From sub-Poissonian to super-Poissonian pumping in the micromaser: Corrections to reservoir theory

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We present a quantum theory of the micromaser which allows for arbitrary (sub- as well as super-Poissonian) fluctuations of the pumping beam. It handles the interaction with the active atoms (gain) and cavity decay (loss) on an equal footing. In conventional reservoir theory the rate of change of the cavity field is a sum of the changes due to separate interactions with the individual reservoirs, i.e., the interactions are uncorrelated. In our approach, corrections to reservoir theory arise. They contain the commutator of the gain and loss operators. The magnitude of these additional terms is characterized by the quantity p/N_{ex} where p is a parameter describing pump-beam fluctuations. The parameter is, in fact, the negative of the Mandel Q parameter of the pump beam so that $p=1$ corresponds to regular pumping, $p=0$ to Poissonian one and $p<0$ to super-Poissonian pump beam fluctuations. N_{ex} is the number of excited atoms passing through the cavity during the lifetime of the intracavity field. Thus we recover the conventional reservoir limit if $p=0$ and/or N_{ex} is large. In all other cases the interactions with the gain and loss reservoirs are correlated. We exploit some of the consequences of the additional terms, presenting analytical as well as numerical results for the steady-state photon statistics (mean photon number and photon-number fluctuations, in particular) without resorting to the p expansion.

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I. INTRODUCTION

One of the main themes in quantum optics in recent years has been the search for ways to reduce the noise in the output light of optical systems. One approach has been to make use of squeezed light [1]. In these schemes, light with known properties (e.g., light from lasers, instantaneously in a coherent state) is converted into squeezed light. The generic model for a converter is the optical parametric oscillator. This approach is based on the realization that active optical systems are just too noisy due to the unavoidable presence of spontaneous emission. Therefore, rather than trying directly to generate squeezed light in an active system, its light is converted into squeezed light in a passive nonlinear system.

Another approach has been to re-examine the sources of noise in active systems in order to determine whether they can be reduced or eliminated. In particular, in lasers and masers it was realized that the pumping process is responsible for much of the output noise [2]. It has subsequently been shown by a number of groups that by making the pump process more regular it is possible to reduce substantially the intensity noise in the output light [3–9].

Here we wish to build on the work of Refs. [5] and [6] to further examine the effects of pump statistics on the field inside a micromaser. A micromaser consists of a high-Q microwave cavity which is pumped by a beam of excited atoms. The beam is sufficiently sparse so that at most one atom is in the cavity at any time. Considerable work, both theoretical and experimental, has been carried out on this system [10,11]. It has been found that the field inside a micromaser can be highly nonclassical. In Refs. [5] and [6] a model for the pump beam was developed in which the statistics can vary continuously from Poissonian to regular, and this model was used to derive a master equation for the field. It was found that by using a regular pump it is possible to reduce the intensity noise to one-half the level produced by a Poissonian pump.

The analysis produced in these papers was based on standard reservoir theory. The gain and loss reservoirs are treated independently and their contributions to the rate of change of the field density matrix $d\rho/dt$ are simply added. Guerra *et al.* have recently found from a numerical analysis that in micromasers the interactions with the gain and loss reservoirs are correlated [12]. This results in corrections to $d\rho/dt$ which depend on the commutator of the gain and loss operators. They argued that the effect of these corrections should be small if the photon number is large.

In order to explore these issues in more detail we develop here a description of a micromaser in terms of a discrete map. This description incorporates our previous model for the pump beam so that we can again examine pump statistics which range continuously from Poissonian to regular. In this approach we take into account fully the terms which correlate the interaction of the system with the gain and loss reservoirs. By comparing the mean photon number and the variance which we calculate from the discrete map to those we obtained using the master equation we are able to see the effect of the gain-loss commutator terms. These corrections are found to be small if the mean number of atoms which pass through the cavity in a cavity damping time is large. The general theory is outlined in Sec. II, its consequences are discussed in Sec. III.

In the next section we derive approximate analytical

expressions for the photon-number distribution of the field. These expressions reduce to a known result in the limit of Poissonian statistics [10]. The locations and the widths of the peaks in the number distribution are found. The locations correspond to the steady-state solutions of the semiclassical theory [5,6]. Individual peaks are relatively narrow; they are, in fact, sub-Poissonian. Therefore, when a single peak dominates the distribution the overall photon statistics are sub-Poissonian as well.

In Sec. V we extend our analysis to consider pumping beams with super-Poissonian fluctuations in the number of excited atoms. We adopt a statistical model for the beam for which the number of atoms which arrive in a given time is given by the negative binomial distribution. This leads to a straightforward generalization of our results for sub-Poissonian beams. In particular, we find that the super-Poissonian case can be covered by extending the pump parameter p from the region $0<p<1$, which corresponds to sub-Poissonian pump beams, to $-\infty<p<0$. With this extension, our results derived for the sub-Poissonian case remain valid and describe now the entire sub- and super-Poissonian domain of pump fluctuations. In Sec. VI we present a numerical analysis of the theory outlined in Secs. III–V and, in particular, compare the "exact" numerical results to the analytical approximations of Sec. IV thereby establishing the limits of validity of the analytical approach.

Finally, we summarize our results in Sec. VII along with a discussion of the implications of our findings for reservoir theory.

II. MODEL

Our basic model is that of Ref. [6]. Three-level atoms, with a level scheme shown in Fig. 1, are first excited to the upper laser level with probability p. They then pass through the cavity where they interact with the field, as shown in Fig. 2. The time T between atomic arrivals is assumed to be uniform; the pump statistics are changed by varying the parameter p. The case $p=1$ corresponds to regular injection statistics, and the case $p\to 0$ and $T\to 0$ with the ratio $r=p/T$ fixed (r is the rate of injection of excited atoms) gives Poissonian statistics.

There are two processes which change the field inside the cavity. The first is the passage of an atom through the cavity. In particular, if ρ is the density matrix of the

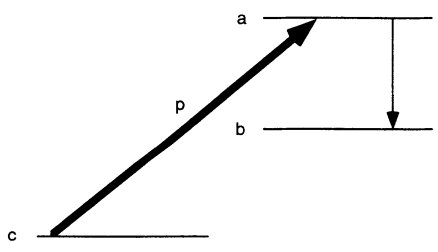

FIG. 1. Level scheme of the active atoms in the micromaser. From a distant ground state $|c\rangle$ the atom is excited to the upper level $|a\rangle$ of the maser transition with probability p. The lower level of the maser transition is $|b\rangle$.

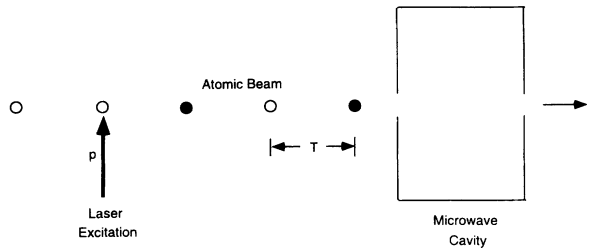

FIG. 2. Scheme of the pumping process in the micromaser. A uniform stream of three-level atoms separated by a time interval T and each in their ground state approaches the micromaser cavity. These atoms are denoted with empty circles. Before entering the cavity some of the atoms are excited to the upper level of the maser transition with probability p. The excited atoms are denoted with dark circles. Thus the probability that at a given arrival time an excited atom will arrive is p and the probability that a ground-state atom will arrive is $1-p$.

single-mode field just before an excited atom is injected, then the density matrix just after the atom has left is given by $M\rho$ where

$$M\rho=\cos(\Lambda\tau)\rho\cos(\Lambda\tau)+g^2a^\dagger\sin(\Lambda\tau)\Lambda^{-1}\rho\Lambda^{-1}\sin(\Lambda\tau)a\ , \tag{2.1}$$

$\Lambda=g\sqrt{aa^\dagger}$, τ is the time the atom spends in the cavity, and g is the atom-field coupling constant. In our model an incoming atom has a probability p of being excited. If the atom is in its ground state, its passage does not affect the field. Therefore, the injection of an atom which has a probability p of being in its upper state causes the field density matrix to go from ρ to $[(1-p)+pM]\rho$. After the passage of K atoms the density matrix is $([(1-p)+pM]^K\rho$. In order to derive a continuous version of the model we note that the number of atoms which arrive in a time t is approximately given by t/T, and so we have that

$$\rho(t)=[(1-p)+pM]^{t/T}\rho(0)\ . \tag{2.2}$$

Taking the time derivative of this equation gives

$$\frac{d\rho}{dt}=(1/T)\ln[1+p(M-1)]\rho\ . \tag{2.3}$$

Between atomic injections the field in the cavity decays due to cavity losses. This is the second process which causes the field to change. If the losses act for a time T the density matrix goes from ρ to $\exp(LT)\rho$ where

$$L\rho=(\gamma/2)(2a\rho a^\dagger-a^\dagger a\rho-\rho a^\dagger a)\ . \tag{2.4}$$

The cavity damping time is given by $1/\gamma$. We shall assume that $\tau\ll T$ so that we can ignore the field decay during the interaction of an atom with the field.

The standard derivation of the master equation for the maser field proceeds from Eqs. (2.3) and (2.4). One assumes that the changes in the density matrix due to the gain process and the loss process can be treated separately, i.e., the interactions with the gain and loss reservoirs

are independent. Therefore, in a time Δt which is small compared to γ^{-1} but much larger than T we have that

$$\Delta\rho=\left[\left[\frac{d\rho}{dt}\right]_{\text{gain}}+\left[\frac{d\rho}{dt}\right]_{\text{loss}}\right]\Delta t\ , \tag{2.5}$$

where $(d\rho/dt)_{\text{gain}}$ is given by the right-hand side of Eq. (2.3) and $(d\rho/dt)_{\text{loss}}=L\rho$. Dividing both sides of Eq. (2.5) by Δt and setting $\Delta\rho/\Delta t$ equal to the coarse-grained time derivative [13] gives us the master equation

$$\frac{d\rho}{dt}=(1/T)\ln[1+p(M-1)]\rho+L\rho\ . \tag{2.6}$$

We can eliminate the auxiliary parameter T from this equation with the substitution $T=p/r$, as discussed at the beginning of this section, which yields the master equation in its final form,

$$\frac{d\rho}{dt}=(r/p)\ln[1+p(M-1)]\rho+L\rho\ . \tag{2.7}$$

This equation was derived and some of its consequences discussed in Refs. [5] and [6]. We will return to the validity of this equation in the discussion following Eq. (2.11).

Let us now step back and consider the evolution of the field density matrix of the micromaser in more detail. What we shall find is that Eq. (2.7), i.e., the assumption of uncorrelated interactions with the reservoirs, is only approximately correct. Let the initial field density matrix be ρ_0. An atom is injected followed by a period of time T before the injection of the next atom. During this time the field decays. Therefore, the field density matrix just before the injection of the second atom is

$$\rho_1=e^{LT}[1+p(M-1)]\rho_0\ . \tag{2.8}$$

Similarly, the density matrix after K atoms have been injected, and just before the injection of the $(K+1)$st, is

$$\rho_K=\{e^{LT}[1+p(M-1)]\}^K\rho_0\ . \tag{2.9}$$

Equation (2.9) completely describes the dynamics of our model. It is this equation which will be used throughout the rest of this paper. For the purpose of comparison with the master equation, Eq. (2.7), it is useful to have a continuous version of Eq. (2.9). This is obtained, as before, by replacing K by t/T. Making this substitution and taking the time derivative of the resulting equation yields

$$\frac{d\rho}{dt}=(1/T)\ln\{e^{LT}[1+p(M-1)]\}\rho\ , \tag{2.10}$$

or, again eliminating T,

$$\frac{d\rho}{dt}=(r/p)\ln\{e^{pL/r}[1+p(M-1)]\}\rho\ . \tag{2.11}$$

Note that this equation is not the same as that derived from the reservoir theory, Eq. (2.7). They would be identical if $[M,L]=0$. This follows from the fact that if two operators, A and B, commute, then $\ln(AB)=\ln A+\ln B$. If this relation is used to evaluate the logarithm in Eq. (2.11), then Eq. (2.7) results. However, as can easily be seen by directly evaluating their commutator, L and M do not commute. Therefore, Eqs. (2.7) and (2.11) differ by terms which are proportional to commutators of L and M. The diagonal elements of $[L,M]$, which enter the photon-number equations, were examined in Ref. [12] and found to be small if the number of atoms injected into the cavity in a damping time $N_{\text{ex}}\equiv r/\gamma$, is large. One would expect, then, that in this regime the differences in photon statistics predicted by Eqs. (2.7) and (2.11) would be small. More generally speaking, the interactions with the gain and loss reservoirs become correlated for sub-Poissonian pumping statistics, as has also been noticed in Refs. [12] and [14]. Although the contribution of the correlation terms quickly decreases with increasing N_{ex} for most cases, it nevertheless remains significant when the photon-number distribution is multipeaked. In particular, the absence of this correlation from Eq. (2.7) may lead to negative probabilities in this case if very high-order terms are kept in the p expansion of the photon statistics [15]. Therefore, Eq. (2.7) can, at best, be regarded as an asymptotic series and the p expansion, suggested in Refs. [5] and [6], should be terminated at a finite order after which the results start to diverge. If this interpretation is adopted, the results of the p-expansion asymptotic series method [16] are virtually indistinguishable from those of the present paper, based on Eqs. (2.9) and (2.11). The purpose of the present paper is to examine the effect of the correlation terms by comparing the properties of the steady states predicted by Eqs. (2.7) and (2.11).

Finally, let us note the following. As has been mentioned earlier, Poissonian injection statistics are obtained in this model by taking the limit $p\to 0$ and $T\to\infty$ with $r=p/T$ fixed. A short calculation shows that in this limit all of the commutator terms vanish and Eq. (2.7) and Eq. (2.11) are identical.

The lesson to be learned from this section is that if the atomic injection statistics are not Poissonian, then the gain and loss terms in the master equation are correlated. This means that the conventional reservoir theory, which treats gain and loss processes as coming from independent reservoirs, must be modified.

III. STEADY STATE

The condition which the field density matrix must satisfy at steady state can be derived in two different ways. The first simply demands that the field be unchanged by the passage of an atom. The second is based on the principle that at steady state the gain must equal the loss. We shall discuss both approaches.

The first approach results from the fact that at steady state the field must be the same just before the passage of the Kth atom as it is just before the passage of the $(K+1)$st. The field density matrices at these two time are related by

$$\rho_{K+1}=e^{LT}[1+p(M-1)]\rho_K\ , \tag{3.1}$$

so that the condition $\rho_{K+1}=\rho_K$ implies that the steady-state density matrix ρ must satisfy

$$\rho=e^{LT}[1+p(M-1)]\rho\ , \tag{3.2}$$

or, equivalently, with $T=p/r$

$$\rho=e^{pL/r}[1+p(M-1)]\rho \ . \tag{3.3}$$

This equation will be used later on in this section to study the steady-state behavior of the system.

The second approach considers gain and loss. Gain is represented by the passage of an excited atom which takes the field density matrix from ρ to $M\rho$. The loss is a result of cavity losses and takes the field from ρ to $e^{Lt}\rho$, where t is the time interval over which the losses operate. In order to obtain the average loss we need to average e^{Lt} over the time intervals between excited atoms. The probability of having n unexcited atoms between two excited atoms, giving a time interval of $(n+1)T$ between excited atoms, is $(1-p)^n p$. The operator describing the average loss between excited atoms is then

$$\sum_{n=0}^{\infty}(1-p)^n pe^{L(n+1)T}=[1-(1-p)e^{LT}]^{-1}pe^{LT} \ . \tag{3.4}$$

The steady-state condition now follows by requiring that a gain event, described by $M\rho$, followed by a mean loss event, described by the operator in Eq. (3.4), returns the field to its original state, i.e.,

$$\rho=[1-(1-p)e^{LT}]^{-1}pe^{LT}M\rho \ . \tag{3.5}$$

After some rearrangement, and with $T=p/r$, this once again yields Eq. (3.3), i.e., the same as the first approach. This approach has the advantage that no averaging of M^n is involved and difficulties with noninteger powers of M can be avoided, in keeping with the spirit of remarks in Ref. [15].

It should be noted that the steady-state density matrix depends on the time, relative to an atomic injection, at which it is defined. We have chosen to examine the density matrix at times just before an atomic injection. We could equally well have chosen to look at times just after an atomic injection. This would give us a different answer. In this system, because of the discrete nature of the injection process, the steady state is characterized by $\rho(xT+nT)=\rho[xT+(n+1)T]$, where $0<x<1$ and n is any positive integer. If $x=0^+$, then we are looking at times just after an atomic injection and if $x=1^-$, we are looking at times just before an injection. What we have is a family of steady-state density matrices characterized by x, $\rho_{ss}(x)$. They are related to each other by $\rho_{ss}(x_1)=e^{(x_1-x_2)TL}\rho_{ss}(x_2)$. If $\gamma T\ll 1$, the differences between density matrices at different values of x are small [17]. This is the situation which we shall consider.

We now want to use Eq. (3.3) to investigate the photon statistics of the steady-state field. The operators M and L (and e^{LT} as well) couple diagonal density matrix elements only to other diagonal density matrix elements in the number representation. For M we have from Eq. (2.1)

$$(M\rho)_{nn}=\cos^2(g\tau\sqrt{n+1})\rho_{nn}+\sin^2(g\tau\sqrt{n})\rho_{n-1,n-1} \ , \tag{3.6}$$

and for e^{LT} [13]

$$(e^{LT}\rho)_{nn}=e^{-n\gamma T}\sum_{m=n}^{\infty}\binom{m}{n}(1-e^{-\gamma T})^{m-n}\rho_{mm} \ . \tag{3.7}$$

Finally, inserting Eqs. (3.6) and (3.7) into Eq. (3.3) yields

$$\rho_n=e^{-n\gamma T}\sum_{m=n}^{\infty}\binom{m}{n}(1-e^{-\gamma T})^{m-n} \times[(1-ps_{m+1}^2)\rho_m+ps_m^2\rho_{m-1}] \ , \tag{3.8}$$

where we have set $\rho_{nn}=\rho_n$ and $s_n=\sin(g\tau\sqrt{n})$ in order to simplify the notation.

It is possible to use Eq. (3.8) to gain information about the mean and the variance of the photon number without first solving for ρ_n. Multiplying both sides of Eq. (3.8) by n and summing gives

$$\begin{aligned}\langle n\rangle&=\sum_{n=0}^{\infty}n\rho_n\\&=\sum_{n=0}^{\infty}\sum_{m=n}^{\infty}n\binom{m}{n}e^{-n\gamma T}(1-e^{-\gamma T})^{m-n}[(1-ps_{m+1}^2)\rho_m+ps_m^2\rho_{m-1}]\\&=\sum_{m=0}^{\infty}[(1-ps_{m+1}^2)\rho_m+ps_m^2\rho_{m-1}]\sum_{n=0}^{\infty}n\binom{m}{n}e^{-n\gamma T}(1-e^{-\gamma T})^{m-n}\\&=e^{-\gamma T}(\langle n\rangle+p\langle s_{n+1}^2\rangle) \ .\end{aligned} \tag{3.9}$$

Solving this for $\langle n\rangle$ gives

$$\langle n\rangle=p\langle s_{n+1}^2\rangle/(e^{\gamma T}-1) \ . \tag{3.10}$$

In order to obtain information about the photon-number variance we multiply both sides of Eq. (3.8) by $n(n-1)$ and sum over n. A calculation similar to that just performed gives

$$\langle n(n-1)\rangle=2p\langle ns_{n+1}^2\rangle/(e^{2\gamma T}-1) \ . \tag{3.11}$$

Let us now assume that the photon-number distribution is strongly peaked about its mean value $n_s=\langle n\rangle$. We then express n as $n=n_s+\Delta n$, where $\langle\Delta n\rangle=0$, and keep only terms of up to second order in Δn. Doing so we find immediately from Eq. (3.10)

$$\begin{aligned}n_s&=p\langle s_{n+1}^2\rangle/(e^{\gamma T}-1)\\&\cong ps_{\langle n\rangle+1}^2/(e^{\gamma T}-1)\\&=p\alpha_{n_s}/(e^{p/N_{ex}}-1) \ ,\end{aligned} \tag{3.12}$$

where, in the last step, we introduced $\alpha_{n_s}=s^2_{\langle n\rangle+1}$ and substituted $\gamma T=p/N_{\rm ex}$. Here, $N_{\rm ex}$ denotes the mean number of excited atoms passing through the cavity in a cavity damping time, which is equal to $p/(\gamma T)\equiv r/\gamma$.

This equation gives a condition for determining n_s. In general, this equation has more than one solution. The solutions are the values of n where the line $[(e^{\gamma T}-1)/pjn$ intersects the curve $\sin^2(g\tau\sqrt{n+1})$. The solutions which correspond to points where $\sin^2(g\tau\sqrt{n+1})$ has a positive slope are unstable and can be ignored [10]. If there is more than one stable solution, then the photon-number distribution can have more than one peak. This will be examined in more detail in the next section. For now, let us assume that

$$(e^{\gamma T}-1)/p>1/[(3\pi/2g\tau)^2-1]\ , \tag{3.13}$$

which guarantees that there is only one stable solution.

Now turning to Eq. (3.11) we have that

$$\langle(n_s+\Delta n)(n_s+\Delta n-1)\rangle = 2p\langle(n_s+\Delta n)(\alpha_{n_s}+\alpha'_{n_s}\Delta n)\rangle/(e^{2\gamma T}-1)\ , \tag{3.14}$$

or upon rearranging

$$v=n_s\{1-n_s\tanh(p/2N_{\rm ex})\}/\{1-(2p/N_{\rm ex})/[\exp(2p/N_{\rm ex})-1]\alpha'_s\}\ , \tag{3.15}$$

where $v=\langle(\Delta n)^2\rangle$. Here we have introduced the notation $\alpha'_s=N_{\rm ex}\alpha'_{n_s}$. Note that if $\alpha'_s<0$ then the photon statistics will be sub-Poissonian.

We wish to compare these results to those derived in Refs. [5] and [6]. Those were derived from Eq. (2.7) which neglects the gain-loss commutator terms. In addition, the logarithm was expanded and only terms of up to first order in p were kept. The results embodied in Eqs. (3.12) and (3.15) contain neither of these approximations. In order to facilitate the comparison with the results of the present paper we shall assume that $\gamma T\ll 1$. Then Eq. (3.12) now becomes

$$n_s=N_{\rm ex}\alpha_{n_s}\ , \tag{3.16}$$

which is identical to the corresponding result in Ref. [10]. The photon-number variance becomes

$$v=n_s\{1-(p/2N_{\rm ex})[1-p^2/(12N_{\rm ex}^2)]n_s\}/\{1-[1-p/N_{\rm ex}]\alpha'_s\}\ . \tag{3.17}$$

The corresponding result in Refs. [5] and [6] is identical except that the terms in the square brackets [ ] are missing from both the numerator and denominator. We see that for $N_{\rm ex}\gg 1$ the difference between the two approaches is of order $1/N_{\rm ex}$ and, therefore, small.

For the purpose of comparing the variance given by Eq. (3.17) with a subsequent result it is useful to evaluate α'_s explicitly. Before doing so let us first introduce some additional notation. If we define $\theta_{\rm int}=\sqrt{N_{\rm ex}}g\tau$, then we can express α_n as $\alpha_n=\sin^2(\theta_{\rm int}\sqrt{n/N_{\rm ex}})$. The parameter $\theta_{\rm int}$ characterizes the strength of the pumping process (i.e., the interaction with the gain reservoir) and is typically of order one to ten. Now differentiating α_n with respect to n and evaluating the result at n_s we find

$$\alpha'_{n_s}=-(\theta_{\rm int}/N_{\rm ex})[1-(n_s/N_{\rm ex})]^{1/2}\ . \tag{3.18}$$

In deriving this result we have made use of Eq. (3.16) and the fact that the slope of s_n^2 is negative at $n=n_s$. Substituting Eq. (3.18) into Eq. (3.17) gives

$$v=n_s\frac{1-(p/2N_{\rm ex})n_s}{1-(1-p/N_{\rm ex})\theta_{\rm int}[1-(n_5/N_{\rm ex})]^{1/2}}\ , \tag{3.19}$$

where we kept only the lowest-order terms in $p/N_{\rm ex}$ in both the numerator and the denominator. It is interesting to note that pump regularity enters the numerator and the denominator in opposing ways. Namely, in the numerator its effect is to reduce photon-number fluctuations whereas in the denominator its effect is to reduce the influence of the micromaser dynamics. Thereby this latter contribution actually increases photon-number fluctuations in the micromaser output.

IV. ANALYTIC APPROXIMATIONS TO THE PHOTON STATISTICS

The analysis in the previous section was based on the assumption that the photon-number distribution has a single dominant peak. In order to determine when this is the case, and to gain further insight into the behavior of the micromaser, we need to find more detailed information about the photon statistics of the cavity field.

The equation for the steady-state density matrix, Eq. (3.8), contains two parameters, p and $N_{\rm ex}$. As we shall see, a natural expansion parameter is $p/N_{\rm ex}=\gamma T$. This means that Eq. (3.8) can be expressed in terms of either $p/N_{\rm ex}$ and p, or $p/N_{\rm ex}$ and $N_{\rm ex}$. Both versions can then be expanded in $p/N_{\rm ex}$, but they lead to different approximations. We shall explore both. We shall find that both approximations agree for small p, as they should, but differ for larger p values. The first of these approximations (with p and $p/N_{\rm ex}$ as independent variables) tends to overestimate the effect of pump regularity, whereas the second (with $N_{\rm ex}$ and $p/N_{\rm ex}$ as independent variables) tends to underestimate it. In general, for single-peaked

situations, the second approximation performs somewhat better.

We first consider the case in which p and p/N_{ex} are the independent parameters. Equation (3.8) is already in a form which is convenient for this purpose as it is expressed in terms of p and $\gamma T = p/N_{ex}$. Keeping terms only up to first order in γT gives

$$\rho_n = (1-n\gamma T)[(1-ps_{n+1}^2)\rho_n + ps_n^2\rho_{n-1}] + (n+1)\gamma T[(1-ps_{n+2}^2)\rho_{n+1} + ps_{n+1}^2\rho_n] . \quad (4.1)$$

This equation satisfies a detailed balance condition. It can be expressed as

$$[1-(n+1)\gamma T]ps_{n+1}^2\rho_n - (n+1)\gamma T(1-ps_{n+2}^2)\rho_{n+1} = (1-n\gamma T)ps_n^2\rho_{n-1} - n\gamma T(1-ps_{n+1}^2)\rho_n . \quad (4.2)$$

A normalizable solution results if the right-hand side of this equation is set equal to zero, i.e.,

$$\rho_n = \frac{(1-n\gamma T)}{n\gamma T}\frac{ps_n^2}{1-ps_{n+1}^2}\rho_{n-1} , \quad (4.3)$$

which yields

$$\rho_n = \rho_0 \prod_{k=1}^{n} \frac{(1-k\gamma T)}{k\gamma T}\frac{ps_k^2}{1-ps_{k+1}^2} , \quad (4.4)$$

or in terms of N_{ex},

$$\rho_n = \rho_0 \prod_{k=1}^{n} \left|\frac{N_{ex}}{k} - p\right| \frac{s_k^2}{1-ps_{k+1}^2} . \quad (4.5)$$

It is clear that this formula is valid only if n is not too large, because for $n > N_{ex}/p$ it gives values of ρ_n which are negative. In the derivation of this result, however, it was assumed that $n\gamma T$ is small. As long as this condition is satisfied the values of ρ_n given by Eq. (4.4) are positive. Note that in the limit $p \to 0$ and $T \to 0$ such that N_{ex} is fixed, which corresponds to Poissonian injection statistics, we obtain the result

$$\rho_n = \rho_0 \prod_{k=1}^{n} (N_{ex}/k)s_k^2 , \quad (4.6)$$

which has been derived previously [10].

One case in which Eq. (4.4) should be a very good approximation is if for some N we have that $s_N = 0$ and $\gamma TN \ll 1$. This corresponds to a certain subset of trapping states. The condition $s_N = 0$ guarantees that $\rho_n = 0$ for $n \geq N$. This means that for all n such that $\rho_n \neq 0$ we have that $\gamma Tn \ll 1$ and, therefore, the expansion in this quantity which was made in deriving Eq. (4.4) is well justified. If it is also the case that $\sqrt{N}g\tau = \pi$, then the photon statistics have only a single peak and the results of the previous section apply.

Let us also note that if $p = 1$, and there is a number state $|n_0\rangle$ such that $s_{n_0+1} = 1$, then the denominator in Eq. (4.3) can vanish. In order to treat this case we go back to the equation which results when the right-hand side of Eq. (4.2) is set equal to zero. We then find that Eq. (4.3) holds for $s_{n+1} \neq 1$, but if $s_{n+1} = 1$, then Eq. (4.3) is replaced by the equation $\rho_{n-1} = 0$.

In general, if Eq. (4.4) is to be a useful approximation to the photon statistics it should be valid for values of n up to N_{ex}. This is the region in which the most important part of the photon-number distribution lies. However, Eq. (4.4) is a good approximation only for those values of n for which $n\gamma T = np/N_{ex} \ll 1$. For this condition to be satisfied around $n \simeq N_{ex}$ we see that p cannot be too large. Our numerical studies, which are discussed in Sec. VI, indicate that Eq. (4.4) is a good approximation for p in the range $0 \leq p \leq 0.3$.

We now wish to examine some of the consequences of Eq. (4.4). In particular it is useful to find the location and width of the peaks in the photon-number distribution which it predicts. These peaks are often the dominant features of the number distribution and knowledge of their properties allows one to understand the behavior of $\langle n \rangle$ and Δn.

The simplest task is to find the location of the peaks. First note that $\rho_n \geq \rho_{n-1}$ if

$$(1-\gamma nT)ps_n^2/[n\gamma T(1-ps_{n-1}^2)] \geq 1 . \quad (4.7)$$

If we assume that $s_{n+1} \cong s_n$, which will be true for $n \gg \theta_{int}/N_{ex}$, then the inequality in Eq. (4.7) will be satisfied if $ps_n^2 \geq n\gamma T$, or in terms of N_{ex},

$$s_n^2 \geq n/N_{ex} . \quad (4.8)$$

For simplicity we shall confine our attention to the regime in which $\theta_{int}^2/N_{ex} \ll 1$ so that the condition $n \gg \theta_{int}^2/N_{ex}$ imposes no restriction. Let us now consider the meaning of Eq. (4.8). If we plot both sides of the inequality versus n we find that n/N_{ex} is a straight line of slope $1/N_{ex}$ and s_n^2 oscillates between 0 and 1. Equation (4.8) will be satisfied between the intersection points of the two curves. In particular, it will be satisfied between an intersection point where s_n^2 has a positive slope (s_n^2 is increasing), and the next intersection point which will occur when s_n^2 has a negative slope (s_n^2 is decreasing). The peak will occur at this latter intersection point. This is because between these intersection points we have that $\rho_n > \rho_{n-1}$, i.e., ρ_n is increasing, but after the second, negative-slope intersection point we have that $\rho_n < \rho_{n-1}$, i.e., ρ_n is decreasing. Therefore, the peaks occur at values of n where Eq. (4.8) is satisfied as an equality, and s_n^2 has a negative slope. Note that this is precisely the condition for finding the value of the mean photon num-

ber n_s which was found in Sec. III from the semiclassical theory [see Eq. (3.16)].

Now that we know where the peaks are let us examine their width. We do so by finding $\ln\rho_n$ and expanding it about the location of a peak. Suppose that a peak is located at $n=n_1$.

We first need to find $\ln\rho_n$ near n_1. We begin by expressing $\ln\rho_n$ as

$$\ln\rho_n=\ln\rho_0+\sum_{k=1}^{n}\ln f(k)\ , \tag{4.9}$$

where

$$f(k)=(1-\gamma kT)ps_k^2/[k\gamma T(1-ps_k^2)]\ . \tag{4.10}$$

Near $k=n_1$ we can approximate $\ln f(k)\cong f(n_1)+\delta k d[\ln f(k)]/dk|_{n_1}$ where $\delta k=k-n_1$. Because n_1 corresponds to a peak we have that $f(n_1)=1$ and $f'(n_1)<0$ from our previous discussion. Therefore, for $\delta n/n_1\ll 1$ we have

$$\begin{aligned}\ln\rho_{n_1+\delta n}&=\ln\rho_{n_1}+\sum_{k=n_1}^{n_1+\delta n}\ln f(k)\\&\cong\ln\rho_{n_1}+f'(n_1)\sum_{\delta k=1}^{\delta n}\delta k\ ,\end{aligned} \tag{4.11}$$

where we have assumed that $\delta n>0$. A similar argument holds for the case $\delta n<0$. Performing the summation we find

$$\begin{aligned}\ln\rho_{n_1+\delta n}&=\ln\rho_{n_1}+[\delta n(\delta n+1)/2]f'(n_1)\\&\cong\ln\rho_{n_1}-|f'(n_1)|(\delta n^2)/2\ ,\end{aligned} \tag{4.12}$$

where we have explicitly put in the sign of $f'(n_1)$ and have dropped δn relative to $(\delta n)^2$. Carrying through the calculations for the case $\delta n<0$ shows that Eq. (4.12) is valid in that case as well. Therefore the peak is approximately a Gaussian with a width given by $1/\sqrt{|f'(n_1)|}$. Evaluating this expression gives

$$1/\sqrt{|f'(n_1)|}=\sqrt{n_1}/\{1+\theta_{\rm int}[1-(n_1/N_{\rm ex})]^{1/2}\}^{1/2}\ . \tag{4.13}$$

Note that width of the peak is less than $\sqrt{n_1}$ so that the peak is narrower than a Poissonian one.

If the peak at n_1 is the dominant feature of the photon-number distribution we would expect that the variance of the distribution would simply be given by the square of Eq. (4.13). However, if a single peak dominates the distribution we should also be able to find the variance by means of Eq. (3.19). If we assume that $n_s/N_{\rm ex}$ is small, which is a necessary condition for Eq. (4.13) to be valid, we find that the two equations give the same result for the variance.

Next we consider the case when $N_{\rm ex}$ and $p/N_{\rm ex}$ are the independent parameters in Eq. (3.8). With the notations $x\equiv p/N_{\rm ex}$ and $S_n\equiv N_{\rm ex}s_n^2$, Eq. (3.8) can be written as

$$\begin{aligned}\rho_n=e^{-nx}\sum_{m=n}^{\infty}\binom{m}{n}(1-e^{-x})^{m-n}\\\times[(1-xS_{m+1})\rho_m+xS_m\rho_{m-1}]\ .\end{aligned} \tag{4.14}$$

Let us now consider the case $x\ll 1$. This condition is certainly satisfied for Poissonian pumping ($p=0$) and we expect that the expansion in terms of x yields a reasonable estimate for the sub-Poissonian domain, $0\le p<1$. Keeping terms up to first order in x in Eq. (4.14) gives

$$\begin{aligned}[x(n+1)(n+2)/2]\rho_{n+2}+(n+1)[1-xS_{n+2}-x(n+1/2)]\rho_{n+1}-[n+S_{n+1}-xn^2/2-x(2n+1)S_{n+1}]\rho_n\\+S_n(1-xn)\rho_{n-1}=0\ .\end{aligned} \tag{4.15}$$

Although more complicated, this equation also satisfies a detailed balance condition, just like Eq. (4.1). It can be expressed as

$$\begin{aligned}[x(n+1)(n+2)/2]\rho_{n+2}+(n+1)[1-xS_{n+2}-x(n+1)/2]\rho_{n+1}-[1-x(n+1)]S_{n+1}\rho_n\\=[xn(n+1)/2]\rho_{n+1}+n[1-xS_{n+1}-xn/2]\rho_n-[1-xn]S_n\rho_{n-1}\ .\end{aligned} \tag{4.16}$$

Again, a normalizable solution results if the right-hand side of this equation is set equal to zero, i.e.,

$$\begin{aligned}[xn(n+1)/2]\rho_{n+1}+n[1-xS_{n+1}-xn/2]\rho_n\\-[1-xn]S_n\rho_{n-1}=0\ ,\end{aligned} \tag{4.17}$$

reducing the general four-term recurrence relation of Eq. (4.15) to a three-term one. It is easy to convince ourselves that in the Poisson limit, i.e., for $x=0$, Eq. (4.17) reduces to the well-known and easily solvable two-term recurrence relation of the micromaser with Poisson pumping [10], as it should.

We now have three different recurrence relations for the photon-number distribution, viz. the exact relationship, Eq. (3.8), and two approximate ones, Eqs. (4.4) and (4.17), corresponding to the two different $p/N_{\rm ex}$ expansions. We relegate a detailed numerical study of these expressions to Sec. VI where we shall present numerical results also for the super-Poissonian pumping to be discussed in the next section. At this point we just want to note that all three recurrence relations predict the same peak positions for the photon-number distribution. However, the width and peak height predictions are different.

In general, Eq. (4.4) tends to overestimate the effect of pump regularity, whereas Eq. (4.17) tends to slightly underestimate it. In addition, Eq. (4.17) yields reliable results for a wider range of p (about $0 \leq p \leq 0.5$) than Eq. (4.4) (the validity of which is limited to $0 \leq p \leq 0.2$), when compared to the exact Eq. (3.18).

It is interesting to compare the three-term recurrence relation of the present approach, Eq. (4.17), to the three-term recurrence relation of Refs. [5] and [6],

$$n\rho_n - [1+xS_n/2]S_n\rho_{n-1} + [xS_nS_{n-1}/2]\rho_{n-2} = 0 , \tag{4.18}$$

which was based on the p expansion of Eq. (2.7). In the limit $x \to 0$ both expressions reduce to the known micromaser formula for Poissonian pumping [10], as expected. The terms proportional to x in Eq. (4.18) describe repeated interaction with the gain reservoir alone. The terms proportional to x in Eq. (4.17) describe repeated interaction with the loss reservoir and, more importantly, correlation terms between the interactions with the gain and loss reservoir. These correlation terms are missing entirely from Eq. (4.18) and they represent the crucial difference between the two approaches.

V. SUPER-POISSONIAN PUMP STATISTICS

In this section we extend our basic model of Sec. II for the case of super-Poisson pump-beam fluctuations. This extension is motivated to a large extent by dye-laser theory [18–23]. It was found there that the assumption of a bunched pumping process (usually the pumping is via another laser or a conventional lamp) is necessary to explain gain and fluctuation properties of the generated field.

In the following we shall show that a simple and straightforward generalization of our model is possible and derive a master equation which holds for bunched as well as antibunched pump fluctuations. Formally, we find that Eq. (2.7) remains valid. However, the parameter p is no longer restricted to positive values but, instead, it can take any value in the interval $-\infty < p \leq 1$. Furthermore, the interval $0 \leq p \leq 1$ corresponds to sub-Poissonian pump fluctuations, as before, whereas the interval $-\infty < p \leq 0$ to super-Poissonian (bunched) pump fluctuations in such a way that $p \to -\infty$ describes, in effect, a thermal pump and $p = 0$ again corresponds to a Poissonian pump. Thus, the sub-Poissonian regime matches continuously with the super-Poissonian regime at the value $p = 0$ of the pump parameter.

A. Model

Our basic model in this section is very similar to that of Refs. [5] and [6] which was summarized briefly in Sec. II. Three-level atoms, with the level scheme shown in Fig. 1, are first excited to the upper laser level with probability p. They then pass through the cavity where they interact with the field. The time T between *possible* atomic arrivals is still uniform. The only difference with respect to the sub-Poissonian case is that now at each arrival time t_i more than one atom, say n ($n = 0, 1, 2, \ldots,$), may arrive simultaneously, as indicated in Fig. 3. In the sub-Poissonian case n was either 0 or 1, i.e., either one excited atom arrived at t_i or none. We shall assume that the probability $p(n)$ of n excited atoms arriving simultaneously is given by

$$p(n) = (1-p_0)p_0^n, \quad (0 < p_0 < 1) . \tag{5.1}$$

As will be discussed this leads to a pump beam with super-Poissonian statistics. We expect that a beam of atoms emerging from an oven will have Poissonian statistics. Thus a method must be found of introducing correlations into the beam. This will be discussed below. At this point, let us just mention that the distribution in Eq. (5.1) is the same as that for the number of photons in a cavity at some specified temperature. We shall henceforth refer to it as thermal distribution.

The assumption of simultaneous arrival of more than one atom at a time requires some clarification. In our present treatment so far we have tacitly assumed that the interaction of an individual atom with the cavity field is a δ-function-type kick at $t = t_i$. If more than one atom arrives the interaction is a sum of nonoverlapping δ functions at $t = t_i$ which means that the atoms are bunched around the arrival time on a time scale much shorter than T but their overlap can still be neglected. In fact, our subsequent treatment can easily be generalized for the case of overlapping atoms but this and other possible generalizations are left for future publications.

This model for the pumping process can be justified in the following way. In Refs. [5] and [6] we presented a model of a pumping beam with sub-Poissonian fluctuations. In Sec. II we gave a simplified version of our original derivation. Let us revisit those considerations here, in more detail. Our model consists of two essential ingredients. First, we had an imaginary beam of ground-state atoms, each separated by a time interval T (or, equivalently, distance $L = vT$ where v is the velocity of the beam). Then, out of this regular flow of ground-state atoms, we excite some atoms to the upper level of the

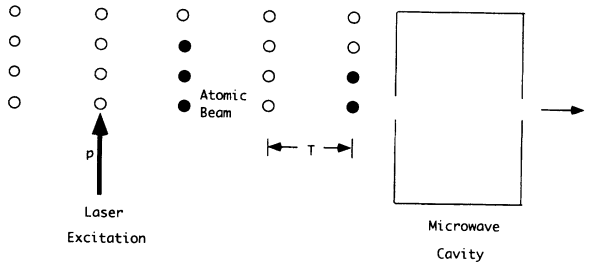

FIG. 3. Scheme of super-Poissonian pumping in the micromaser. A bunched stream of ground-state atoms separated by the time interval T approaches the micromaser cavity. These atoms are denoted by empty circles. The number of ground-state atoms in each bunch is itself a stochastic variable. Before entering the cavity some of the ground-state atoms in each bunch are excited to the upper level of the maser transition with probability p. The excited atoms are denoted with dark circles. Thus the probability that at a given arrival time precisely n excited atoms will arrive ($n = 0, 1, 2, \ldots,$) is given by $(1-p)p^n$.

maser transition (cf. Fig. 2). The exact mechanism of excitation need not be specified. It is enough to merely assume that the probability of excitation is p. Then the probability $p(n,N)$ of finding precisely n excited atoms in the time interval (t_1,t_N), where $t_N=t_1+(N-1)T$, is given by the binomial distribution

$$p(n,N)=\binom{N}{n}p^n(1-p)^{N-n} .$$

Applying Eq. (2.1) n times we obtain that if the field interacts with n atoms in a fixed time interval (t_1,t_N) then $\rho^{(n)}=M^n\rho$. Of course, here n is a fluctuating quantity. If we take the average of $\rho^{(n)}$ with respect to the above $p(n,N)$ we again obtain (2.2) and the rest of the analysis of Sec. II follows (Refs. [5] and [6]). Now, in order to incorporate super-Poissonian fluctuations we introduce the following modifications into our model. We assume a beam of ground-state atoms where the separation T between arrival times is uniform. However, at each arrival time t_i any number of ground-state atoms may simultaneously arrive, in the way described at the beginning of this section. Their number N_1 follows the thermal distribution $p(N_1)=(1-p_1)p_1^{N_1}$. We now excite atoms, with probability p_2 to the upper level of the lasing transition out of this thermal beam of ground-state atoms. For a fixed number N_1 of ground-state atoms the probability $p(n,N_1)$ of exciting n atoms is

$$p(n,N_1)=\binom{N_1}{n}p_2^n(1-p_2)^{N_1-n} .$$

Since N_1 itself is stochastic variable the probability of n excited atoms arriving simultaneously (simultaneously in the sense discussed above) is the convolution of $p(n,N_1)$ with $p(N_1)$, i.e.,

$$p(n)=\sum_{N_1=n}^{\infty}\binom{N_1}{n}p_2^n(1-p_2)^{N_1-n}(1-p_1)p_1^{N_1}$$

$$=[1+(p_1p_2)/(1-p_1)]^{-1}[p_1p_2/(1-p_1+p_1p_2)]^n .$$

If we introduce p_0 as $p_0=(p_1p_2)/(1-p_1+p_1p_2)$ then the above distribution can be written in the form given by Eq. (5.1). This is the underlying model of our thermal beam of excited atoms. It is interesting to note that if both $p_1,p_2\ll 1$, then $p_0\simeq p_1p_2$, i.e., the elementary probability in (5.1) is a product of two independent events: the thermalization of the ground state and excitation. The above equation expresses a well-known result in the theory of probability: the convolution of a thermal distribution with a binomial one is a thermal distribution.

Next we want to obtain the probability $p(n,N)$ of finding n excited atoms in a time interval $t=(N-1)T$ given that at each arrival time $t_i=iT(i=0,1,\ldots,N-1)$ any number of excited atoms, say n_i, may arrive. Clearly, $\sum_i n_i=n$ and from (5.1) we easily find that

$$p(n,N)=\binom{N+n-1}{n}p_0^n(1-p_0)^N , \tag{5.2}$$

where the binomial factor arises from combinatorial considerations.

The above probability distribution function is known as the negative binomial distribution since expansion of the normalized binomial expression, $q_0^N(1-p_0)^{-N}$, with exponent $-N$ (negative binomial) in powers of p_0 (and with $q_0=1-p_0$ substituted at the end) leads to (5.2). One of the properties of this distribution is

$$\langle n\rangle=\sum_{n=0}^{\infty}np(n,N)=(p_0/q_0)N .$$

Since, by definition of the pumping rate r, $\langle n\rangle=rt$, and $N=t/T$ we find that $r=(p_0/q_0T)$ or, conversely, the auxiliary parameter T can be expressed with the physical r as $T=(p_0/q_0r)$. In particular, for $p_0>\frac{1}{2}$ the pumping rate $r>1/T$ or, rather, the interval between successive arrivals T is longer than the average injection time $T>1/r$.

The second feature of interest for our purposes is the mean-square fluctuation of the negative binomial distribution for which we find

$$v=\langle(n-\langle n\rangle)^2\rangle=(p_0/q_0)N[1+(p_0/q_0)]$$

$$=\langle n\rangle[1+(p_0/q_0)] .$$

In other words, the fluctuations are super-Poissonian or the beam is bunched. For short times or, equivalently, for $N=1$ the beam is thermal, as it should be. For longer times (or larger N) it is subthermal but still super-Poissonian. From the above expression we find the Mandel Q parameter of the pump beam, which is defined as $Q=v/\langle n\rangle-1$, to be $Q=p_0/q_0>1$.

B. Master equation in the reservoir limit

Now we are in the position to work out the generalization of Eqs. (2.3)–(2.7) for super-Poissonian pump fluctuations. If the initial density operator of the cavity field is given by $\rho(0)$ then after interacting with one atom we obtain $\rho(1)=M\rho(0)$ where M is the superoperator given in (2.1), describing the effect of a single atom on the field. For more general models it is straightforward to obtain more general expressions for M. After interacting with n atoms in a time interval $t=(N-1)T$ we easily find that

$$\rho^{(n)}=M^n\rho(0) . \tag{5.3}$$

The average density operator after a time interval $t=(N-1)T$ can be obtained by taking the average of Eq. (5.3) with respect to the distribution of n, Eq. (5.2). This procedure yields

$$\rho(N)=\sum_{n=0}^{\infty}p(n,N)\rho(n)$$

$$=\sum_{n=0}^{\infty}\binom{N+n-1}{n}p_0^n(1-p_0)^NM^n\rho(0) . \tag{5.4}$$

The summation can easily be carried out resulting in

$$\rho(N)=(1-p_0)^N(1-p_0M)^{-N}\rho(0)$$

$$=[1-(p_0/q_0)(M-1)]^{-N}\rho(0) . \tag{5.5}$$

Going from the discrete variable N to the continuous t we

can write

$$\rho(t)=[1-(p_0/q_0)(M-1)]^{-t/T}\rho(0)$$
$$=[1-(p_0/q_0)(M-1)]^{-q_0rt/p_0}\rho(0)\ , \qquad (5.6)$$

where in the last step we substituted $(1/T)=(q_0/p_0)r$, which was justified above. By taking the time derivative of both sides we finally arrive at the master equation

$$\frac{d\rho}{dt}=-r(q_0/p_0)\ln[1-(p_0/q_0)(M-1)]\rho\ . \qquad (5.7)$$

This expression depends only on the particular combination $-(p_0/q_0)$. If $0<p_0<1$ then this parameter changes between 0 and $-\infty$. By formally denoting $-(p_0/q_0)$ with p, Eq. (5.7) becomes identical with the gain term in Eq. (2.7). We can again add the loss term to the above equation to make the analogy with Eq. (2.7) complete. The only difference is that the validity of Eq. (2.7) is now extended from the interval $0<p<1$ to the entire $-\infty<p<1$ interval. Positive values of the parameter p correspond to sub-Poissonian fluctuations, as before, whereas negative values of the parameter correspond to super-Poissonian fluctuations. Furthermore, as we have seen in the previous subsection, the parameter p is just -1 times the Mandel Q parameter of the pumping beam. This relationship between p and Q holds in both the sub- and the super-Poissonian case.

Adding the term by hand, which represents the interaction with the loss reservoir, assumes that we can neglect the commutator between the gain and loss terms. The validity of this assumption has already been discussed in connection with Eq. (2.7). Therefore, we leave this line of inquiry at this point. It should be noted, however, that this master equation is worth considering in its own right and we plan to do so in a future publication. Here, instead, we want to develop the analog of the discrete map approach presented in Eqs. (2.8)–(2.11) and Sec. III for super-Poissonian injection statistics. This approach, which takes into account the noncommutative character of the gain and loss superoperators, will yield the generalization of Eq. (3.3).

C. Corrections to reservoir theory

Let us now step back and consider the evolution of the density matrix in more detail. We shall find that Eq. (5.7) is only approximately correct. To find the analog of (2.6)–(2.11) we must proceed with a little caution. In the treatment so far $\rho(0)$ was always taken *before* the first injection time. In order to fully exploit the analytical results of Secs. III and IV, however, in the case of super-Poissonian pump fluctuations we are forced to develop a theory which follows the discrete evolution of the density operator immediately *after* the injection times. In the following treatment let $\rho(0)$ be the density operator of the cavity field *after* the first injection time, $t=0$. Until the next injection time the field just decays. At time $t_1=T$, n excited atoms ($n=0,1,2,\ldots,$) may enter the cavity. Therefore, the field density operator just after t is

$$\rho_1^{(n)}=M^n e^{LT}\rho(0)\ . \qquad (5.8)$$

The probability that at a given injection time precisely n excited atoms pass through the cavity is given by (5.1). Therefore, the average density operator after t_1 is

$$\rho_1=\sum_{n=0}^{\infty}p(n)\rho_1^{(n)}=[1-(p_0/q_0)(M-1)]^{-1}e^{LT}\rho(0)\ . \qquad (5.9)$$

Similarly, right after N injection times ($t=t_N=NT$) we obtain for the average density operator

$$\rho_N=\{[1-(p_0/q_0)(M-1)]^{-1}e^{LT}\}^N\rho(0)$$
$$=\{e^{-LT}[1-(p_0/q_0)(M-1)]\}^{-N}\rho(0)\ . \qquad (5.10)$$

This equation completely describes the dynamics of our model. It is this equation which will be used in the rest of this section. For the purpose of comparison with the master equation, Eq. (2.7), which we now know holds for the super-Poissonian case with negative pump parameter p, it is useful to have a continuous version of (5.10). This is obtained, as before by replacing N by t/T. Making this substitution and taking the time derivative of the resulting equation yields

$$\frac{d\rho}{dt}=-(1/T)\ln\{e^{-LT}[1-(p_0/q_0)(M-1)]\}\rho\ . \qquad (5.11)$$

In the discussion following Eq. (5.2) we found that the auxiliary parameter T can be expressed in terms of the physical injection rate r as $T=(p_0/q_0r)$. Substituting this expression for T in the above equation we finally obtain

$$\frac{d\rho}{dt}=-r(p_0/q_0)\ln\{e^{-(p_0/rq_0)L}[1-(p_0/q_0)(M-1)]\}\rho\ , \qquad (5.12)$$

which is to be compared to Eq. (2.11). Just as we noticed in connection with (5.7) this expression also depends only on the particular combination $-(p_0/q_0)$. If $0<p_0<1$ then this parameter changes between 0 and $-\infty$. By formally denoting $-(p_0/q_0)$ with p, Eq. (5.12) becomes identical with the generalized master equation, Eq. (2.11), which we found to be valid when the commutator of the gain and loss superoperators is not negligible. The only difference is that the validity of (2.11) is now extended from the interval $0<p<1$ to the interval $-\infty<p<1$. Positive values of the parameter p correspond to sub-Poissonian fluctuations, as before, whereas negative values of the parameter correspond to super-Poissonian fluctuations. So our generalized master equation reads as

$$\frac{d\rho}{dt}=(r/p)\ln\{e^{(p/r)L}[1+p(M-1)]\}\rho\ , \qquad (5.13)$$

same as Eq. (2.11) but valid in the entire $-\infty<p<1$ interval and for arbitrary pump fluctuations. Since the above equation is identical with (2.11) we can, without repeating the algebra, just borrow the results of Sec. III and apply them to the discussion of the steady-state properties of our model.

D. Steady-state intensity and fluctuations

The condition which the field density matrix must satisfy at steady state can be derived from the following considerations. At steady state the field must be the same just after the $(N+1)$st injection time, $t_{N+1}=(N+1)T$, as it is just after the Nth, at $t_N=NT$. The field intensity matrices at these two times are related by [cf. Eq. (5.10) with the exponent replaced by 1]

$$\rho_{N+1}=\{[1-(p_0/q_0)(M-1)]^{-1}e^{LT}\}\rho_N$$
$$=\{e^{-LT}[1-(p_0/q_0)(M-1)]\}^{-1}\rho_N \ . \quad (5.14)$$

In the limit of N large we may reasonably expect that ρ_N tends to its steady-state limit ρ and becomes independent of N. The condition $\rho_{N+1}=\rho_N$ implies that the steady-state density matrix ρ must satisfy

$$\rho=\{e^{-LT}[1-(p_0/q_0)(M-1)]\}^{-1}\rho \ . \quad (5.15)$$

Applying the operation $\{e^{-LT}[1-(p_0/q_0)(M-1)]\}$ to both sides of this equation we obtain

$$\rho=\{e^{-LT}[1-(p_0/q_0)(M-1)]\}\rho \ . \quad (5.16)$$

We can now eliminate T with the substitution $T=(p_0/q_0r)$ and then formally denote $-p_0/q_0$ with p, yielding

$$\rho=\{e^{(p/r)L}[1+p(M-1)]\}\rho \ . \quad (5.17)$$

This equation constitutes the central result of this section. It is formally identical with Eq. (3.3), except p is now defined in the interval $-\infty<p<0$. To proceed, we note that in the analytical treatment presented in Sec. III the actual value of p never really played a role. In particular, no p expansion has ever been introduced, in sharp contrast with the method employed in Refs. [5] and [6]. Therefore the analytical results, Eqs. (3.12) and (3.16) for the average intensity and Eq. (3.15) for the photon-number variance, remain valid. Taking explicitly into account that the parameter p is negative, we can write the expression for the mean photon number (average intensity in the cavity) for the super-Poissonian case under consideration as follows:

$$n_s=|p|\alpha_{n_s}/(1-e^{-|p|/N_{\rm ex}}) \ . \quad (5.18)$$

In the limit $|p|/N_{\rm ex}\ll 1$ we again obtain equation (3.16). This time, however, we do so by approaching the Poissonian limit from the direction of the super-Poissonian domain. Therefore the first moment, i.e., average intensity, is insensitive to the statistics of the pumping beam for a wide range of parameter values. Namely, the condition $|p|/N_{\rm ex}\ll 1$ can be expressed in terms of the original excitation parameter p_0 as $|p|=p_0/(1-p_0)$, i.e.,

$$p_0/(1-p_0)\ll N_{\rm ex} \quad \text{or} \quad p_0\ll N_{\rm exp}/(1+N_{\rm ex}) \ . \quad (5.19)$$

Even for very modest pump intensities ($N_{\rm ex}\leq 10$) this condition is well satisfied, since the excitation probability of individual atoms is small under normal circumstances. In order to see the effect of pump statistics on the first moment one then needs extremely low pump-beam intensities and a very large excitation probability ($p_0\simeq 1$). In this limit $\exp(-|p|/N_{\rm ex})$ can be neglected in the denominator of Eq. (5.18) and the expression then reduces to

$$n_s=|p|\alpha_{n_s} \ . \quad (5.20)$$

Both (5.18) and the above expression predict that the (micromaser as well as laser) threshold is shifted towards lower values of the pump parameter θ. In laserlike systems this threshold shift is also accompanied by an increase in the mean number of photons for the same value of the pump parameter, indicating that, under very specific conditions, the super-Poissonian pumping can be more efficient than the ordinary Poissonian one. This conclusion is well confirmed in dye lasers [18–23]. The situation is more complicated in the micromaser where, due to the multipeaked structure of the photon-number distribution, the super-Poissonian pump reduces the mean photon number, as we will show in the next section. In terms of the excitation probability p_0 [where, as before $|p|=p_0/(1-p_0)$], the condition for Eq. (5.20) to hold now reads as

$$p_0/(1-p_0)>N_{\rm ex} \quad \text{or} \quad p_0>N_{\rm ex}/(1+N_{\rm ex}) \ , \quad (5.21)$$

which is just the opposite of Eq. (5.19). This can only be met at very low pump-beam intensities (e.g., $N_{\rm ex}<10$) and even then $p_0>0.9$ is required by this condition, a formidable experimental task in itself.

Much more pronounced is the effect of the super-Poissonian pump fluctuations on second moments. We can again borrow the analytical expression, Eq. (3.15), and apply it to negative values of p. For the normalized photon-number variance we obtain

$$v=n_s\{1+n_s\tanh(|p|/2N_{\rm ex})\}/$$
$$\times\{1-(2|p|/N_{\rm ex})/(1-e^{-2|p|/N_{\rm ex}})\alpha_s'\} \ , \quad (5.22)$$

where we have already taken into account that $p=-|p|$ for negative values of the pump parameter. The most important modification with respect to the sub-Poissonian case, Eq. (3.15), is the change of sign of $\tanh(|p|/2N_{\rm ex})$ multiplying the term n_s in the numerator of Eq. (5.22). Thus, the photon-number fluctuations become super-Poissonian in character, at least for $|p|/N_{\rm ex}<1$, when the numerator dominates the overall behavior of the above expression. Note, however, that while the numerator depends only weakly on $|p|$ [the term multiplying n_s, i.e., $\tanh(|p|/2N_{\rm ex})$ changes between 0 and 1 as $|p|$ varies from 0 to ∞] the denominator is a monotonic function of $|p|$ and depends on the sign of α_s'. Therefore we can distinguish two qualitatively different types of behavior. For laserlike devices, on the one hand, $\alpha_s'\geq 0$ ($=0$ at saturation) and the denominator enhances the super-Poissonian character of the photon number fluctuations. At the saturated regime of operation the denominator of (5.22) is always 1. The limit of $\tanh(|p|/2N_{\rm ex})$ for large values of the argument is also 1. In this limit Eq. (5.22) reduces to

$$v=n_s\{1+n_s\} \ , \quad (5.23)$$

a remarkable result, indeed, showing that for thermal pumps the output of the laser approaches thermal behavior. The conditions of this regime, however, were given in Eq. (5.21) and it is clear that a unit excitation probability is required. Therefore, even the most noisy pumping mechanism conceivable will never quite turn a laser into a thermal source of radiation (but it comes close to it).

For the micromaser, on the other hand, $\alpha_s' < 0$ and the denominator in Eq. (5.22) always exceeds unity. In fact, it is an increasing function of $|p|$. This may compensate for the tendency to produce super-Poissonian behavior which stems from the numerator and the micromaser may produce sub-Poissonian photon-number fluctuations even for highly super-Poissonian pump beam fluctuations. The limiting expressions for the micromaser case are

$$v = n_s\{1+(|p|/2N_{\rm ex})n_s\}/\{1-[1+(|p|/N_{\rm ex})]\alpha_s'\}\ , \tag{5.24}$$

for the case $|p|/N_{\rm ex} \ll 1$, and

$$v = n_s\{1+n_s\}/\{1-(2|p|/N_{\rm ex})\alpha_s'\}\ , \tag{5.25}$$

for the case $|p|/N_{\rm ex} \geq 1$. Equation (5.24) is to be compared to the corresponding expression for the sub-Poissonian case, Eq. (3.17). Both of the above expressions clearly reflect the competition between the tendency to super-Poissonian behavior, which stems from the pump-beam statistics (numerator), and the tendency to the sub-Poissonian one, which is characteristic of the micromaser dynamics (denominator).

VI. STEADY-STATE PHOTON DISTRIBUTION: NUMERICAL RESULTS

A. Sub-Poissonian pump

In Sec. III we derived an exact recurrence relation for the photon distribution, Eq. (3.8), and in Sec. IV we gave two approximate expressions, Eqs. (4.4) and (4.17), for the same quantity. The advantage of the approximate expressions is in their relative simplicity, i.e., it is much easier to handle them analytically as well as numerically than the exact equation.

In Figs. 4–6 we plot the photon-number distributions, resulting from these expressions, for various values of the parameters θ, $N_{\rm ex}$, and p, so that we can compare them for single-peaked, double-peaked, and triple-peaked situations.

In Fig. 4 we have chosen $N_{\rm ex}=20$ and $\theta=0.8\pi$ which yields a single-peaked distribution. What can be noted from these figures is that the validity of the approximation, given by the two-term recurrence relation, Eq. (4.4), is rather limited. In fact, for $p \geq 0.3$, the distribution resulting from (4.4) overestimates the effect of pump regularity, inasmuch as it predicts a much narrower and, consequently, much higher peak and, thus, more squeezing than the exact result. The other approximation given by the three-term recurrence, Eq. (4.17), performs much better. In fact, it agrees with the exact result very well all

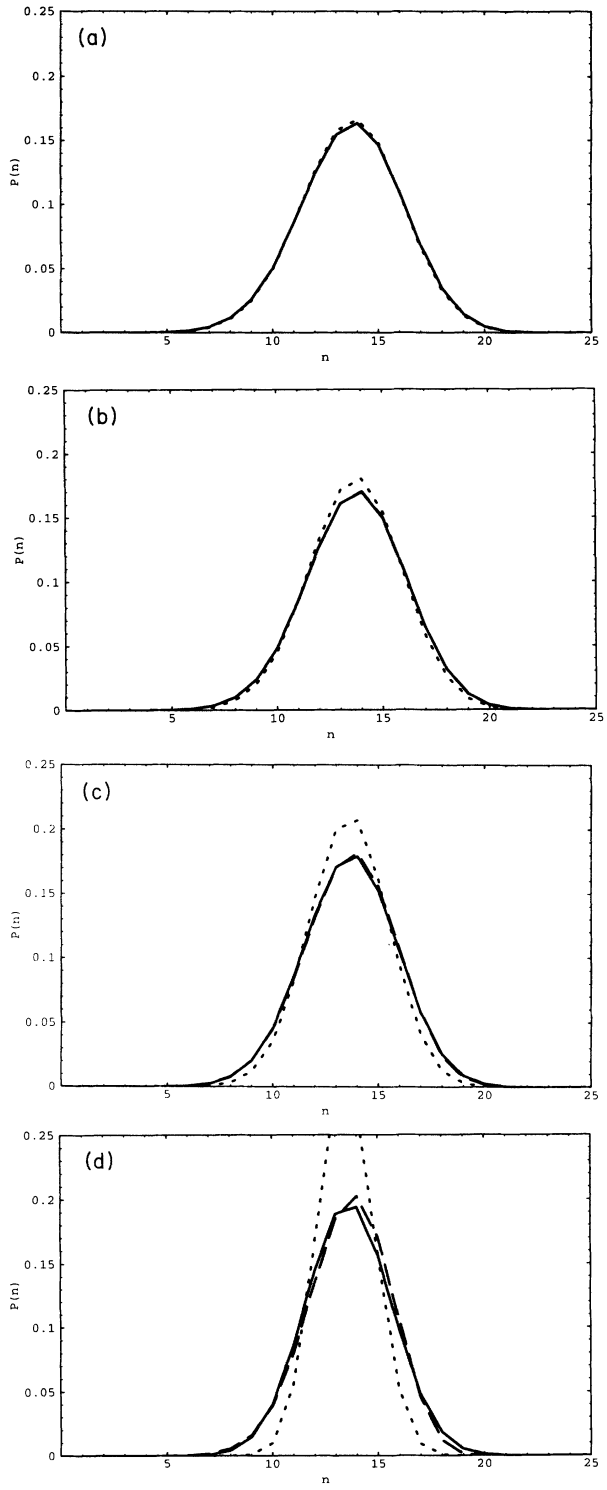

FIG. 4. Photon-number distribution for pump parameter $\theta=0.8\pi$, zero thermal background, and $N_{\rm ex}=20$, as a function of the pump regularity parameter p for sub-Poissonian pump: (a) $p=0.1$ (nearly Poisson pump); (b) $p=0.3$; (c) $p=0.6$; and (d) $p=1$ (regular pump). Pump regularity increases with increasing p. The solid line corresponds to the exact distribution, Eq. (3.8), the dotted line to the two-term approximation, Eq. (4.4), and the dashed line to the three-term approximation, Eq. (4.17). The steady narrowing of the peak with increasing pump regularity can clearly be seen while the peak position remains largely unaffected as p changes.

the way up to $p=1$.

While the limit of validity (which can be set to be $p=0.2$) of the two-term approximation does not depend on the number of peaks in the distribution function, the same statement does not hold for (4.17). In Fig. 5 we have chosen $N_{ex}=50$ and $\theta=2.1\pi$ which yields a two-peaked distribution. Equation (4.4) still starts to overestimate the effect of pump regularity around $p=0.3$. Equation (4.17), on the other hand, noticeably underestimates it for $p \geq 0.6$. The plots in Fig. 5 reveal that the effect of pump regularity is twofold. First, it makes the individual peaks narrower. In a single-peaked situation this ex-

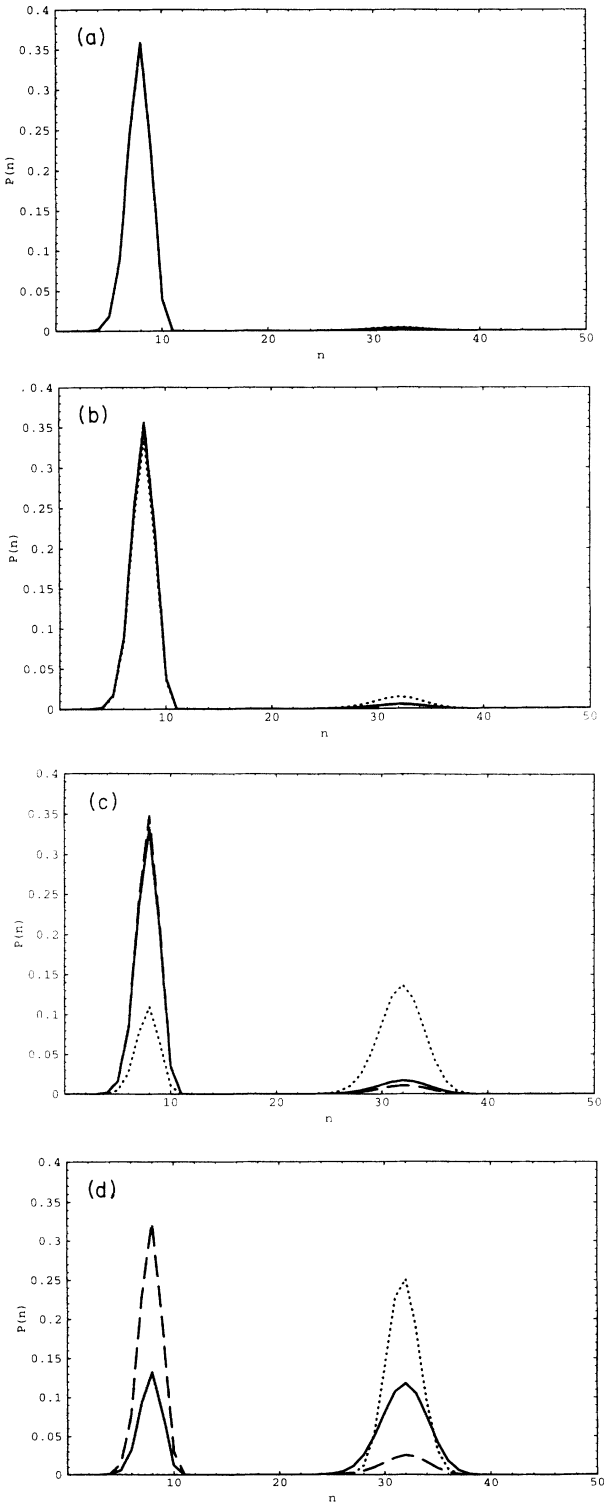

FIG. 5. As for Fig. 4, except $\theta=2.1\pi$ and $N_{ex}=50$.

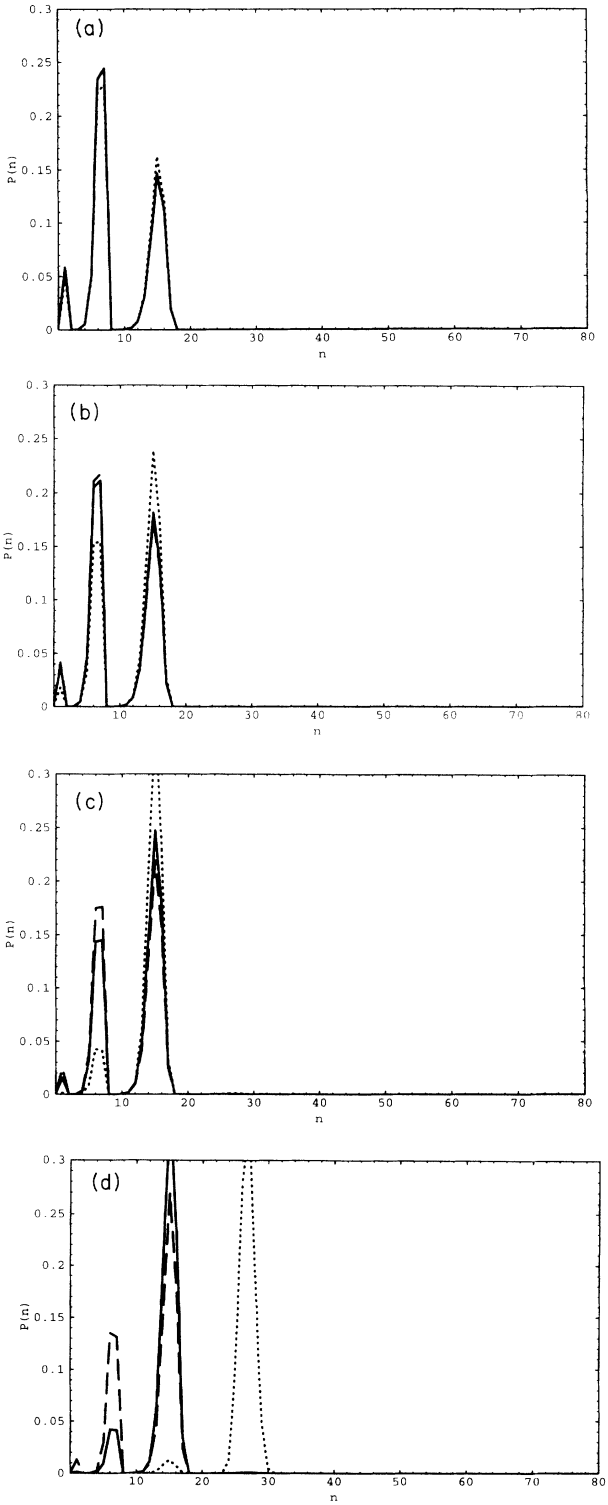

FIG. 6. As for Fig. 4, except $\theta=5.01\pi$ and $N_{ex}=50$.

plains immediately why the output fluctuations are reduced, sometimes even below the shot-noise level. Second, it enhances the contribution of the peaks at larger photon numbers. This might make the whole distribution noisier with regular pump than with a Poissonian one, in good agreement with the general conclusions of Ref. [16] which were based on the more limited Eq. (2.7). This enhancement of the higher lying peaks is overestimated by Eq. (4.4) and underestimated by Eq. (4.17) when compared to the exact (3.8). The tendency in the behavior of the approximate formulas is even more striking in Fig. 6 which depicts the photon distribution for $N_{ex}=50$ and $\theta=5.01\pi$, yielding a three-peaked distribution. In fact, Eq. (4.4) predicts a high peak at $n=27$ for $p\simeq 1$ which is barely visible in the exact formula and missed entirely by Eq. (4.17). On the other hand, the two peaks at $n=1$ and $n=6,7$ are missed completely by (4.4) and overestimated by (4.17). Again, Eq. (4.4) is valid up to $p\simeq 0.2$ and Eq. (4.17) up to $p\simeq 0.5$.

B. Super-Poissonian pump

We illustrate the effect of a super-Poissonian pump beam in Fig. 7, using the exact expression (3.8), with $N_{ex}=50$ and $\theta=5.01\pi$. Thus, the figure can be regarded as the extension of Fig. 6 into the super-Poissonian regime. The trend noticed in Fig. 6 continues: making the pump more random ($|p|\simeq N_{ex}$) suppresses the peaks at higher photon numbers and makes the distribution settle to the peak at the lowest n (in the present case it is $n=1$) in Fig. 7(c). Making the beam even more random ($|p|\gg N_{ex}$) wipes out the effect of the pump beam altogether and settles the system to the thermal background [which is zero in our case, so Fig. 7(d) corresponds to vacuum, i.e., a single peak at $n=0$]. This can easily be understood in the following way. In such a superbunched beam the average time between the arrival of atomic bursts is several cavity lifetimes. Hence the system is, most of the time, in equilibrium with the thermal background (vacuum in our case). The weight of those intervals when it is not is inversely proportional to $|p|/N_{ex}$ so when $|p|\gg N_{ex}$ the effect of the pumping becomes altogether negligible and the photon distribution becomes identical with that of the thermal background.

It should be noted at this point that the case of a super-Poissonian pump has been investigated in a recent work using a different approach [24]. While the overall trends are in good agreement in both approaches, the details depends on the specific model employed. This point requires further clarification which, however, is left to a future work.

For the sake of completeness, we also present the results using Eqs. (4.4) and (4.17), in Figs. 8 and 9, respectively. Since, in obtaining those equations, the expansion parameter is $|p|/N_{ex}$, we only display the results for $|p|=2$ ($|p|/N_{ex}=0.04$) and 20 ($|p|/N_{ex}=0.4$). Up to these values of $|p|$ the two approximations agree well with Figs. 7(a) and 7(b).

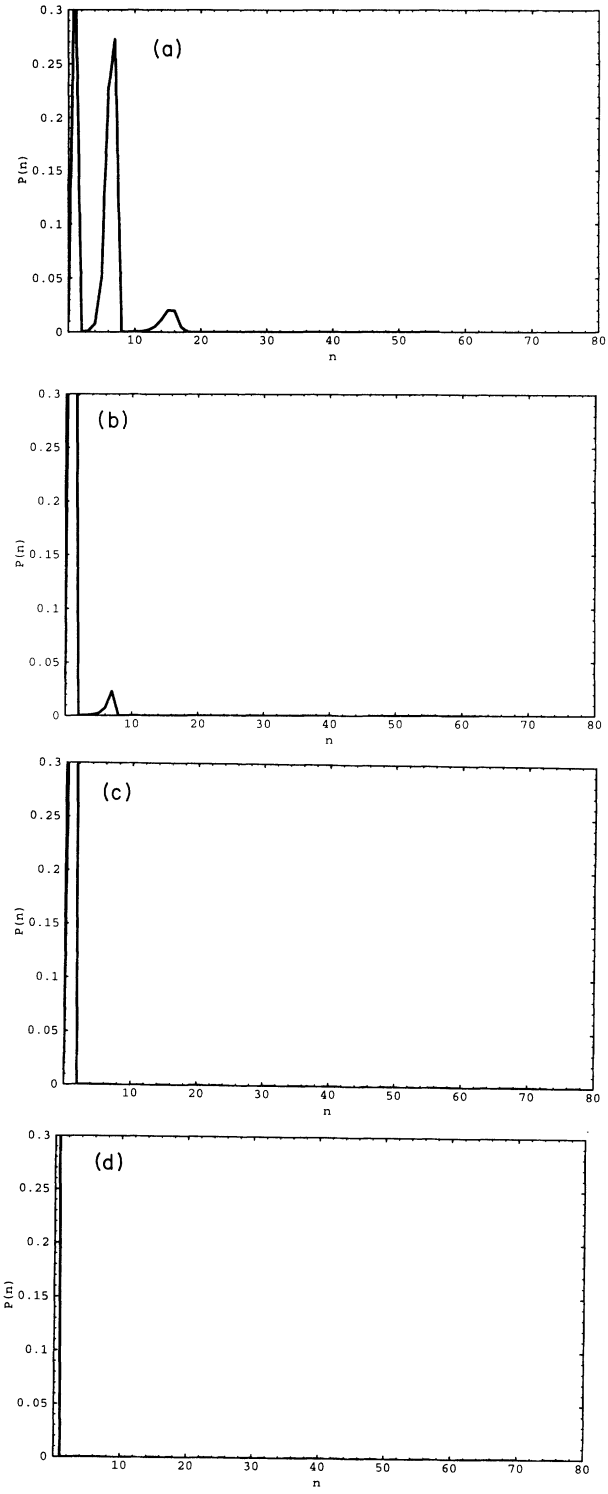

FIG. 7. Photon-number distribution, from the exact equation (3.8), for pump parameter $\theta=5.01\pi$, zero thermal background, and $N_{ex}=50$ as a function of the bunching parameter $|p|=p$ for super-Poissonian pump: (a) $|p|=2$ ($|p|/N_{ex}=0.04$); (b) $|p|=20$ ($|p|/N_{ex}=0.4$); (c) $|p|=200$ ($|p|/N_{ex}=4$); and (d) $|p|=2000$ ($|p|/N_{ex}=40$). Bunching of the pump beam increases with increasing $|p|$. First, (a)–(c), the peaks at higher photon numbers get suppressed. Then for $|p|/N_{ex}\gg 1$, in (d), the effect of the pump is washed out and the distribution settles to that of the thermal background (vacuum, a single peak at $n=0$, in our case).

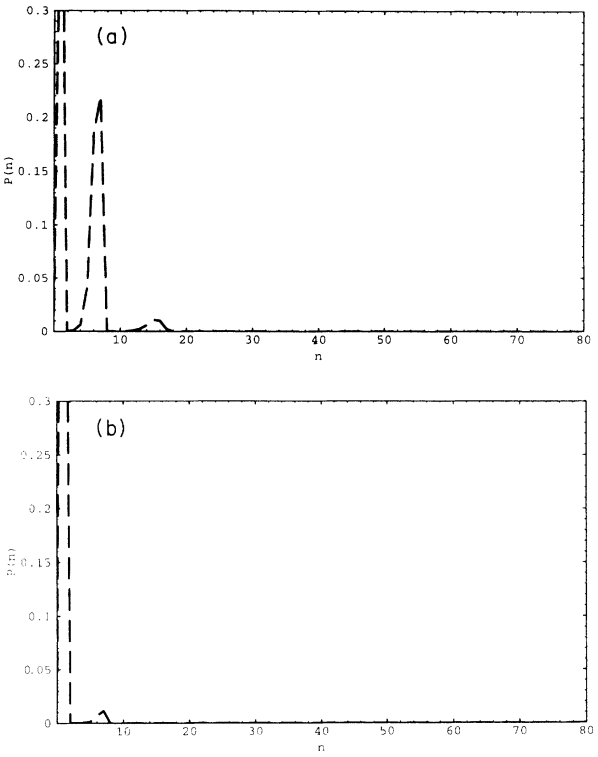

FIG. 8. (a) and (b). Same as for 7(a) and 7(b), except from Eq. (4.17).

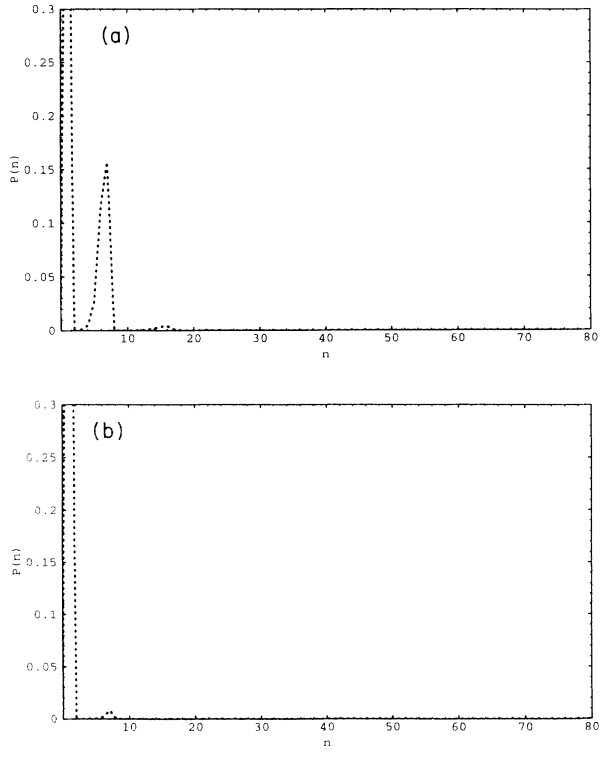

FIG. 9. (a) and (b). Same as for 7(a) and 7(b), except from Eq. (4.4).

VII. SUMMARY AND CONCLUSION

In the preceding sections we have presented a rather detailed theory of the micromaser with specific attention to the fluctuations in the pumping beam. In Sec. II we derived an alternative master equation for the density matrix of the field in the micromaser cavity. This equation, Eq. (2.11), differs from the one previously derived by us, Eq. (2.7), by terms involving the commutators of the gain and loss superoperators [see also Refs. [5] and [6] for a more detailed theory leading to Eq. (2.7)]. These commutator terms are negligible when the number of atoms, N_{ex}, passing through the cavity in a cavity lifetime is large and/or the pumping statistics is essentially Poissonian. In other cases, however, the presence of these new terms leads to correlation between the interactions of the cavity field with the gain and loss reservoirs. Traditional reservoir theory, which builds on the assumption that the total time rate of change of the density matrix is a sum of the individual changes due to uncorrelated interactions with the separate reservoirs, needs to be modified. Our master equation, Eq. (2.11), accomplishes this goal. The present contributions found with its help become significant when the parameter p/N_{ex} is relatively large. Here, p is a parameter describing fluctuations in the pump beam. It is essentially -1 times the Mandel Q parameter of the beam. So for p/N_{ex} to become appreciable one needs a low intensity pump beam and a pump parameter p different from zero, i.e., a non-Poissonian pumping process. This parameter, through N_{ex}, characterizes the strength of the interaction with both the gain and loss reservoirs. In order to see deviations from conventional reservoir theory one needs a small value of N_{ex}. Therefore, lacking a better name, we refer to the case when Eq. (2.11) is different from Eq. (2.7) as the small reservoir limit (admittedly a contradiction in itself). By this we simply mean that in this limit the interactions with the different reservoirs are correlated. Of course, in the limit of large reservoirs, i.e., when p/N_{ex} is small Eq. (2.11) reduces to the conventional reservoir limit, Eq. (2.7).

In Sec. III we have exploited some of the consequences of the present master equation. Using the steady-state condition, Eq. (3.3), we have derived analytical expressions for the steady-state photon-number (average intensity of the cavity field), Eqs. (3.12) and (3.16), and the photon-number fluctuation, Eq. (3.15). Since this steady-state condition is a direct consequence of the discrete mapping type dynamics of the micromaser, Eq. (3.10), it is independent of the assumption of continuous time which is inherent in the derivation of the master equation (2.11). Nevertheless, it coincides with the steady-state condition of the master equation itself. In our earlier work (Refs. [5] and [6]) we have introduced the so-called p expansion. The theory presented in this section is accurate, no expansion in p/N_{ex} is involved. Therefore its range of validity is expected to be much larger than that of Refs. [5] and [6]. In particular it is expected to hold in the small reservoir limit, in the sense discussed above. It should also be noted at this point that all the results reduce to those of the large-reservoir limit obtained in [5]

and [6] when p/N_{ex} is small. The lowest-order contribution to conventional reservoir theory, i.e., the first direct consequence of the correlation between the interactions with the gain and loss reservoirs is contained in Eq. (3.17). It is represented by the term $\alpha_s'(p/N_{ex})$ in the denominator. This term can counteract the dynamical intensity squeezing in the micromaser when the photon-number distribution is single peaked and can enhance squeezing when more than one peak contributes to the steady-state behavior, in agreement with the findings of Refs. [12], [14], and [16] which were based on numerical simulations. Thus the results are not limited to particular value ranges of this parameter. Instead, they are limited by the validity of the assumption about the photon-number distribution. In the derivation we have assumed that the photon-number distribution is essentially dominated by a single peak although some of the consequences of the results in Sec. III remain valid even when this assumption breaks down (see the effect of the term $\alpha_s'(p/N_{ex})$, in particular, as discussed above). In Sec. IV we take a closer look at the validity of this assumption. We present an analytical theory of the position and width of the peaks of the photon-number distribution and find conditions for single peakedness of the distribution function. This is based on the two approximate expressions, Eqs. (4.5) and (4.17), for the photon-number distribution obtained by expanding the exact (3.8) in terms of p/N_{ex} in two different ways. Since, in general, in the micromaser $p \ll N_{ex}$ the expressions are expected to have a rather large range of applicability. The findings of this section concerning the first and second moment of the distribution confirm those of Sec. III and provide a firm foundation of the analytical results.

In Sec. V we extend our investigations to the case of super-Poissonian pump fluctuations. This is accomplished by extending our simple model of the sub-Poissonian pumping beam, which was explained in Sec. II, to the case when we allow for bunching, i.e., super-Poissonian fluctuations in the pump beam. The most interesting result of this section is, perhaps, Eq. (5.13). It formally coincides with our master equation, Eq. (2.11), but the range of the pumping parameter is now extended from the interval $0<p<1$ to the interval $-\infty<p<1$. When p is between 0 and 1 it still describes sub-Poissonian pump-beam fluctuations. When $-\infty<p<0$ it describes super-Poissonian pump fluctuations. In all these cases p is -1 times the Mandel Q parameter of the pumping beam. Then we analyze the steady state predictions of the extended master equation. Since the analytical theory of Sec. III was not based on a p expansion the expressions for the mean number of photons and the photon-number variance remain valid for negative values of p and we just simply carry them over to the case of negative p. With the help of these expressions we have been able to show that the average number of photons is rather insensitive to pumping statistics for a large range of parameters although an extremely bunched pump beam appears to be more efficient in laserlike systems than the Poissonian one and shifts the lasing threshold towards lower values. The variance is super-Poissonian and in the limit of extreme bunching it approaches thermal behavior. The findings of this section are in very good agreement with dye laser theory (Refs. [18–23]) where the introduction of a bunched pumping process via the assumption of multiplicative noise was necessary in order to explain the extra fluctuations. Our method leads to multiplicative noise quite naturally [cf. Eq. (5.22) where the dependence of the photon number variance on the pump parameter p is explicit) and, thus, provides a uniform basis to treat arbitrary (sub- as well as super-Poissonian] pump fluctuations.

In Sec. VI we presented numerical results for the steady-state photon distribution for single-, double-, and triple-peaked distribution functions both for sub-Poissonian and super-Poissonian pump-beam fluctuations. In the limiting cases our results agree with those of similar recent investigations (Refs. [16] and [24], in particular). The method can easily be generalized from the micromaser to other, laser and maser, cases by generalizing the operator M of Eq. (2.1) which describes the effect of a single atom on the cavity field. This and other possible lines of inquiries, including a detailed numerical study of the master equation and the steady-state distribution, are left for future publications.

ACKNOWLEDGMENTS

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