PHYSICAL REVIEW A VOLUME 48, NUMBER 2 AUGUST 1993

Experimental determination of quantum-phase distributions using optical homodyne tomography

M. Beck, D. T. Smithey, and M. G. Raymer
Department of Physics and Chemical Physics Institute, University of Oregon, Eugene, Oregon 97403
(Received 31 March 1993)

From the experimental measurement of probability distributions of quadrature-field amplitudes, followed by numerical inversion (optical homodyne tomography), we have determined distributions and/or moments of the optical phase of small-photon-number fields for several definitions of the phase variable, including those based on Hermitian operators and on quasiprobability distributions. These measurements were performed on a vacuum field, and a weakly squeezed field. It is found that each definition of phase yields different distributions and/or moments for the experimental data. In addition, all of the experimentally determined quantities agree with the corresponding theoretical predictions.

PACS number(s): 42.50.Wm, 03.65.Bz, 42.50.Dv

The quantum-mechanical description of the phase of an electromagnetic field has long been a subject of interest [1–3]. Since Dirac's earliest proposal for a phase operator, there have been many theoretical proposals put forth for operators that for intense fields reproduce some aspect of the statistics of the classical phase of the field. The proposed operators may give widely different statistics, however, for fields that contain only a few photons [3]. Other approaches that do not require the definition of a Hermitian phase operator have also been proposed [4–6]. The weak-field regime is of fundamental interest because it is here that interesting quantum-mechanical effects can most easily be studied, such as the uncertainty principle applied to phase and photon-number variables [7].

Despite this interest, there have been few experimental attempts to measure the phase of fields with small photon number [6,8,9]. Early measurements of phase were carried out by Gerhardt, Büchler, and Litfin, who measured the phase of a coherent state with as few as three photons [8]. This measurement has been compared with predictions using several different phase operators [10,11]. Noh, Fougères, and Mandel have performed measurements that determine the relative phase between two coherent-state fields with low mean photon number, and the results agree well with a phase operator defined for this specific experiment [6].

Previously there have been no methods proposed for experimentally determining either the Pegg-Barnett [3], or the marginal Wigner [4] phase distributions. Here we demonstrate an experimental technique, based on state measurement, for inferring these distributions without the need to make individual measurements of the corresponding phase variable. We note that the Wigner phase variable is unmeasurable, even in principle, on individual trials, due to the fact that the Wigner phase distribution can be negative [12].

We apply this technique to both a vacuum state and a weakly quadrature-squeezed state of a single spatial-temporal mode of the field, containing on average less than one photon. The results for the squeezed state represent experimentally determined phase distributions for a nonclassical state of the electromagnetic field, i.e., a state that cannot be represented by a classical mixture of coherent states. The density matrix and Wigner distribution that we used to determine the phase distributions were obtained using the technique of optical homodyne tomography [13]. In the standard interpretation of quantum mechanics, the density matrix contains all knowable information about a given quantum system. Thus, from the density matrix for the measured mode, we are able to calculate distributions and/or moments for any appropriately defined phase operator, including those of Pegg and Barnett [3], and Susskind and Glogower [2]. We can also calculate phase distributions that are not directly associated with operators, such as the Wigner phase distribution defined by Schleich, Horowicz, and Varro [4].

Our results illustrate that it is possible to obtain phase distributions without making individual phase measurements (or even estimates). Using homodyne detection, we measure an ensemble of field-quadrature amplitudes and mathematically construct phase distributions from the data. In the language of Ref. [6], our phase distributions would thus be "inferred" rather than "measured," because individual phase measurements are not made. We find that the experimental data, when analyzed according to each of the various theories for quantum phase, yields significantly different distributions and/or moments. Nevertheless, each distribution or moment agrees quite well with the corresponding theoretical prediction. In a sense, ours is the opposite point of view of that taken by Noh, Fougères, and Mandel, who propose that it may be necessary to define a particular phase operator for a given experimental arrangement [6]. If one is willing to sacrifice phase measurements on individual realizations, it is possible to obtain several phase distributions from one apparatus that measures the state of the system.

The first useful phase operators were defined by Susskind and Glogower. They defined exponential phase operators in terms of the number states as [2]

$$\widehat{\exp}_{SG}(+i\phi)=\sum_{n=0}^{\infty}|n\rangle\langle n+1| \ , \tag{1a}$$

$$\widehat{\exp}_{SG}(-i\phi)=\sum_{n=0}^{\infty}|n+1\rangle\langle n| \ . \tag{1b}$$

These operators are not unitary, however, and as such they are not functions of a unique Hermitian phase operator. From these operators, Hermitian sine and cosine operators can be defined as

$$\widehat{\cos}_{SG}=\tfrac{1}{2}[\widehat{\exp}_{SG}(+i\phi)+\widehat{\exp}_{SG}(-i\phi)] , \quad (2a)$$

$$\widehat{\sin}_{SG}=\frac{1}{2i}[\widehat{\exp}_{SG}(+i\phi)-\widehat{\exp}_{SG}(-i\phi)] . \quad (2b)$$

The statistical moments of $\widehat{\cos}_{SG}$ and $\widehat{\sin}_{SG}$ can be calculated from the density matrix in the number-state representation. Since the operators contain no explicit value for a phase reference, phase shifts must be performed by multiplying Eqs. (1) by complex exponentials [5].

In retrospect, part of the problem with defining a Hermitian phase operator is found to be due to the fact that the operators in Eqs. (1) operate in an infinite dimensional Hilbert space. By restricting the operation of a phase operator to a finite (but arbitrarily large) dimensional Hilbert space, Pegg and Barnett (PB) have defined a Hermitian phase operator [3]. In an $(s+1)$-dimensional Hilbert space the phase states are defined as

$$|\phi\rangle=\frac{1}{\sqrt{s+1}}\sum_{n=0}^{s}e^{in\phi}|n\rangle . \quad (3)$$

This Hilbert space is spanned by a complete orthonormal set of basis phase states $|\phi_m\rangle$, where

$$\phi_m=\phi_0+\frac{2\pi m}{s+1} , \quad m=0,1,\ldots,s , \quad (4)$$

and ϕ_0 is a reference phase. In terms of the states $|\phi_m\rangle$ the Hermitian phase operator is

$$\hat{\phi}=\sum_{m=0}^{s}\phi_m|\phi_m\rangle\langle\phi_m| . \quad (5)$$

With this definition, the phase states are the eigenstates of the phase operator, i.e., $\hat{\phi}|\phi\rangle=\phi|\phi\rangle$. Since $\hat{\phi}$ is Hermitian, it is possible to create $\cos(\hat{\phi})$ and $\sin(\hat{\phi})$ operators in terms of Taylor expansions.

Using the PB formalism, one can define a probability distribution for the phase. For a system in a state described by a density operator $\hat{\rho}$, the probability of measuring a particular value of the phase is $P_{PB}(\phi)=[(s+1)/2\pi]\langle\phi|\hat{\rho}|\phi\rangle$. In terms of the number-state basis, this distribution is

$$P_{PB}(\phi)=\frac{1}{2\pi}\sum_{n,m=0}^{s}e^{i(m-n)\phi}\langle n|\hat{\rho}|m\rangle . \quad (6)$$

Another phase distribution is the Wigner phase distribution, proposed by Schleich, Horowicz, and Varro [4]. This is defined in terms of the Wigner quasiprobability density $W(x,p)$, which for a single degree of freedom is defined as

$$W(x,p)=\frac{1}{\pi}\int_{-\infty}^{\infty}\langle x+x'|\hat{\rho}|x-x'\rangle e^{-2ipx'}dx' , \quad (7)$$

where $|x\rangle$ is an eigenstate of the operator $\hat{x}$, which obeys $[\hat{x},\hat{p}]=i$ with its conjugate variable $\hat{p}$. For a light mode with annihilation operator $\hat{a}$, the operators $\hat{x}$ and $\hat{p}$ are defined as $\hat{x}=(\hat{a}+\hat{a}^\dagger)/\sqrt{2}$ and $\hat{p}=(\hat{a}-\hat{a}^\dagger)/i\sqrt{2}$. The Wigner phase distribution $P_W(\phi)$ is given by the overlap in the x-p phase space of the Wigner distribution and a narrow "wedge-shaped" region, having an angular width $\Delta\phi$ and making an angle ϕ with respect to the x axis. This yields the marginal distribution

$$P_W(\phi)\Delta\phi=\int_{\phi}^{\phi+\Delta\phi}d\phi'\int_0^{\infty}dr\, rW(r\cos\phi',r\sin\phi') . \quad (8)$$

Our measurement apparatus is a balanced homodyne detector [14]. In this detector a pulsed signal field E_s is superposed by a 50-50 beam splitter with a pulsed coherent-state field E_{LO} called the local oscillator (LO), with phase θ. The local oscillator is an approximately 400-ps duration, near transform-limited pulse, with a wavelength of 1064 nm. The LO pulse contains an average of 4×10^6 photons. The two resulting fields are detected with high-quantum-efficiency ($\sim 85\%$) photodiodes, and the resulting current pulses are temporally integrated and subtracted with a low-noise preamplifier. This yields the total photoelectron difference number N_θ. Recently we showed that our apparatus allows the measurement of distributions of the photoelectron difference number in the macroscopic domain [15]. The subtraction eliminates classical-like fluctuations of the LO (but not of the signal field). Assuming the LO to be much stronger than the signal, the operator $\hat{N}_\theta$ for the total photon-difference number is proportional to the quadrature-amplitude operator, defined with respect to the LO phase θ by $\hat{x}_\theta=\hat{x}\cos\theta+\hat{p}\sin\theta=\hat{N}_\theta/(2\bar{n}_{LO})^{1/2}$, where $\bar{n}_{LO}$ is the mean photoelectron number in the LO pulse. Thus, for a given local oscillator phase, properly normalized distributions of N_θ yield $P_\theta(x_\theta)$ distributions of the quadrature amplitude.

The quadrature-squeezed field was generated by a type-II phase-matched, walkoff-compensated KTP optical parametric amplifier with no injected signal, i.e., a "squeezed vacuum." With this field, we find that the field quadrature with minimum variance has a variance that is 25% below the shot noise level [13]. Since in this experiment the local oscillator is pulsed, the operator $\hat{a}$ corresponds to the "spatial-temporal mode" of the local oscillator. Further experimental details are found in [15,13].

The distribution $P_\theta(x_\theta)$ is related to the Wigner function by the relation

$$P_\theta(x_\theta)=\int_{-\infty}^{\infty}W(x_\theta\cos\theta-p_\theta\sin\theta,x_\theta\sin\theta+p_\theta\cos\theta)dp_\theta . \quad (9)$$

Given a set of distributions $P_\theta(x_\theta)$ for values of θ between 0 and π, it is possible to invert Eq. (9) to obtain $W(x,p)$ [16]. This is done numerically by using the inverse radon transformation familiar in tomographic imaging. Experimentally, we make measurements at 27 equally spaced values of θ; each distribution $P_\theta(x_\theta)$ is obtained from 4000 measurements of the photoelectron difference number in the balance homodyne detector. To obtain the density matrix in the x representation from $W(x,p)$, all that is necessary is to perform a one-dimensional Fourier transform of $W(x,p)$:

$$\langle x+x'|\hat{\rho}|x-x'\rangle=\int_{-\infty}^{\infty}W(x,p)e^{2ipx'}dp . \quad (10)$$

In Ref. [13] we demonstrated the ability to obtain $W(x,p)$ and $\langle x|\hat{\rho}|x'\rangle$ from the measured distributions $P_\theta(x_\theta)$. We refer to this technique as optical homodyne tomography. Since in this technique, we use a strong coherent-state local oscillator, our phase distributions are referenced to the phase of this field.

In order to obtain $P_{PB}(\phi)$, or the moments of $\widehat{\cos}_{SG}$ and $\widehat{\sin}_{SG}$, it is necessary to transform the density matrix from the x representation into the number-state representation. This is done by the transformation

$$\langle n|\hat{\rho}|m\rangle=\left[\frac{1}{\pi 2^n 2^m n!m!}\right]^{1/2}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dx' e^{-(x^2+x'^2)/2}H_n(x)H_m(x')\langle x|\hat{\rho}|x'\rangle\ , \tag{11}$$

where the functions $H_i(x)$ are Hermite polynomials. Thus, simple integration of our experimentally determined density matrix in the x representation yields the density matrix in the number-state representation. An indication of the experimental accuracy with which we can calculate the matrix elements is determined by evaluating $\langle n|\hat{\rho}|m\rangle$ for large values of n and m, where we expect $\langle n|\hat{\rho}|m\rangle$ to approach zero. We find that for values of n and m greater than about 6, the value of $|\langle n|\hat{\rho}|m\rangle|$ is typically less than 0.005.

It should be noted that the diagonal terms $\langle n|\hat{\rho}|n\rangle$ yield the probability $P(n)$ of having n photons in the signal mode after losses. For the experimentally measured vacuum signal, we find $P(0)=0.99$, and all other $P(n)$'s to be less than 0.006. For the measured squeezed state we find $P(0)=0.94$, $P(1)=0.042$, $P(2)=0.010$, and all other $P(n)$'s less than 0.005. For a pure squeezed vacuum state without losses, the probability of measuring an odd number of photons is zero. However, when losses are introduced it is possible to measure odd numbers of photons, and we believe that the high probability of seeing a single photon in our experiment is due to losses (both the quantum efficiency of the photodiodes, and the homodyne efficiency due to the spatial-temporal overlap of the local oscillator with the squeezed field).

Shown in Fig. 1 are the experimentally determined Pegg-Barnett and Wigner phase distributions, for both a vacuum signal and a squeezed-vacuum signal. Because the phase distribution of the vacuum is assumed to be uniform, we infer that the deviation from a uniform distribution is an indication of our experimental error. As the number of photoelectron difference number measurements increases, the determination of the distributions $P_\theta(x_\theta)$ will become better and the noise on the phase distributions should decrease. The 27 phase angles used in the reconstruction of the Wigner function is more than enough to accurately determine it; so increasing the angular resolution should not significantly alter these phase distributions [13]. It is found that $P_{PB}(\phi)$ is not sensitive to the choice of the Hilbert-space dimension, as long as the dimension is not too small; the results of Fig. 1(b) are essentially identical if a 10-, 25-, or 100-dimensional space is used.

It is seen in Fig. 1(b) that the experimental phase distribution of the squeezed state has two maxima, located at approximately $\pi/2$ and $3\pi/2$. The theoretical results shown in Fig. 1(c), which are for a pure squeezed state with 25% squeezing in the variance, are in agreement with the experimental results, for both Pegg-Barnett and Wigner phases. The theoretical results were obtained by using the same algorithm as used for the experimental results, but by using a theoretical Wigner function. In both the theoretical and experimental plots, the reference phase ϕ_{ref} was taken to be $\pi/2$. The choice of ϕ_{ref} is arbitrary, and we have made a choice that minimizes the variance of ϕ for the theoretical data [3]. Precise quantitative agreement between the theoretical and experimental distributions is not expected, because the theory is for a pure squeezed state, while the experimental data indicate that the measured state is not pure. The slight discrepancy of the location of the peak near $3\pi/2$ in Fig. 1(b) may be due to drift in our interferometer, which is only passively stabilized, as well as statistical error, due to the finite sample numbers. Note that the theoretical results for the Wigner phase distribution are more sharp-

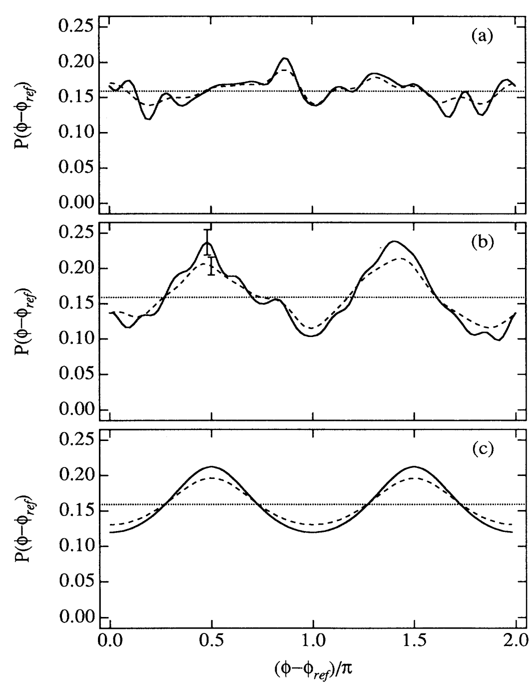

FIG. 1. The Wigner (solid lines) and Pegg-Barnett (dashed lines) phase distributions are plotted over a 2π phase window. The Wigner phase distributions are obtained from Eq. (8), while the Pegg-Barnett distributions are obtained from Eqs. (6) and (11). The dotted lines represent a uniform, random-phase distribution. (a) shows the distributions for the experimentally measured vacuum signal (the deviations from a uniform, random distribution are due to experimental noise); (b) shows the distributions for the experimentally measured squeezed-vacuum signal, the error bars represent the rms error estimated from (a); (c) shows the distributions for a theoretical squeezed vacuum, whose minimum variance is 25% below the shot noise level.

TABLE I. Experimental and theoretical moments of functions of phase, for several different definitions of phase. The mean and variance of f are signified by $\langle f \rangle$ and $\langle \Delta f^2 \rangle$, respectively. These moments are for a vacuum signal state.

	Pegg-Barnett		Wigner		Susskind-Glogower	
	Expt.	Theory	Expt.	Theory	Expt.	Theory
$\langle \phi - \phi_{\rm ref} \rangle$	3.10	3.11	3.11	3.11		
$\langle \Delta(\phi - \phi_{\rm ref})^2 \rangle$	3.19	3.29	3.15	3.29		
$\langle \cos(\phi - \phi_{\rm ref}) \rangle$	−0.028	0	−0.036	0	−0.028	0
$\langle \Delta \cos^2(\phi - \phi_{\rm ref}) \rangle$	0.50	0.5	0.50	0.5	0.25	0.25
$\langle \sin(\phi - \phi_{\rm ref}) \rangle$	0.003	0	0.003	0	0.003	0
$\langle \Delta \sin^2(\phi - \phi_{\rm ref}) \rangle$	0.50	0.5	0.50	0.5	0.25	0.25

TABLE II. Same as Table I, but for a squeezed-vacuum signal state.

	Pegg-Barnett		Wigner		Susskind-Glogower	
	Expt.	Theory	Expt.	Theory	Expt.	Theory
$\langle \phi - \phi_{\rm ref} \rangle$	3.06	3.11	3.05	3.11		
$\langle \Delta(\phi - \phi_{\rm ref})^2 \rangle$	3.07	3.19	3.00	3.15		
$\langle \cos(\phi - \phi_{\rm ref}) \rangle$	−0.029	0	−0.036	0	−0.029	0
$\langle \Delta \cos^2(\phi - \phi_{\rm ref}) \rangle$	0.44	0.45	0.42	0.43	0.21	0.20
$\langle \sin(\phi - \phi_{\rm ref}) \rangle$	0.003	0	0.004	0	0.003	0
$\langle \Delta \sin^2(\phi - \phi_{\rm ref}) \rangle$	0.56	0.55	0.58	0.57	0.32	0.30

ly peaked than those for the PB phase distribution; this fact is also reflected in the experimental results.

Tables I and II give a summary of various statistical moments calculated using several different definitions of phase. Table I shows the moments for a vacuum signal, while Table II shows the moments for the squeezed signal. Generally the different definitions of phase yield the same mean values, but different values for higher moments. The values of the PB and Wigner moments are similar, with the phase variance of the Wigner phase distribution being slightly smaller, as expected from the above discussion. The Susskind-Glogower moments are dissimilar from the others. The experimental moments agree well in all cases with the theoretical moments. This emphasizes that any of these phase definitions may be taken to be a useful characterization of phase properties of the state.

Since our technique for inferring phase distributions is based on measurements of the state of the system, not on direct phase measurements, the results do not favor one phase definition over any other. We believe this to be a strength, not a weakness, of the technique, because it is possible to compare the results for many different phase definitions from a single set of data. If, however, one wishes to obtain phase information on individual measurements, our technique is not useful. In many circumstances what is important are phase *distributions*. In any method for determining a distribution many repeated measurements are required, whether they be direct phase measurements or measurements of some other quantity (here, quadrature amplitude). So, measuring a quantity other than phase is not necessarily a disadvantage.

We have demonstrated that it is possible to obtain distributions and/or moments for the phase of a mode of an electromagnetic field using optical homodyne tomography. The obtained phase information is referenced to the phase of a strong coherent state, and is intrinsically characteristic of the measured field; it is not tied to any particular measurement apparatus. These results have been demonstrated here for the case of a vacuum and squeezed states, but the technique will in principle work for arbitrary states of the field. It is notable that this information can be obtained without directly making measurements, or even estimates, of the phase of the field on individual measurements.

We wish to thank J. Cooper and W. Schleich for providing important insights into this work. This research is supported by the National Science Foundation (Grant No. PHY 8921709-01).

[1] P. A. M. Dirac, Proc. R. Soc. London, Ser. A **114**, 243 (1927).
[2] L. Susskind and J. Glogower, Physics **1**, 49 (1964).
[3] D. T. Pegg and S. M. Barnett, Phys. Rev. A **39**, 1665 (1989); S. M. Barnett and D. T. Pegg, J. Mod. Opt. **36**, 7 (1989).
[4] W. Schleich, R. J. Horowicz, and S. Varro, Phys. Rev. A **40**, 7405 (1989).
[5] J. H. Shapiro and S. R. Shepard, Phys. Rev. A **43**, 3795 (1991).
[6] J. W. Noh, A. Fougères, and L. Mandel, Phys. Rev. Lett. **67**, 1426 (1991); Phys. Rev. A **46**, 2840 (1992).
[7] J. A. Vaccaro and D. T. Pegg, J. Mod. Opt. **37**, 17 (1990).
[8] H. Gerhardt, U. Büchler, and G. Litfin, Phys. Lett. **49A**, 119 (1974).
[9] D. R. Matthys and E. T. Jaynes, J. Opt. Soc. Am. **70**, 263 (1980).
[10] M. M. Nieto, Phys. Lett. **60A**, 401 (1977).
[11] R. Lynch, Phys. Rev. A **41**, 2841 (1990).
[12] B. M. Garraway and P. L. Knight, Phys. Rev. A **46**, R5346 (1992).
[13] D. T. Smithey, M. Beck, A. Faridani, and M. G. Raymer, Phys. Rev. Lett. **70**, 1244 (1993).
[14] H. P. Yuen and V. W. S. Chan, Opt. Lett. **8**, 177 (1983).
[15] D. T. Smithey, M. Beck, M. Belsley, and M. G. Raymer, Phys. Rev. Lett. **69**, 2650 (1992).
[16] K. Vogel and H. Risken, Phys. Rev. A **40**, 2847 (1989).