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Necessity of sine-cosine joint measurement

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The quantum measurement of trigonometric variables is revisited. We show that the probability distributions of the sine and cosine operators of Susskind and Glogower [Physics **1**, 49 (1964)] suffer unphysical features for nonclassical states. We suggest that any measurement of a trigonometric variable needs necessarily a joint measurement of the two cosine-sine phase quadratures. In this way unphysical quantum statistics are avoided, and no violation of the trigonometric calculus occurs for expected values. We show that this trigonometric measurement can be defined in general terms in the framework of quantum estimation theory.

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What should a *cosine* probability distribution $d\mu(c)$ behave like? Let us consider the simplest situation of a state with uniformly distributed (i.e., perfectly random) phase. A constant phase distribution $d\mu(\phi) = d\phi/2\pi$ trivially does not correspond to a constant cosine distribution, as the Radon-Nikodym derivative leads to

$$d\mu(c) = \frac{1}{\pi}\frac{1}{\sqrt{1-c^2}}dc\ . \tag{1}$$

As illustrated in Fig. 1, the probability distribution has fixed positive curvature, and the $c = \pm 1$ bounds of the cosine range have infinite probability density, because they correspond to stationary points of the cosine. In the quantum-mechanical description of the harmonic oscillator the simplest example of random-phase states is provided by the number states $|n\rangle$. The outcomes of a set of physical measurements of the cosine for an oscillator in a number state should be randomly distributed according to the probability (1).

An ideal *cosine* measurement is customarily described in terms of the observable $\hat{c}$ introduced by Susskind and Glogower [1],

$$\hat{c} = \tfrac{1}{2}(\hat{e}^- + \hat{e}^+)\ , \tag{2}$$

where $\hat{e}^{\pm}$ denote the raising and lowering operators $\hat{e}_+|n\rangle = |n+1\rangle$, $\hat{e}_- \equiv (\hat{e}_+)^\dagger$. In a similar fashion one can also define a *sine* quadrature for the phase

$$\hat{s} = \frac{1}{2i}(\hat{e}^- - \hat{e}^+)\ . \tag{3}$$

The two operators $\hat{c}$ and $\hat{s}$ do not commute, even though they are thought of as functions of the same phase "variable." In fact, one has

$$[\hat{c},\hat{s}] = \frac{i}{2}|0\rangle\langle 0|\ . \tag{4}$$

As a consequence of Eq. (4) it is generally accepted that the above sine and cosine operators are well defined for states with a negligible vacuum component. This argument certainly holds true for semiclassical highly excited coherent states, where, however, the main quantum features are lost. On the other hand, the expectation values of the commutator (4) on number states $|n\rangle$ rigorously vanish for $n > 0$, but, as we will see, for such nonclassical states the probability distributions of the two observables suffer unphysical features also in the limit of large n.

Let us focus attention on the cosine operator (the case of the sine is similar). The probability distribution of $\hat{c}$ is given by

$$d\mu(c) = \langle c|\hat{\rho}|c\rangle dc\ , \tag{5}$$

where $\hat{\rho}$ is the density matrix of the states and $|c\rangle$ denotes the eigenstate of $\hat{c}$ corresponding to the eigenvalue c, namely [1, 2]

$$|c\rangle = \sqrt{\frac{2}{\pi}}(1-c^2)^{-1/4}\sum_{n=0}^{\infty}\sin[(n+1)\arccos c]|n\rangle\ . \tag{6}$$

One can simply check that the states (6) for $c \in [-1,1]$ correctly provide the resolution of identity, thus ensuring that (5) is a genuine probability distribution. For a number state $\hat{\rho} = |n\rangle\langle n|$, the latter is given by

$$\frac{d\mu(c)}{dc} = \frac{2}{\pi}\frac{\sin^2[(n+1)\arccos c]}{\sqrt{1-c^2}}\ . \tag{7}$$

In Fig. 2 the function (7) is plotted for $n = 0, 10, 50$, and is compared with the probability density (1). The dif-

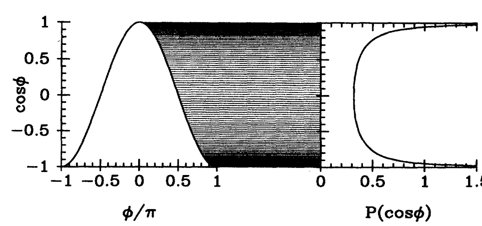

FIG. 1. Illustration of the probability distribution of the cosine for a perfectly random-phase state.

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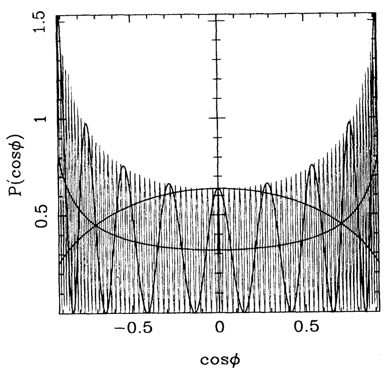

FIG. 2. The Susskind-Glogower cosine probability distribution for number states in comparison with the random-phase distribution of Fig. 1. The convex distribution refers to the vacuum; the slowly oscillating distribution is for $n = 10$, the rapidly oscillating one is for $n = 50$.

ferences with the random-phase probability (1) are striking: the vacuum distribution has the wrong curvature, whereas the distributions for number states depend on the number of photons n and oscillate rapidly around the function (1). Do such probabilities have an actual meaning? And what is the physical meaning of the observable $\hat{c}$ for meaningless probability distributions?

Despite the operator, (2) being the only self-adjoint one that has been devised for a quantum analog of the cosine, we assert that it cannot describe an actual cosine measurement. Here we show that an alternative way to define a cosine measurement can be naturally given in general terms in the framework of quantum estimation theory [3]. As we will see, the resulting expectation values are identical to those obtained by averaging the operator $\hat{c}$, but the probability distribution of the measurement outcomes is very different from Eq. (7), and does not suffer the above unphysical features. We analyze the cosine measurement both in the ideal and in a feasible scheme: in the ideal case the same results could also be equivalently obtained using the Pegg-Barnett approach [4].

The main ingredient of the quantum estimation theory of Helstrom [3] is the probability operator measure (or POM) $d\hat{\mu}(\phi)$, which gives the probability distribution $d\mu(\phi)$ of the phase ϕ for any state $\hat{\rho}$ according to the rule

$$d\mu(\phi) = \mathrm{Tr}[\hat{\rho}\, d\hat{\mu}(\phi)] \,. \tag{8}$$

The explicit form of the POM $d\hat{\mu}(\phi)$ in Eq. (8) depends on the particular detection scheme. At a purely abstract level, however, the theory provides the POM of the ideal detection, with the following form:

$$d\hat{\mu}(\phi) = \frac{d\phi}{2\pi} |e^{i\phi}\rangle\langle e^{i\phi}| \,. \tag{9}$$

In Eq. (9) $|e^{i\phi}\rangle$ denote the Susskind-Glogower phase states

$$|e^{i\phi}\rangle = \sum_{n=0}^{\infty} e^{in\phi} |n\rangle \,, \tag{10}$$

which form a nonorthogonal complete set of states for ϕ ranging in a chosen 2π interval (in the following we use $\phi \in [-\pi, \pi)$). From the POM (9) one can define any operator functions of the phase as follows:

$$\widehat{f(\phi)} = \int_{-\pi}^{\pi} d\hat{\mu}(\phi) f(\phi) \,. \tag{11}$$

Nonorthogonality of states $|e^{i\phi}\rangle$ leads in general to violations of the operator function calculus, namely, one has

$$\widehat{f(\phi)} \neq f(\hat{\phi}) \,, \tag{12}$$

where $\hat{\phi}$ is defined again as in Eq. (11) for $f(\phi) = 0$, the identical function. The cosine and sine operators which result from Eq. (11) coincide with those in Eqs. (2) and (3), and hence the same expectation values are obtained. However, due to violation (12) the POM operators for products (or higher powers) of sine and cosine are different from the corresponding products of operators (2) and (3), thus leading to different nonlinear moments and different probability distributions. Moreover, as a consequence of definition (11) the expectation values of all phase functions obey the customary trigonometric calculus, whereas the Susskind-Glogower operators lead to violations. For example, one has that

$$\mathrm{Tr}[\hat{\rho}(\hat{c}^2 + \hat{s}^2)] = 1 - \tfrac{1}{2}\langle 0|\hat{\rho}|0\rangle \,, \tag{13}$$

whereas in the POM description one obtains

$$\mathrm{Tr}[\hat{\rho}(\widehat{\sin^2\phi} + \widehat{\cos^2\phi})] = 1 \,. \tag{14}$$

One should notice that violations of the function calculus do not suffer interpretational problems when occurring at the operator level, whereas they lead to unavoidable difficulties when occurring at the expected-value level.

The probability distribution of the cosine in the present context is obtained with the rule

$$d\mu(c) = \mathrm{Tr}[\hat{\rho}\, d\hat{\mu}(c)] \,, \tag{15}$$

where $d\hat{\mu}(c)$ is the Radon-Nikodym derivative of the POM (9)

$$d\hat{\mu}(c) = \frac{dc}{\pi} \frac{1}{\sqrt{1-c^2}} |e^{i \arccos c}\rangle\langle e^{i \arccos c}| \,. \tag{16}$$

One can see that for any number state the probability (15) correctly corresponds to the random-phase distribution (1).

The present measurement scheme represents an ideal joint measurement of both sine-cosine phase quadratures [a thorough analysis of the physical interpretation of the POM (9) can be found in Ref. [5]]. As this measurement is only an ideal limit and does not correspond to any feasible detection scheme, here we also discuss an actual scheme, which corresponds to the joint measurement of a couple of conjugated field quadratures (obviously, being nonideal, this scheme will exhibit more noise than the ideal one).

The detection apparatus is the double-homodyne scheme of Ref. [6] (the quantum oscillator is represented by a selected mode of the electromagnetic field). In short,

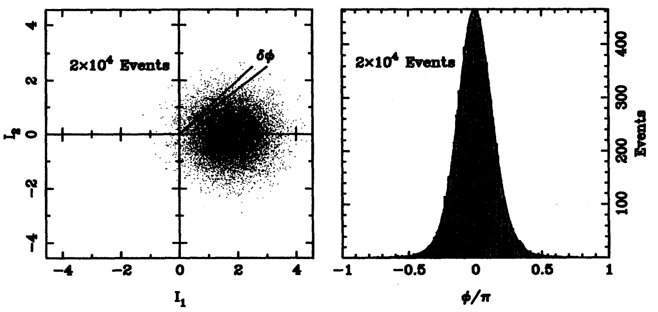

FIG. 3. Computer simulation of the double-homodyne experimental procedure for a coherent state. The experimental histogram (2×10^4 events) is compared with the theoretical result from the POM in Eq. (9). The angle $\delta\phi$ is a (amplified) representation of an angular bin used to construct the "experimental" histogram.

one measures two output commuting photocurrents $\hat{I}_1$ and $\hat{I}_2$ from two balanced homodyne detectors and, in the limit of strong local reference oscillator, the two currents have the same probability distribution of a joint measurement of the two field quadratures [7]. From the outcomes of the complex current $I = I_1 + iI_2$ (representing the amplitude of the field), one can infer the phase itself and the cosine-sine couple. In Fig. 3 the experimental procedure is illustrated on the basis of a computer simulation. The probability distribution of the phase is obtained in the framework of the quantum estimation theory using the following POM [8]:

$$d\hat{\mu}(\phi) = \frac{d\phi}{2\pi} \int_0^\infty \rho d\rho \, |\rho e^{i\phi}\rangle\langle\rho e^{i\phi}| \, , \tag{17}$$

where $|z\rangle$ ($z \in \mathbb{C}$) denote customary coherent states. For comparison in Fig. 3 the probability distribution from the POM (17) is superimposed onto the simulated histogram. The Radon-Nikodym derivative of the POM (17) and the evaluation of the integral lead to the POM for the cosine

$$d\hat{\mu}(c) = \frac{dc}{\pi} \frac{1}{\sqrt{1-c^2}} \times \sum_{n,m=0}^{\infty} \frac{\Gamma(1+(n+m)/2)}{\sqrt{n!m!}} e^{i(n-m)\arccos c} |n\rangle\langle m| \, . \tag{18}$$

For number states one correctly obtains the random-phase distributions (1), as in the ideal case. In Fig. 4 we compare the cosine probability for coherent states from POM's (16) and (18) with that from Eq. (5). One can see that the differences between the ideal POM distribution and the Susskind-Glogower one—which are dramatic for the vacuum state—become less and less relevant for highly excited coherent states. On the other hand, as previously announced, the double-homodyne (nonideal) distribution exhibits additional "instrumental" noise with respect to the ideal joint phase-quadrature measurement. We stress again that the high-energy agreement between the ideal and the Susskind-Glogower probabilities is found only for coherent—i.e., semiclassical—states, whereas for truly nonclassical ones (number states or highly squeezed states) the Susskind-Glogower distribution exhibits undesired oscillations, whereas the double-homodyne probability is qualitatively similar to the ideal one. Furthermore, as in the ideal case, the double-homodyne measurement does not lead to violations of the trigonometric calculus for expected values, because it corresponds again to a joint detection of the two phase quadratures.

In conclusion, we have seen that a customary quantum description of trigonometric measurements based on

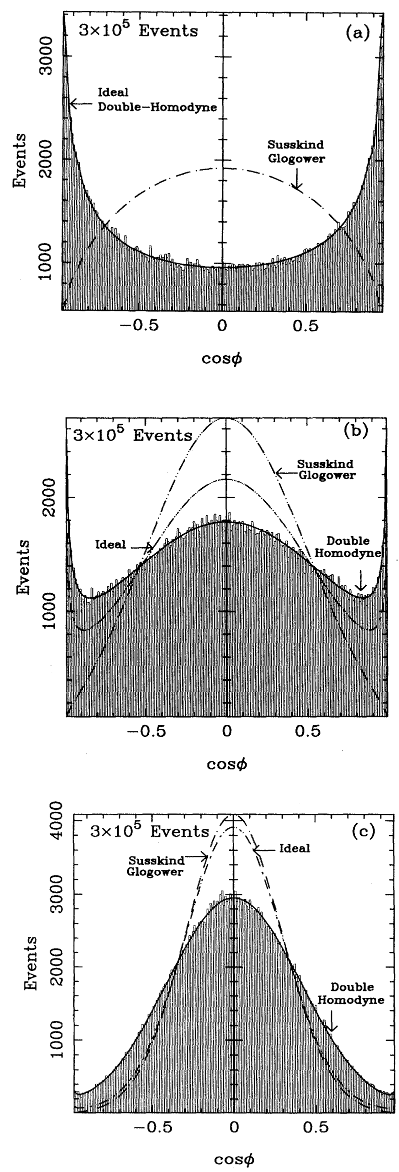

FIG. 4. Comparison for various coherent states of the three different cosine probability distributions: (i) the Susskind-Glogower distribution (5); (ii) the ideal distribution (15); and (iii) the double-homodyne distribution from POM (18). The histograms give the results of a simulated experiment for different value of average number of photons $\langle n\rangle$. (a) $\langle n\rangle = 0$; (b) $\langle n\rangle = 1$; (c) $\langle n\rangle = 3$.

the self-adjoint operators of Susskind and Glogower suffers unphysical probability distributions for nonclassical states, and leads to violations of the trigonometric calculus for expected values. In contrast, a correct quantum analysis of the phase detection, which consists of a joint measurement of both sine-cosine quadratures, does not suffer such unphysical features either in the ideal or in a nonideal feasible scheme.

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