}}\begin{bmatrix}\hat{q}\\ \hat{p}\end{bmatrix}e^{-i\hat{H}}=S^{(r)}(\theta)\begin{bmatrix}\hat{q}\\ \hat{p}\end{bmatrix} ,$$
$$e^{i\hat{H}}\begin{bmatrix}\hat{a}\\ \hat{a}^\dagger\end{bmatrix}e^{-i\hat{H}}=S^{(c)}(\theta)\begin{bmatrix}\hat{a}\\ \hat{a}^\dagger\end{bmatrix} ,$$
$$S^{(r)}(\theta)=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix} ,$$
$$S^{(c)}(\theta)=\begin{bmatrix}e^{-i\theta} & 0\\ 0 & e^{i\theta}\end{bmatrix} . \tag{3.4}$$

These are phase space rotations: $S^{(r)}(\theta)\in SO(2)\subset Sp(2,\mathbb{R})$, which is the same as changing $\hat{a}$ by the U(1) phase $e^{-i\theta}$. The second type of Hamiltonians consists of a pair of generators labeled by a complex number z:

$$\hat{H}(z)=\tfrac{1}{4}(z\hat{a}^{\dagger 2}+z^*\hat{a}^2) \ . \tag{3.5}$$

Such Hamiltonians give rise to scaling (squeezing) transformations in phase space. The three independent Hermitian generators contained in Eqs. (3.3) and (3.5) together build up, on exponentiation, the well-known squeezing group $Sp(2,\mathbb{R})=SL(2,\mathbb{R})=SU(1,1)$ for the single-mode case. [More precisely, they lead to the metaplectic group Mp(2)].

In technical terms the two types of generators described above are, respectively, compact and noncompact ones. In physical terms, and more importantly for us, the first (compact) type (3.3) conserves photon number and hence correspond to *passive* systems. Examples of such systems are free evolution, and action by lossless beam splitters in homodyne detection. The second (noncompact) type of Hamiltonians (3.5), does not conserve photon number, and hence correspond to *active* systems.

Now in this single-mode case squeezing is defined modulo passive transformations of the type (3.4). That is, one considers not only $\hat{q}$ and $\hat{p}$, but also the rotated versions $\hat{q}_\theta$ and $\hat{p}_\theta$ for all θ, where $(\hat{q}_\theta+i\hat{p}_\theta)/\sqrt{2}=\hat{a}e^{-i\theta}$. A state is deemed to be squeezed if, for some θ, either $\hat{q}_\theta$ or $\hat{p}_\theta$ is squeezed. Actually it is sufficient to consider just $\hat{q}_\theta$ since $\hat{p}_\theta=\hat{q}_{\theta+\pi/2}$. We regard $\hat{q}_\theta$ and $\hat{p}_\theta$ for every θ in the full range $-\pi\leq\theta\leq\pi$ of SO(2) to be just as good quadrature components as $\hat{q}$ and $\hat{p}$ since they are related by passive transformations.

In terms of $V^{(r)}$ this means that we look at all the matrices

$$V^{(r)}(\theta)=S^{(r)}(\theta)V^{(r)}S^{(r)}(\theta)^T , \tag{3.6}$$

obtained from $V^{(r)}$ by Eq. (2.26), with $S^{(r)}(\theta)$ going over the (maximal) compact subgroup SO(2) = U(1) of $Sp(2,\mathbb{R})$. We see whether for some $\theta=\theta_0$, any one of the diagonal elements of $V^{(r)}(\theta)$ falls below $\frac{1}{2}$ and so shows manifest squeezing. (In fact it is enough to fix one's attention on a specific diagonal element as θ is varied.) If the answer is in the affirmative, then the state giving rise to the original $V^{(r)}$ is deemed to be squeezed, the squeezed component being $\hat{q}_{\theta_0}$ or $\hat{p}_{\theta_0}$ as the case may be.

For the example (3.1) we find that when $\theta_0=+\pi/4$,

$$V^{(r)}(\theta_0)=\frac{1}{2}\begin{bmatrix}e^{-\eta} & 0\\ 0 & e^{\eta}\end{bmatrix} . \tag{3.7}$$

Thus $\hat{q}_{\pi/4}=(\hat{q}+\hat{p})/\sqrt{2}$ is indeed squeezed by a factor $e^{\eta/2}$ below the vacuum fluctuation of $1/\sqrt{2}$, and this is consistent with (3.2).

To re-emphasize the point: the definition of squeezing in the single-mode case is set up so as to be U(1) invariant. The variance matrices in (3.1) and (3.7) are U(1) related, and one views the state with variances (3.1) to be as much squeezed as the state leading to (3.7), notwithstanding the fact that the latter alone shows manifest squeezing.

These considerations generalize to the n-mode case governed by the $n(2n+1)$-dimensional group $Sp(2n,\mathbb{R})$. The Hermitian quadratic Hamiltonians which are compact generators are combinations of the following n^2 independent ones:

$$\begin{aligned}&\tfrac{1}{4}(\hat{a}_j^\dagger\hat{a}_j+\hat{a}_j\hat{a}_j^\dagger)\ ,\quad j=1,2,\ldots,n\ ;\\&\tfrac{1}{4}(\hat{a}_j^\dagger\hat{a}_k+\hat{a}_k^\dagger\hat{a}_j)\ ,\\&\frac{i}{4}(\hat{a}_j^\dagger\hat{a}_k-\hat{a}_k^\dagger\hat{a}_j)\ ,\quad j<k=2,\ldots,n\ .\end{aligned}\tag{3.8}$$

All of these n^2 operators conserve the total photon number since they commute with the total number operator $\hat{N}$:

$$\hat{N}=\sum_{j=1}^{n}\hat{a}_j^\dagger\hat{a}_j\ ,\tag{3.9}$$

so they correspond to passive systems. The subgroup of $Sp(2n,\mathbb{R})$ generated by these compact-type Hamiltonians is the n^2-dimensional unitary group $U(n)$, as can be seen from the action on $\hat{\xi}^{(c)}$. This is the maximal compact subgroup of $Sp(2n,\mathbb{R})$. Stated in another way, these generators produce the (maximal) rotation subgroup $K(n)=SO(2n)\cap Sp(2n,\mathbb{R})$ of $Sp(2n,\mathbb{R})$, as can be seen from the action on $\hat{\xi}^{(r)}$.

The remaining $n(n+1)$ linearly independent Hermitian quadratic generators are of the noncompact, nonphoton number conserving type; we can take them to be

$$\begin{aligned}&\tfrac{1}{4}(\hat{a}_j^\dagger\hat{a}_k^\dagger+\hat{a}_k\hat{a}_j)\ ,\\&\frac{i}{4}(\hat{a}_j^\dagger\hat{a}_k^\dagger-\hat{a}_k\hat{a}_j)\ ,\quad j\le k=1,2,\ldots,n\ .\end{aligned}\tag{3.10}$$

These generators as Hamiltonians correspond to active systems. Taken together, the expressions in (3.8) and (3.10) account for the $n(2n+1)$ generators of $Sp(2n,\mathbb{R})$.

We now display the manner in which the maximal compact subgroup of $Sp(2n,\mathbb{R})$ can be explicitly *seen* in the defining representation of $Sp(2n,\mathbb{R})$. Consider a real $2n\times 2n$ matrix of the form

$$S^{(r)}(X,Y)=\begin{bmatrix}X & Y\\ -Y & X\end{bmatrix}\ ,\tag{3.11}$$

where X and Y are real $n\times n$ matrices obeying

$$\begin{aligned}&XX^T+YY^T=1\ ,\\&XY^T=YX^T\ .\end{aligned}\tag{3.12}$$

We immediately verify that

$$\begin{aligned}&S^{(r)}(X,Y)S^{r}(X,Y)^T=1\ ,\\&S^{(r)}(X,Y)\beta S^{(r)}(X,Y)^T=\beta\ .\end{aligned}\tag{3.13}$$

Thus such a matrix is both orthogonal (in $2n$ dimensions) and symplectic. It is therefore contained in the intersection

$$K(n)=SO(2n)\cap Sp(2n,\mathbb{R})\ .\tag{3.14}$$

Conversely, every element of $K(n)$, the maximal compact subgroup of $Sp(2n,\mathbb{R})$, can be written in the form $S^{(r)}(X,Y)$ with X and Y obeying conditions (3.12).

From Eq. (2.15) the complex representation of $S^{(r)}(X,Y)$ is

$$\begin{aligned}S^{(c)}(X,Y)&=\Omega S^{(r)}(X,Y)\Omega^\dagger\\&=\begin{bmatrix}U & 0\\ 0 & U^*\end{bmatrix}=S^{(c)}(U)\ ,\\U&=X-iY\ .\end{aligned}\tag{3.15}$$

The conditions (3.12) transcribed in terms of U read

$$UU^\dagger=1\ .\tag{3.16}$$

Thus we have established the isomorphism between $K(n)$ and $U(n)$:

$$K(n)=SO(2n)\cap Sp(2n,\mathbb{R})=U(n)\ .\tag{3.17}$$

It is necessary to emphasize again that not all rotations in the $2n$-dimensional phase space [elements of $SO(2n)$] are canonical transformations. Only those which have the special form (3.11) are canonical. Therefore we may refer to $K(n)\subset SO(2n)$ as the subgroup of canonical rotations.

Let us for a moment view the $2n$-dimensional phase space as $\mathbb{R}^{2n}$ and define basis and general unit vectors

$$\begin{aligned}&e_\mu=(0,\ldots,0,1,0,\ldots,0)^T\ ,\\&x=(x_1,x_2,\ldots,x_{2n})^T\ ,\\&x^Tx=1\ .\end{aligned}\tag{3.18}$$

Here e_μ is the unit vector in the μth direction (q_μ or $p_{\mu-n}$ as the case may be), its column having unity at the μth position and zero elsewhere. The set of all unit vectors x constitutes the unit sphere S^{2n-1}. Now it is well known that $SO(2n)$ acts transitively on S^{2n-1}. For later application we now ask whether the much smaller group $K(n)$ also acts transitively on S^{2n-1}. To answer this question we shall compute the orbit of some unit vector, say, e_μ, under $K(n)$, and see whether it exhausts all of S^{2n-1}. [The orbit of e_μ under $K(n)$ is the set of all vectors $x\in S^{2n-1}$ which are obtained from e_μ by action with elements of $K(n)$; the group automatically acts transitively thereon.]

From Eq. (3.11) it is clear that if a canonical rotation $S^{(r)}(X,Y)$ leaves a particular $\hat{q}_j$ unchanged, it necessarily also leaves the conjugate $\hat{p}_j$ unchanged, because the former property means that in the jth row of $S^{(r)}(X,Y)$ we have unity at the jth column and zero elsewhere. The specific structure of $S^{(r)}(X,Y)$ then means that in the $(j+n)$th row also we have unity at the $(j+n)$th column and zero elsewhere. Thus the stability group of e_μ is just $K(n-1)$, the group of canonical rotations in the (remaining) $2n-2$ dimensions involving $\hat{q}_k,\hat{p}_k$ for $k\neq j$. This means that the orbit of e_μ under $K(n)$ is the coset space $U(n)/U(n-1)$, but this is known to be S^{2n-1}:

$$\frac{K(n)}{K(n-1)}=\frac{U(n)}{U(n-1)}=\mathcal{S}^{2n-1}\ .\tag{3.19}$$

We have thus proved that not only SO($2n$) but also (the much smaller subgroup) K(n) acts transitively on the unit sphere S^{2n-1}. The reasons in the two cases are different: in the first case it is because the coset space SO($2n$)/SO($2n-1$)$=S^{2n-1}$, in the second because the coset space U(n)/U($n-1$)$=S^{2n-1}$.

We now have all the necessary tools to handle the n-mode squeezing criterion. Generalizing from the single-mode case, we have seen that the subgroup of Sp($2n,\mathbb{R}$) corresponding to passive systems is K(n). Therefore, as far as squeezing is concerned, we must treat every component of $\hat{\xi}^{(r)}$, and also every component of $\hat{\xi}^{(r)\prime}=S^{(r)}(X,Y)\hat{\xi}^{(r)}$ for every $S^{(r)}(X,Y)\in$K(n), as equally good quadrature components. Therefore a state is to be deemed as squeezed if the fluctuation of some component of $\hat{\xi}^{(r)\prime}$, for some $S^{(r)}(X,Y)$, is less than $\frac{1}{2}$ in the concerned state. Now the fluctuation in $\hat{\xi}^{(r)\prime}_{\mu}$ is simply the μth diagonal element in the transformed variance matrix $V^{(r)\prime}=S^{(r)}(X,Y)V^{(r)}S^{(r)}(X,Y)^T$. Thus our explicitly K($n$)-invariant n-mode squeezing criterion reads:

$$V^{(r)} \text{ is squeezed} \iff \min_{S^{(r)}(X,Y)\in \mathrm{K}(n)} [S^{(r)}(X,Y)V^{(r)}S^{(r)}(X,Y)^T]_{\mu\mu}<\tfrac{1}{2}\ , \quad \mu\in(1,2,\ldots,2n)\ . \tag{3.20}$$

This is just the same as the statement

$$V^{(r)} \text{ is squeezed} \iff \min_{S^{(r)}(X,Y)\in \mathrm{K}(n)} [\{S^{(r)}(X,Y)^T\mathbf{e}_\mu\}^T V^{(r)}\{S^{(r)}(X,Y)^T\mathbf{e}_\mu\}]<\tfrac{1}{2}\ , \quad \mu\in(1,2,\ldots,2n)\ ,$$
$$\iff \min_{x\in S^{2n-1}}[x^TV^{(r)}x]<\tfrac{1}{2}\ . \tag{3.21}$$

It is here in the last step that we made use of the result embodied in Eq. (3.19), namely, that K(n) acts transitively on S^{2n-1}. Finally, we see immediately that this last form of our squeezing criterion can be expressed in terms of the eigenvalue spectrum of $V^{(r)}$, namely, we have the following theorem.

Theorem 4. A state with variance matrix $V^{(r)}$ is squeezed according to the criterion (3.20) if and only if the least eigenvalue $l(V^{(r)})$ of $V^{(r)}$ obeys

$$l(V^{(r)})<\tfrac{1}{2}\ , \tag{3.22}$$

and conversely.

We stress that our squeezing criterion has been set up based on the reasonable premise that all quadrature components related by elements of the U(n) subgroup of Sp($2n,\mathbb{R}$) should be treated on equal footing, since these compact elements conserve total photon number, and hence correspond to passive optical systems, while the rest of Sp($2n,\mathbb{R}$) corresponds to active systems. As a result we have a U(n)-invariant squeezing criterion. Further, we have been aware all along of the fact that we do not have at our disposal the entire rotation group SO($2n$), with whose help any $V^{(r)}$ could have been diagonalized, but only the subgroup K(n) of canonical rotations. Hence, as one would have rightly suspected, diagonalization of $V^{(r)}$ just using elements of K(n) is in general not possible. It is therefore our result on the transitive action of K(n) on S^{2n-1} that has nevertheless allowed us to express our squeezing criterion in terms of the smallest eigenvalue of $V^{(r)}$.

IV. CANONICAL FORMS FOR VARIANCE MATRICES

Now we examine the question of canonical forms for variance matrices in the context of our squeezing criterion. To begin, we note two obvious simple forms which, however, for reasons to be made clear, are not suitable for the present purpose.

Since a variance matrix $V^{(r)}$ is always real symmetric, it can definitely be transformed to a diagonal matrix by conjugation (similarity transformation) with a suitable SO($2n$) rotation. The resulting diagonal elements will be the eigenvalues of $V^{(r)}$. However, as already noted, general elements of SO($2n$) are not canonical transformations, so the diagonal form achieved in this way is not relevant to the squeezing problem. Indeed, an SO($2n$) transform of a variance matrix may well fail to be a bona fide variance matrix. We therefore do not consider this SO($2n$) normal form any further.

There is yet another normal form which is provided by the Williamson theorem referred to in Sec. II. This form obtains because in addition to being real symmetric, $V^{(r)}$ is also *positive definite*. Therefore by a suitable element of Sp($2n,\mathbb{R}$) the symmetric symplectic transform of $V^{(r)}$ can be made diagonal. Recognizing that after achieving diagonal form there is the freedom to make reciprocal scale changes in $\hat{q}_j$ and $\hat{p}_j$ independently for each canonical pair, we can achieve Eq. (2.27) and call this the Williamson normal form. In contrast to the diagonal form considered in the preceding paragraph, the present one is achieved through canonical transformations. Even so this is not suitable for our present purpose, for two reasons. First, the diagonal entries in the Williamson normal form are generally not the eigenvalues of the original $V^{(r)}$. Second, passage to this form makes use in principle of the full group Sp($2n,\mathbb{R}$) inclusive of squeezing transformations, whereas our squeezing criterion is only K(n)$\equiv$U(n) invariant.

Against this background we may ask: What is the most natural and simplest canonical form into which a general $V^{(r)}$ can be cast, if we use only transformations by elements of the K(n) group of canonical rotations? It is clear at the outset that this form cannot generically be diagonal, because a diagonal matrix has $2n$ free parameters, while K(n) is an n^2-parameter group. Thus matrices $V^{(r)}$ that can be brought to diagonal form using K(n) action can at the most be an $n(n+2)$-parameter family. But the totality of all $V^{(r)}$ constitutes an $n(2n+1)$-parameter family, being restricted only by symmetry; and for $n \geq 2$ this is a much larger family. (The positive definiteness conditions and the uncertainty principles are all inequalities and so do not cut down the number of parameters).

In this way we are led also to ask the supplementary question: Is it possible to characterize in a concise manner the family of variance matrices which can be brought to diagonal form using $\mathrm{K}(n)$ transformations alone? It turns out that this can be done, and in a rather elegant way. We shall denote this family of variance matrices by $\mathcal{S}_K$, reserving for the set of all variance matrices the symbol $\mathcal{S}$. But before giving a characterization of $\mathcal{S}_K$, we first discuss two subfamilies of $\mathcal{S}_K$, to be denoted by $\mathcal{S}_G$ and $\mathcal{S}_H$. Each of these, then, consists of certain kinds of $V^{(r)}$, diagonalizable within $\mathrm{K}(n)$. After dealing with $\mathcal{S}_G$ and $\mathcal{S}_H$, we take up $\mathcal{S}_K$, and thereafter explore possible canonical forms for general $V^{(r)}$ in $\mathcal{S}$.

A. The family $\mathcal{S}_G$

This family consists of those $V^{(r)}$ which apart from a factor of $\frac{1}{2}$ are also elements of $\mathrm{Sp}(2n,\mathbb{R})$:

$$\mathcal{S}_G=\{V^{(r)}=\tfrac{1}{2}S\,|\,S\in \mathrm{Sp}(2n,\mathbb{R}), S^T=S, S>0\} \ . \quad (4.1)$$

It is known that symmetric positive-definite symplectic matrices correspond one-to-one to points of the coset space $\mathcal{O}=\mathrm{Sp}(2n,\mathbb{R})/\mathrm{K}(n)$. (This is seen, for instance, from the polar decomposition.) It is a fact that the $V^{(r)}$ occurring in the definition (4.1) do obey the uncertainty principles; this is most easily seen by applying Williamson's theorem to S. Finally, it is also known that this family of $\mathrm{Sp}(2n,\mathbb{R})$ matrices can be parametrized globally and smoothly by two real symmetric $n\times n$ matrices u and v, with u being positive definite:

$$\begin{aligned} &S\in \mathrm{Sp}(2n,\mathbb{R}),\quad S^T=S\ ,\quad S>0\\ &\Longrightarrow S=S(u,v)=\begin{bmatrix} u^{-1} & -uv\\ -vu & u+vu^{-1}v\end{bmatrix}\ , \quad (4.2)\\ &u^T=u>0\ ,\quad v^T=v\ . \end{aligned}$$

All this is consistent with the dimensionality of the coset space $\mathcal{O}=\mathrm{Sp}(2n,\mathbb{R})/\mathrm{K}(n)$, and so of the family $\mathcal{S}_G$, being $n(n+1)$.

That $\mathcal{S}_G$ indeed lies within $\mathcal{S}_K$ is a consequence of the property that every matrix of the type $S(u,v)$ can be diagonalized through conjugation by a suitable element of $\mathrm{K}(n)$:

$$\begin{aligned} &S(u,v)=R^T\Lambda R\ ,\\ &R\in \mathrm{K}(n)\ , \quad (4.3)\\ &\Lambda=\mathrm{diag}(s_1^2,s_2^2,\dots,s_n^2,s_1^{-2},s_2^{-2},\dots,s_n^{-2})>0\ . \end{aligned}$$

[For ease in writing we have denoted by R rather than by $S^{(r)}(X,Y)$ the necessary element of $\mathrm{K}(n)$ here.] Clearly, the diagonal entries in Λ are the eigenvalues of $S(u,v)$, and we see that they occur in reciprocal pairs. It now follows that the $\mathrm{K}(n)$ canonical form of $V^{(r)}\in\mathcal{S}_G$ is diagonal; for any such $V^{(r)}$, there is a suitable $R\in \mathrm{K}(n)$ such that

$$\begin{aligned}(V_0)_G&=RV^{(r)}R^T\\ &=\tfrac{1}{2}\mathrm{diag}(s_1^2,s_2^2,\dots,s_n^2,s_1^{-2},s_2^{-2},\dots,s_n^{-2})\ . \quad (4.4)\end{aligned}$$

It is now again evident that the elements of $\mathcal{S}_G$ do satisfy, in fact they saturate, the uncertainty principles (2.32), and thus are indeed bona fide variance matrices. We also see that except for the isolated case $V^{(r)}=\frac{1}{2}1\in\mathcal{S}_G$, every other $V^{(r)}\in\mathcal{S}_G$ *is squeezed by our criterion.*

In passing we may note that this $n(n+1)$-parameter family can be trivially extended to an $(n(n+1)+1)$-parameter family of variance matrices of the form $\sigma V^{(r)}$, with $V^{(r)}\in\mathcal{S}_G$ and σ real positive. The uncertainty principle (2.32) imposes the condition $1\leq\sigma<\infty$, and then $\sigma V^{(r)}$ is physically realizable.

B. The family $\mathcal{S}_H$

In this second special family each element is fully determined by an $n\times n$ Hermitian positive-definite matrix H subject to further conditions to be derived below:

$$\begin{aligned} &H=H^\dagger=C+iD\ ,\quad H>0\ ,\\ &V^{(r)}(H)=\begin{bmatrix} C & -D\\ D & C\end{bmatrix}\ . \end{aligned} \quad (4.5)$$

The real matrices C and D are, respectively, symmetric and antisymmetric. The complex form of this variance matrix is given by Eq. (2.38):

$$V^{(c)}(H)=\begin{bmatrix} H & O\\ O & H^*\end{bmatrix}\ . \quad (4.6)$$

This is a particular case of the general structure (2.37), with vanishing B. Now, according to Eqs. (2.44) and (3.15), if $U=X-iY$ is any element of $\mathrm{U}(n)$, the effect on $V^{(c)}(H)$ is simply given by

$$\begin{aligned} V^{(c)}(H)&\to S^{(c)}(U)V^{(c)}(H)S^{(c)}(U)^\dagger\\ &=\begin{bmatrix} UHU^\dagger & 0\\ 0 & U^*H^*U^T\end{bmatrix}\ . \end{aligned} \quad (4.7)$$

It is this $\mathrm{K}(n)$ transformation law, together with the fact that any Hermitian matrix can be diagonalized by a unitary transformation, that has motivated the definition of this family $\mathcal{S}_H$. Thus the $\mathrm{K}(n)$ canonical forms of $V^{(r)}(H)$ and $V^{(c)}(H)$ are obtained by choosing $U\in\mathrm{U}(n)$ so that $UHU^\dagger$ is diagonal, and so are themselves diagonal:

$$\begin{aligned} &U=X-iY\in U(n)\ ,\\ &UHU^\dagger=\mathrm{diag}(\kappa_1,\kappa_2,\dots,\kappa_n)\ : \quad (4.8)\\ &(V_0)_H=S^{(c)}(U)V^{(c)}(H)S^{(c)}(U)^\dagger\\ &\quad=S^{(r)}(X,Y)V^{(r)}(H)S^{(r)}(X,Y)^T\\ &\quad=\mathrm{diag}(\kappa_1,\kappa_2,\dots,\kappa_n,\kappa_1,\kappa_2,\dots,\kappa_n)\ . \end{aligned}$$

We see that the eigenvalues κ_j, $j=1,2,\dots,n$, of H are also the doubly degenerate eigenvalues of $V^{(r)}(H)$ [and of

$V^{(c)}(H)$], and the uncertainty principle (2.32) imposes the condition $\kappa_j \geq \frac{1}{2}$, $j=1,\ldots,n$; that is, it demands $H\geq(\frac{1}{2})\mathbb{1}$. The definition of $\mathcal{S}_H$ can now be given in full detail as

$$\mathcal{S}_H=\left\{ V^{(r)}(H)=\begin{bmatrix} C & -D \\ D & C \end{bmatrix} \middle| H=C+iD=H^\dagger\geq\tfrac{1}{2} \right\} . \tag{4.9}$$

It is evident that by our criterion *no* element of $\mathcal{S}_H$ is squeezed.

The normal form (4.8) of an element in $\mathcal{S}_H$ has n independent parameters. The collection $\mathcal{S}_H$ is the union of the orbits under $K(n)$ of all such variance matrices in normal form. However, $\mathcal{S}_H$ is an n^2-parameter family, rather than an $n(n+1)$-parameter family, of matrices. This happens because a variance matrix in the diagonal form (4.8) generically possesses invariance under the subgroup $SO(2)\times SO(2)\times\cdots\times SO(2)\subset K(n)$, the n factors being independent phase-space rotations in the q_j and p_j planes for $j=1,2,\ldots,n$. This is precisely how n parameters are *lost*.

It is easy to examine to what extent the two special families $\mathcal{S}_G$ and $\mathcal{S}_H$ intersect. We have seen above that *every* $V^{(r)}\in\mathcal{S}_H$ is *not* squeezed, while every $V^{(r)}\in\mathcal{S}_G$ other than $(\frac{1}{2})\mathbb{1}$ *is* squeezed. Now the matrix $(\frac{1}{2})\mathbb{1}$ is present in $\mathcal{S}_H$, too, so we conclude

$$\mathcal{S}_G\cap\mathcal{S}_H=\{\tfrac{1}{2}\mathbb{1}\} . \tag{4.10}$$

This common element is the variance matrix of any n-mode coherent state.

C. The family $\mathcal{S}_K$

We now move up to consideration of this case, having seen how to characterize $\mathcal{S}_G,\mathcal{S}_H\subset\mathcal{S}_K$. The definition is

$$\mathcal{S}_K=\{V^{(r)}\in\mathcal{S}\,|\,R^TV^{(r)}R=(\text{diagonal}) ,\ \text{suitable } R\in K(n)\} . \tag{4.11}$$

We can see from Eqs. (2.18), (2.37), and (2.40) that a variance matrix $V^{(r)}$ being diagonal corresponds exactly to the submatrices A and B entering the complex form $V^{(c)}$ being simultaneously real diagonal. It turns out that a complete characterization of elements $V^{(r)}\in\mathcal{S}_K$ is most concisely stated in terms of A and B entering $V^{(c)}$. But we first state a property of complex symmetric matrices which is needed for this purpose.

Lemma 1. If M is a complex symmetric matrix of dimension n, $M^T=M$, it possesses an Euler decomposition of the form

$$M=U^TM_0U , \tag{4.12}$$

where M_0 is real diagonal positive-semidefinite and $U\in U(n)$.

The proof of this lemma is a straightforward analysis of the process of diagonalizing the Hermitian matrices $MM^\dagger=MM^*$ and $M^\dagger M=M^*M$. Note that while M_0 in (4.12) is unique up to the sequence of its diagonal elements, U is in general not unique. Its arbitrariness is to the extent of $\mathcal{O}(n_1)\times\mathcal{O}(n_2)\times\cdots\times\mathcal{O}(n_k)$ transformations, where $n_1,n_2,\ldots,n_k$ are the multiplicities of the distinct eigenvalues of the diagonal M_0, $\sum_{\alpha=1}^{k}n_\alpha=n$.

With the help of this lemma, we can characterize the elements of $\mathcal{S}_K$ very simply and elegantly. We have the result

Theorem 5. The $K(n)$ canonical form of a variance matrix $V^{(r)}\in\mathcal{S}$ is diagonal, $V^{(r)}\in\mathcal{S}_K$, if and only if its complex form $V^{(c)}$ has blocks A and B obeying $AB=BA^*$.

Note that the properties $A^\dagger=A$, $B^T=B$ allow us to express the condition $AB=BA^*$ in the equivalent form $AB=(AB)^T$.

First we prove the necessity of this condition, and next the sufficiency (which is where the lemma comes in). Under the action by $U\in K(n)=U(n)$, we know from Eq. (2.44) and (3.15) that $A\to UAU^\dagger$, $B\to UBU^T$. Thus the rule for AB is $AB\to UABU^T$. Further, when $V^{(r)}$ is diagonal, we have noted above that A and B are simultaneously real diagonal, so AB is real diagonal, hence symmetric. Hence by the transformation rule just given for AB, the necessity is proved.

The sufficiency involves a moderate amount of effort. We begin with Hermitian A and symmetric B obeying $(AB)^T=AB$. Choose $U_1\in U(n)$ to diagonalize A.

$$\begin{aligned} A\to A'&=U_1AU_1^\dagger \\ &=\text{diag}(a_1,a_2,\ldots,a_n) ,\quad a_j>0 . \end{aligned} \tag{4.13}$$

If the eigenvalues of A are nondegenerate, then the condition $(A'B')^T=A'B'$, where $B'=U_1BU_1^T$, implies that B' is also diagonal. However this could continue to be complex:

$$\begin{aligned} B'&=U_1BU_1^T \\ &=\text{diag}(b_1e^{i\phi_1},b_2e^{i\phi_2},\ldots,b_ne^{i\phi_n}) ,\quad b_j\geq 0 . \end{aligned} \tag{4.14}$$

Now follow up the U_1 action with the action by

$$U_2=\text{diag}(\pm e^{-i\phi_1/2},\pm e^{-i\phi_2/2},\ldots,\pm e^{-i\phi_n/2})\in U(n) , \tag{4.15}$$

where the sign at each entry may be chosen independently. Then we see that

$$\begin{aligned} A'&\to A''=U_2A'U_2^\dagger=A' , \\ B'&\to B''=U_2B'U_2^T=\text{diag}(b_1,\ldots,b_n) . \end{aligned} \tag{4.16}$$

This completes the proof of sufficiency in this nondegenerate case: $U=U_2U_1\in U(n)$ carries A and B to A'' and B'', which are both real diagonal; hence the corresponding $S^{(r)}(X,Y)$ (where $U=X-iY$) takes $V^{(r)}$ to diagonal form:

$$V^{(r)} \to (V_0)_K = S^{(r)}(X,Y)V^{(r)}S^{(r)}(X,Y)^T$$
$$= \tfrac{1}{2}\,\mathrm{diag}(a_1+b_1,a_2+b_2,\ldots,a_n+b_n,a_1-b_1,a_2-b_2,\ldots,a_n-b_n)\ , \quad a_j > b_j \geq 0\ , \quad j=1,2,\ldots,n\ . \tag{4.17}$$

It is clear that in this nondegenerate case there is a $Z_2 \times Z_2 \times \cdots \times Z_2$ freedom in the choice of U_2 (apart from the freedom in U_1 corresponding to the ordering of the diagonal elements of A'), so this is the nature of the stability group of the canonical form.

On the other hand, if A has degenerate eigenvalues, we modify the argument as follows. Let the distinct eigenvalues $(a_1,\ldots,a_k)$ have respective multiplicities $n_1,n_2,\ldots,n_k$ such that $\sum_{\alpha=1}^{k} n_\alpha = n$. When we pass from A to its diagonal form A' through Eq. (4.13), the condition $(AB)^T = AB$ forces B' into a block-diagonal form:

$$A' = U_1 A U_1^\dagger$$
$$= \mathrm{diag}(a_1 \cdots a_1 a_2 \cdots a_2 \cdots a_k \cdots a_k)$$
$$\Longrightarrow B' = U_1 B U_1^T$$
$$= \mathrm{block-diag}(B^{(1)}B^{(2)} \cdots B^{(k)})\ ,$$
$$\dim B^{(\alpha)} = n_\alpha\ , \quad \alpha = 1,2,\ldots,k\ . \tag{4.18}$$

Each submatrix $B^{(\alpha)}$ along the diagonal is complex symmetric. Now we appeal to the lemma stated above and choose $U_2 \in \mathrm{U}(n)$ also in block diagonal form so as to bring each $B^{(\alpha)}$ into real diagonal positive-semidefinite form:

$$U_2 = \mathrm{block-diag}(U_2^{(1)}U_2^{(2)} \cdots U_2^{(k)}) \in \mathrm{U}(n)\ ,$$
$$U_2^{(\alpha)} \in \mathrm{U}(n_\alpha)\ ,$$
$$U_2^{(\alpha)} B(\alpha) U_2^{(\alpha)T} = \mathrm{diag}(b_1^{(\alpha)}, b_2^{(\alpha)}, \ldots, b_{n_\alpha}^{(\alpha)})\ , \tag{4.19}$$
$$b_\beta^{(\alpha)} \geq 0\ , \quad \beta = 1,2,\ldots,n_\alpha\ , \quad \alpha = 1,2,\ldots,k\ .$$

Under the action of U_2, A' remains unaltered and diagonal, while B' has been carried to B'' which is diagonal with real non-negative entries. Hence $S^{(r)}(X,Y)$ corresponding to $U = U_2 U_1 = X - iY \in U(n)$ diagonalizes $V^{(r)}$, and the proof is complete.

One can analyze the extent to which each $U_2^{(\alpha)}$ used in Eq. (4.19) is nonunique by counting the degeneracies in the diagonal elements $b_1^{(\alpha)}, b_2^{(\alpha)}, \ldots, b_{n_\alpha}^{(\alpha)}$ when $B^{(\alpha)}$ is carried to diagonal form. This will then disclose the stability group of $V^{(r)}$, but we forego the details. It is evident that the generic situation in $\mathcal{S}_K$ is the nondegenerate one described by Eq. (4.17), and then the stability group of the K(n) canonical diagonal form of $V^{(r)}$ is discrete (we may assume for definiteness that the nondegenerate eigenvalues of A have been ordered, say, as an increasing sequence $a_1 < a_2 < \cdots < a_n$). Since, again generically, we have $2n$ independent eigenvalues for $V^{(r)}$ appearing in Eq. (4.17), and n^2 parameters in K(n), we see that $\mathcal{S}_K$ is an $n(n+2)$-parameter family.

For $V^{(r)} \in \mathcal{S}_G$ and $V^{(r)} \in \mathcal{S}_H$, respectively, we have expressed the diagonal forms of $V^{(r)}$ as in Eqs. (4.4) and (4.8). Since $\mathcal{S}_G, \mathcal{S}_H \subset \mathcal{S}_K$, these are special instances of $V^{(r)} \in \mathcal{S}_K$. It is therefore natural to rewrite the K(n) canonical diagonal form (4.17) for a general $V^{(r)} \in \mathcal{S}_K$ in this way:

$$(V_0)_K = \mathrm{diag}(s_1^2\kappa_1, \ldots, s_n^2\kappa_n, s_1^{-2}\kappa_1, \ldots, s_n^{-2}\kappa_n)\ , \tag{4.20}$$

with real positive κ_j, s_j. Now the uncertainty principle (2.32) translates into $\kappa_j \geq \frac{1}{2}$, and does not involve s_j at all. This is to be expected since the s_j do not enter the Williamson normal form (2.27) at all. Indeed, starting from (4.20) and scaling away the s_j we have

$$\mathring{V}^{(r)} = \mathrm{diag}(\kappa_1, \ldots, \kappa_n, \kappa_1, \ldots, \kappa_n)\ . \tag{4.21}$$

Returning to Eq. (4.20), we see that the $\mathcal{S}_G$ subfamily corresponds to $\kappa_j = \frac{1}{2}$, and the $\mathcal{S}_H$ subfamily to $s_j = 1$.

It is important to appreciate that the condition $(AB)^T = AB$ is a very stringent requirement on the blocks of the complex variance matrix $V^{(c)}$. Even quite simple looking variance matrices may fail to obey it. For instance, in the two-mode case, the following variance matrix has recently [20] played an important role in the study of a new kind of twisted state:

$$V^{(r)} = \frac{1}{2}\begin{pmatrix} a & 0 & 0 & c \\ 0 & a & -c & 0 \\ 0 & -c & b & 0 \\ c & 0 & 0 & b \end{pmatrix}\ , \quad a,b > 0\ . \tag{4.22}$$

The uncertainty principle (2.32) requires $ab - c^2 \geq 1$. It may be checked that in this example, the further demand $(AB)^T = AB$ forces the equality of a and b.

To conclude this discussion of the family $\mathcal{S}_K$, we may ask how the subfamilies $\mathcal{S}_G$ and $\mathcal{S}_H$ obey the condition $(AB)^T = AB$. For $\mathcal{S}_H$, from Eq. (4.6) the submatrix B vanishes, so it is a trivial situation. In the case of $\mathcal{S}_G$, it is a consequence of the symplectic condition (2.41a).

D. The family $\mathcal{S}$

Finally, we turn our attention to $\mathcal{S}$, the full set of all physically realizable n-mode variance matrices. We want to find K(n) canonical forms for general $V^{(r)} \in \mathcal{S}$. We have shown already through consideration of dimensionality that such a form cannot generically be diagonal, for K(n) is too small a group and $\mathcal{S}$ too large a family. We now develop two interesting canonical forms.

1. *First canonical form*

Take a general complex variance matrix $V^{(c)}$ with block matrices A and B. Choose $U_1 \in \mathrm{U}(n)$ to put $A' = U_1 A U_1^\dagger$ into diagonal form. Since we no longer have the symmetry $(AB)^T = AB$, the resulting $B' = U_1 B U_1^T$ does not have any specific form, though it will be complex symmetric. Let the diagonal elements of B' (when nonzero) have phases $e^{i\varphi_1}, e^{i\varphi_2}, \ldots, e^{i\varphi_n}$. Now choose a diagonal $U_2 \in \mathrm{U}(n)$, as in Eq. (4.15), to make the diagonal elements of $B''=U_2 B' U_2^T$ real non-negative. In the process $A''=U_2 A' U_2^\dagger = A'$ is unchanged. Then the combined transformation $U=U_2U_1 \in \mathrm{U}(n)$ takes the original $V^{(c)}$ to a new $V^{(c)''}$ for which A'' is real diagonal positive definite and B'' is complex symmetric with real non-negative diagonal elements. Through Eqs. (2.40) this implies for the real variance matrix $V^{(r)''}$:

$$\begin{aligned} &V_1''+V_3''=(\text{diagonal}) , \\ &V_2''^T=V_2'' , \\ &(V_2'')_{jj}=0 , \quad j=1,2,\ldots,n . \end{aligned} \tag{4.23}$$

The number of independent parameters left here can be counted as n in $V_1''+V_3''$; $n(n+1)/2$ in $V_1''-V_3''$; and $n(n-1)/2$ in V_2''. This adds up to $n(n+1)$, exactly the difference between $n(2n+1)$, the dimension of $\mathcal{S}$, and n^2, the dimension of $\mathrm{K}(n)$. Therefore we indeed have in Eq. (4.23) a $\mathrm{K}(n)$ canonical form for general $V^{(r)} \in \mathcal{S}$. However, none of the diagonal entries in this form is expected to be an eigenvalue of $V^{(r)}$, so we turn our attention to another possibility.

2. Second canonical form

This is constructed by repeated appeal to the fact, proved in Sec. III, that $\mathrm{K}(m)$ acts transitively on S^{2m-1}, for every m. Given any $V^{(r)} \in \mathcal{S}$, choose any one of its eigenvalues , λ_1, say, and a corresponding normalized real eigenvector $x^{(1)}$:

$$\begin{aligned} &V^{(r)}x^{(1)}=\lambda_1 x^{(1)} , \quad \lambda_1>0 , \\ &x^{(1)T}x^{(1)}=1 . \end{aligned} \tag{4.24}$$

Transitive $\mathrm{K}(n)$ action on S^{2n-1} means that there exists an element $R_1 \in \mathrm{K}(n)$ [in fact, an entire $\mathrm{K}(n-1)$ coset of elements] such that $R_1 x^{(1)}=e_1$, the first unit basis vector defined in Eq. (3.18). So, transformation by R_1 reduces $V^{(r)}$ to this form:

$$R_1 V^{(r)} R_1^T = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ 0 & & & & \square & \cdots & \cdots & \\ \vdots & & & & \vdots & & & \\ 0 & & & & \square & & & \\ 0 & \square & \cdots & \square & \square & \cdots & \cdots & \square \\ 0 & & \cdots & & & & \cdots & \\ \vdots & & & & \vdots & & & \\ 0 & & & & \square & & & \end{bmatrix} . \tag{4.25}$$

Here the leading matrix element is the chosen eigenvalue λ_1, and all other entries in the first row and first column vanish.

Now we iterate,and perform a second canonical rotation R_2 leaving $x^{(1)}$, the *direction* of the new q_1, unaffected. As noted in Sec. III, it is a property of canonical rotations that R_2 will necessarily leave the new p_1 also unaffected. That is, the elements in Eq. (4.25) marked $\square$ are left unchanged by R_2. Thus we isolate an $(2n-2)$-dimensional symmetrical submatrix in $R_1 V^{(r)} R_1^T$ by dropping the first and $(n+1)$th rows, and the first and $(n+1)$th columns. We can choose R_2 now to attain

$$R_2 R_1 V^{(r)} R_1^T R_2^T = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_2' & 0 & \cdots & 0 & \square & 0 & \cdots & \cdots & 0 \\ 0 & 0 & & & & \square & \square & & & \\ \vdots & \vdots & & & & \vdots & \vdots & & & \\ 0 & 0 & & & & \square & \square & & & \\ 0 & \square & \square & \cdots & \square & \square & \square & \cdots & \cdots & \square \\ 0 & 0 & \square & \cdots & \square & \square & \square & \cdots & \cdots & \square \\ \vdots & \vdots & & & & \vdots & \vdots & & & \\ 0 & 0 & & & & \square & \square & & & \end{bmatrix} . \tag{4.26}$$

It will be realized that λ_2' is not an eigenvalue of the original $V^{(r)}$ (in general), but of the $(2n-2)$-dimensional submatrix of $R_1 V^{(r)} R_1^T$ referred to above.

This process can be repeated n times and then one has $V^{(r)}$ reduced to the canonical form

$$R_n \cdots R_1 V^{(r)} R_1^T \cdots R_n^T = \begin{bmatrix} \lambda_1 & & & 0 & & \\ & \lambda_2' & \mathbf{0} & & \ddots & \mathbf{0} \\ & \mathbf{0} & \lambda_3'' & & & \ddots \\ & & & \ddots & & 0 \\ 0 & & & & & \\ & \ddots & & & & \\ \mathbf{0} & & \ddots & & & \\ & & & 0 & & \end{bmatrix} . \tag{4.27}$$

That is, in this form the submatrix V_1 is diagonal, while V_2 is lower diagonal with zeros along the diagonal. That this is a canonical form can be checked by counting the number of free parameters left: n in the diagonal V_1, $n(n-1)/2$ in the lower diagonal V_2, and $n(n+1)/2$ in the symmetric V_3, adding up to $n(n+1)$.

While the diagonal elements λ_2', λ_3'', ... in Eq. (4.27) are not expected to be eigenvalues of $V^{(r)}$, λ_1 is one of its eigenvalues. For squeezing problems it is natural to choose λ_1 to be the least eigenvalue of $V^{(r)}$, so that from the canonical form (4.27) one can decide by inspection whether the given $V^{(r)}$ corresponds to a squeezed state or not.

V. PHYSICAL EXAMPLES FOR CANONICAL FORM VARIANCE MATRICES

It is instructive to construct explicit physical examples to illustrate the various canonical forms for variance matrices derived in Sec. IV. All of them can be reproduced using ordinary or squeezed thermal states.

Let us consider first the single-mode case. The thermal state density operator for an oscillator is

$$\hat{\rho}=(1-e^{-\beta})\exp(-\beta\hat{a}^\dagger\hat{a}) . \tag{5.1}$$

with $\beta=\hbar\omega/kT$ the usual thermal parameter. We can rewrite $\hat{\rho}$ in terms of $\bar{n}$, the mean occupation number:

$$\begin{aligned}\bar{n}&=\mathrm{Tr}(\hat{\rho}\hat{a}^\dagger\hat{a})=\frac{1}{e^{\beta}-1} ,\\ \hat{\rho}&=\frac{1}{1+\bar{n}}\left[\frac{\bar{n}}{1+\bar{n}}\right]^{\hat{a}^\dagger\hat{a}} .\end{aligned} \tag{5.2}$$

Since $\hat{\rho}$ is diagonal in the number representation, both $\hat{\xi}(r)$ and $\hat{\xi}(c)$ have zero means, and further $V^{(r)}$ has a simple form:

$$\begin{aligned}&\langle\hat{\xi}^{(r)}\rangle=\langle\hat{\xi}^{(c)}\rangle=0 ,\\ &V^{(r)}=\mathrm{diag}(\bar{n}+\tfrac{1}{2},\bar{n}+\tfrac{1}{2}) .\end{aligned} \tag{5.3}$$

Conversely, given a variance matrix of the canonical form

$$V^{(r)}=\mathrm{diag}(\kappa,\kappa) , \tag{5.4}$$

we can associate it with a thermal state of mean photon number $\bar{n}=\kappa-\frac{1}{2}$ and thermal parameter

$$\begin{aligned}\beta&=\ln[(\kappa+\tfrac{1}{2})/(\kappa-\tfrac{1}{2})]\\ &=2\coth^{-1}(2\kappa) .\end{aligned} \tag{5.5}$$

The uncertainty principle limit $\kappa\geq\frac{1}{2}$ ensures that β is well defined and non-negative.

The fact that (5.4) is the variance matrix of a thermal state means that

$$V^{(r)\prime}=\mathrm{diag}(s^2\kappa,s^{-2}\kappa) \tag{5.6}$$

is the variance matrix of a squeezed thermal state with squeeze factor s and the same thermal factor (5.5) as before. Indeed, since

$$\begin{aligned}&V^{(r)\prime}=S^{(r)}V^{(r)}S^{(r)T} ,\\ &S^{(r)}=\mathrm{diag}\left[s,\frac{1}{s}\right]\in\mathrm{Sp}(2,\mathbb{R}) ,\\ &S^{(c)}=\begin{bmatrix}\cosh\eta & \sinh\eta\\ \sinh\eta & \cosh\eta\end{bmatrix} ,\\ &s=e^{\eta} ,\end{aligned} \tag{5.7}$$

where $S^{(c)}$ is the complex representation (2.15) of the scaling transformation $S^{(r)}$ connecting $V^{(r)}$ and $V^{(r)\prime}$, we see that $V^{(r)\prime}$ is produced by the density operator

$$\begin{aligned}\hat{\rho}'&=\hat{U}(S^{(r)})\hat{\rho}\hat{U}(S^{(r)})^\dagger\\ &=(1-e^{-\beta})\exp[-\beta\{(\cosh^2\eta)\hat{a}^\dagger\hat{a}+(\sinh^2\eta)\hat{a}\hat{a}^\dagger\\ &\qquad+(\cosh\eta)(\sinh\eta)(\hat{a}^2+\hat{a}^{\dagger 2})\}] .\end{aligned} \tag{5.8}$$

In the single-mode case governed by $\mathrm{Sp}(2,\mathbb{R})=\mathrm{SU}(1,1)$, the maximal compact subgroup is $\mathrm{SO}(2)\sim\mathrm{U}(1)$. Thus *all* phase-space rotations are canonical, $\mathrm{SO}(2)\cap\mathrm{Sp}(2,\mathbb{R})=\mathrm{SO}(2)$, a situation that does not generalize for $n>1$. The K(1) canonical form for a single-mode variance matrix is always diagonal.

The variance matrix (5.6) arose from (5.4) by squeezing along $\hat{q}$ or $\hat{p}$. If the squeezing was done along a general phase-space direction, we would get a *rotated version* $V^{(r)\prime\prime}$ of $V^{(r)\prime}$ in (5.6): it would be symmetric positive definite but nondiagonal. Even for such a general squeezed thermal state it is true that $(\det V^{(r)\prime\prime})^{1/2}$ plays the role of κ in fixing the thermal parameter, while the fourth root of the ratio of the eigenvalues of $V^{(r)\prime\prime}$ would determine the squeeze factor. In this rotated case $\hat{a}^2+\hat{a}^{\dagger 2}$ in the exponent in Eq. (5.8) is replaced by $e^{2i\varphi}\hat{a}^2+e^{-2i\varphi}\hat{a}^{\dagger 2}$, so the operator in the exponent multiplying $-\beta$ would be the most general Hermitian quadratic expression in $\hat{a}$ and $\hat{a}^\dagger$ consistent with positive definiteness.

We can summarize these considerations for the single-mode case by saying that the allowed variance matrices and squeezed thermal states are in one-to-one correspondence. This connection helps us in the multimode case to which we now turn. We consider first the family $\mathcal{S}_G$.

A. The family $\mathcal{S}_G$

From the canonical form (4.4) we know that these variance matrices saturate the uncertainty principle. It is well known that the only pure states that do so are the squeezed coherent states, including the squeezed vacuum as a special case.

Let $\mathcal{A}$ be the n-parameter Abelian subgroup of $\mathrm{Sp}(2n,\mathbb{R})$ consisting of diagonal positive-definite matrices. Since any multimode coherent state $|\underline{\alpha}\rangle=|\alpha_1,\alpha_2,\ldots,\alpha_n\rangle$ has $V^{(r)}=\frac{1}{2}\mathbb{1}$, since the canonical form (4.4) can be written as

$$\begin{aligned}&(V_0)_G=S_d(\tfrac{1}{2}\mathbb{1})S_d^T ,\\ &S_d=\mathrm{diag}(s_1,\ldots,s_n,s_1^{-1},\ldots,s_n^{-1})\in\mathcal{A} ,\end{aligned} \tag{5.9}$$

and since S_d represents reciprocal scalings along $\hat{q}_j$ and $\hat{p}_j$ independently for each $j=1,2,\ldots,n$, we see that $(V_0)_G$ can be produced by a squeezed coherent state with squeezing along $\hat{q}_j$ or $\hat{p}_j$ for each j.

The fact that $(V_0)_G$ is diagonal leads to

$$\begin{aligned} V^{(r)}\in\mathcal{S}_G \Longrightarrow V^{(r)} &= R(V_0)_G R^T\ ,\quad \text{some } R\in\mathrm{K}(n)\ ,\\ &= \tfrac{1}{2}RS_dR^TRS_dR^T\\ &= \tfrac{1}{2}[S^{(r)}]^2\ ,\\ S^{(r)} = RS_dR^T &= S^{(r)T}\in\mathrm{Sp}(2n,\mathbb{R})\ . \end{aligned} \tag{5.10}$$

Thus the symmetric positive-definite squeeze operator $S^{(r)}\in\mathrm{Sp}(2n,\mathbb{R})$ produces the squeezed coherent state variance matrix $V^{(r)}\in\mathcal{S}_G$ from the coherent state value $V^{(r)}=\frac{1}{2}\mathbb{1}$. The last line of Eq. (5.10) is just the Euler decomposition of the squeeze operator. We have thus proved that the variance matrix of a squeezed coherent state is one-half of the square of the unique symmetric positive-definite squeezing matrix in $\mathrm{Sp}(2n,\mathbb{R})$ which produces that squeezed coherent state from a coherent state. Since squaring is a one-to-one on to map of the family of squeezing matrices, we are reassured that the elements in $\mathcal{S}_G$ and (symmetric) squeezing matrices are indeed in one-to-one correspondence. Each of these families corresponds to the coset space $\mathrm{Sp}(2n,\mathbb{R})/\mathrm{K}(n)$.

B. The family $\mathcal{S}_H$

In this case the normal form (4.8) in the multimode situation corresponds to each mode having a variance of the form (5.4), with no correlation between different modes. Thus $(V_0)_H$ of Eq. (4.8) is produced by a multimode density operator which is a product of operators of the form (5.1), one factor per mode:

$$\hat{\rho}=\prod_{j=1}^{n}(1-e^{-\beta_j})\exp(-\beta_j\hat{a}_j^\dagger\hat{a}_j)\ . \tag{5.11}$$

The thermal parameters are

$$\beta_j(\kappa_j)=\ln[(\kappa_j+\tfrac{1}{2})/(\kappa_j-\tfrac{1}{2})]\ ,\quad j=1,2,\ldots,n\ . \tag{5.12}$$

Different values for the different β_j need not mean different temperatures for the various modes, since their frequencies could differ. Going back to general $V^{(r)}\in\mathcal{S}_H$, we see that variance matrices in this family correspond to $\mathrm{K}(n)$ transforms of thermal states.

C. The family $\mathcal{S}_K$

Comparing the canonical form $(V_0)_K$ in Eq. (4.20) with the single-mode case (5.6) we see that $(V_0)_K$ corresponds to the *uncorrelated* multimode squeezed thermal state. Each mode with thermal parameter $\beta_j(\kappa_j)$ has squeeze parameter s_j; the modes of the thermal state are single-mode squeezed by independent amounts and hence remain uncorrelated even after squeezing. The corresponding density operator, based on Eq. (5.8), is

$$\begin{aligned} \hat{\rho}=\prod_{j=1}^{n}(1-e^{-\beta_j})\exp[-\beta_j\{&(\cosh^2\eta_j)\hat{a}_j^\dagger\hat{a}_j\\ &+(\sinh^2\eta_j)\hat{a}_j\hat{a}_j^\dagger\\ &+(\cosh\eta_j)(\sinh\eta_j)\\ &\times(\hat{a}_j^2+\hat{a}_j^{\dagger 2})\}]\ ,\\ \beta_j(\kappa_j)=\ln[(\kappa_j+\tfrac{1}{2})&/(\kappa_j-\tfrac{1}{2})]\ ,\\ \eta_j=\ln s_j\ .& \end{aligned} \tag{5.13}$$

Thus the family $\mathcal{S}_K$ can be viewed as arising from $\mathrm{K}(n)$ transforms of uncorrelated squeezed thermal states.

D. The family $\mathcal{S}$

Finally we come to the family $\mathcal{S}$ of all possible bona fide variance matrices $V^{(r)}$. As already seen in Sec. IV, elements in $\mathcal{S}$ outside of $\mathcal{S}_K$ [being an $n(n+2)$-parameter family, $\mathcal{S}_K$ is a subset of measure zero within the $n(2n+1)$-parameter family $\mathcal{S}$] do not admit a diagonal $\mathrm{K}(n)$ canonical form. We have also noted that any $V^{(r)}\in\mathcal{S}$ admits the diagonal Williamson canonical form (2.27):

$$\begin{aligned} &V^{(r)}\in\mathcal{S}\Longrightarrow V^{(r)}=(S^{(r)})^{-1}\mathring{V}^{(r)}((S^{(r)})^{-1})^T\ ,\\ &\mathring{V}^{(r)}=\mathrm{diag}(\kappa_1,\kappa_2,\ldots,\kappa_n,\kappa_1,\kappa_2,\ldots,\kappa_n)\ ,\\ &S^{(r)}\in\mathrm{Sp}(2n,\mathbb{R})\ . \end{aligned} \tag{5.14}$$

Comparing this $\mathring{V}^{(r)}$ with $(V_0)_H$ in Eq. (4.8), we see that the Williamson canonical form of $V^{(r)}$ is simply a thermal state with thermal parameters $\beta_j(\kappa_j)$. However, the family $\mathcal{S}$ differs from (and is much larger than) the family $\mathcal{S}_H$ because now we allow the *full* symplectic group $\mathrm{Sp}(2n,\mathbb{R})$ rather than just the canonical rotation subgroup $\mathrm{K}(n)$. Thus $\mathcal{S}$ can be viewed as the set of variance matrices of all states obtained from the thermal state $(V_0)_H$ by all possible elements of $\mathrm{Sp}(2n,\mathbb{R})$.

Starting from $(V_0)_H$'s and performing all $\mathrm{K}(n)$ transformations we obtain, of course, the family $\mathcal{S}_H$. If again starting from $(V_0)_H$'s we perform all transformations of the subset (not subgroup) $\Sigma\subset\mathrm{Sp}(2n,\mathbb{R})$ defined by

$$\Sigma=\left\{\frac{S^{(r)}\in\mathrm{Sp}(2n,\mathbb{R})}{S^{(r)}=RS_d\ ,\quad R\in\mathrm{K}(n)\ ,\quad S_d\in\mathcal{A}}\right\}\ , \tag{5.15}$$

then clearly we obtain $\mathcal{S}_K$. That the $\mathrm{K}(n)$ canonical form of a general $V^{(r)}\in\mathcal{S}$ is not diagonal can now be understood as arising from the facts that a general $S^{(r)}\in\mathrm{Sp}(2n,\mathbb{R})$ has the decomposition $S^{(r)}=RS_dR'$, R and $R'\in\mathrm{K}(n)$, $S_d\in\mathcal{A}$, rather than $S^{(r)}=RS_d$ alone, and that $R'\in\mathrm{K}(n)$ does not generically commute with a $(V_0)_H$ and hence acting on $(V_0)_H$ produces correlations between the modes.

It is in this respect that the family $\mathcal{S}_G$ is special. The Williamson normal form of its elements consists of a fixed matrix $\mathring{V}^{(r)}=\frac{1}{2}\mathbb{1}$, corresponding to any coherent state, so every $R'\in\mathrm{K}(n)$ commutes with this $\mathring{V}^{(r)}$. Thus we see why the $\mathrm{K}(n)$ canonical form of any $V^{(r)}\in\mathcal{S}_G$ is diagonal, even though the full group $\mathrm{Sp}(2n,\mathbb{R})$ acts transitively

TABLE I. Summary of normal forms of variance matrices.

Family	Dimension	$K(n)$ normal form	Stability groups of $K(n)$ normal	Williamson normal form	Transitive action by
$\mathcal{S}_G$	$n(n+1)$	$\frac{1}{2}\text{diag}(\ldots,s_j^2,\ldots,\ldots,s_j^{-2},\ldots)$	discrete	$\frac{1}{2}1$	$Sp(2n,\mathbb{R})$
$\mathcal{S}_H$	n^2	$\text{diag}(\ldots,\kappa_j,\ldots,\ldots,\kappa_j,\ldots)$	$U(1)\times U(1)\times\cdots\times U(1)$	$\text{diag}(\ldots,\kappa_j,\ldots,\ldots,\kappa_j,\ldots)$	$K(n)$
$\mathcal{S}_K$	$n(n+2)$	$\text{diag}(\ldots,s_j^2\kappa_j,\ldots,\ldots,s_j^{-2}\kappa_j,\ldots)$	discrete	$\text{diag}(\ldots,\kappa_j,\ldots,\ldots,\kappa_j,\ldots)$	$K(n)$
$\mathcal{S}$	$n(2n+1)$	nondiagonal	discrete	$\text{diag}(\ldots,\kappa_j,\ldots,\ldots,\kappa_j,\ldots)$	$Sp(2n,\mathbb{R})$

on this family.

To see our results at a glance, we have collected them in Table I. To conclude this section, we note that the examples we have given are zero-mean Gaussian states, and we have provided illustrative density operators for every conceivable variance matrix. It is the zero-mean and Gaussian conditions that make our examples unique. But the variance matrix $V^{(r)}$ by itself cannot specify the state. For instance, the multimode displacement operator changes the mean (first moment) of $\hat{\xi}^{(r)}$ and hence the state, without affecting $V^{(r)}$ at all. Beyond this, two states with the same means and variances can still differ in their higher moments. This is yet another respect in which the family $\mathcal{S}_G$ enjoys a special status. Specification of a $V^{(r)}\in\mathcal{S}_G$ determines the state up to a displacement (first moments). This is so because every $V^{(r)}\in\mathcal{S}_G$ saturates the uncertainty principle and hence has to necessarily correspond to some Gaussian (squeezed coherent or squeezed vacuum) state. Gaussian states are fully determined by first and second moments. Thus there are no non-Gaussian states with $V^{(r)}\in\mathcal{S}_G$.

VI. CONCLUDING REMARKS

We have presented in this paper a comprehensive analysis of variance matrices and squeezing properties of arbitrary states, pure or mixed, of n-mode quantum systems. The symplectic group $Sp(2n,\mathbb{R})$ underlying the kinematics and dynamics of such systems has been exploited, and the special role played by the maximal compact subgroup $K(n)$ in squeezing problems brought out.

The necessary and sufficient conditions on a given $2n\times2n$ matrix for it to be physically realizable as the variance matrix of some state have been derived and expressed as simple matrix inequalities. A $K(n)$-invariant squeezing criterion has been motivated, and its surprising connection with the eigenvalues of the variance matrix made clear. The $K(n)$ canonical forms for general variance matrices have been worked out, and the subfamily of those matrices diagonalizable within $K(n)$ has been characterized in a concise manner. Squeezed thermal states have been given as examples to illustrate multimode variance matrices and their $K(n)$ canonical forms.

The entire treatment has been given in such a manner as to apply to the real representation in terms of the quadrature components $\hat{q}_j,\hat{p}_j$ as well as to the complex representation in terms of $\hat{a}_j,\hat{a}_j^\dagger$. Formulas for passing between these representations, in a simple form, have been given. It may be of interest to note that this passage is strictly analogous to that between the linear and the circular polarization bases in polarization optics.

We have shown that a multimode state is squeezed if and only if the least eigenvalue $l(V^{(r)})$ of its variance matrix $V^{(r)}$ is less than the coherent state or vacuum fluctuation limit of $\frac{1}{2}$. When this happens we can define the squeeze factor as

$$(\text{squeeze factor})=\frac{1}{\sqrt{2l(V^{(r)})}}\ . \tag{6.1}$$

It is also of interest to ask: How many linearly independent quadrature components are squeezed? The uncertainty principle demands that $m\le n$. Squeezed coherent states generically saturate this inequality [see Eq. (4.3)]. It should be noted, however, that there are also other states which do so. In particular, we can have squeezed thermal states for which $m=n$ [see the canonical form (4.20)].

We have shown that for any given variance matrix we can construct a unique zero-mean Gaussian (squeezed thermal) state which reproduces it. Since such states have Gaussian Wigner distributions, it follows that with every variance matrix there is associated a unique (modulo rigid phase-space displacements) Gaussian Wigner distribution. Squeezed thermal states correspond to distributions centered at the origin, and displaced states to displaced distributions. It should be appreciated that these are one-to-one onto correspondences.

In this paper we have only considered quadrature squeezing. It is of interest to extend this analysis to higher orders. The next leading one involves the fourth moments. It may at first appear that analysis of these (and higher) moments would be considerably more complex than that of the variance matrix. We believe, however, that judicious exploitation of the $Sp(2n,\mathbb{R})$ structure underlying the problem in the spirit of the present work, and due appreciation of the special role played by the maximal compact subgroup $K(n)$, may render the problem tractable. We plan to return to this elsewhere.

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