PHYSICAL REVIEW A

ATOMIC, MOLECULAR, AND OPTICAL PHYSICS

THIRD SERIES, VOLUME 48, NUMBER 5 NOVEMBER 1993

RAPID COMMUNICATIONS

The Rapid Communications section is intended for the accelerated publication of important new results. Since manuscripts submitted to this section are given priority treatment both in the editorial office and in production, authors should explain in their submittal letter why the work justifies this special handling. A Rapid Communication should be no longer than 4 printed pages and must be accompanied by an abstract. Page proofs are sent to authors.

Quantum-algebraic description of quantum superintegrable systems in two dimensions

Dennis Bonatsos
Institute of Nuclear Physics, Nuclear Research Centre "Demokritos," GR-15310 Aghia Paraskevi, Attiki, Greece

C. Daskaloyannis and K. Kokkotas
Department of Physics, Aristotle University of Thessaloniki, GR-54006 Thessaloniki, Greece
(Received 10 May 1993)

An alternative method for the description of quantum superintegrable systems in two dimensions through the use of quantum algebraic techniques is introduced. It is suggested that such systems can be described in terms of a generalized deformed oscillator, characterized by a structure function specific to the system. The energy eigenvalues corresponding to a state with finite-dimensional degeneracy can then be determined directly from the properties of the relevant structure function. The validity of the method is demonstrated in the case of the isotropic harmonic oscillator in a space with constant curvature. The method can be used for constructing the quantum versions of several classical superintegrable systems, the Holt potential being given as an example.

PACS number(s): 03.65.Fd, 11.30.Na

I. INTRODUCTION

Quantum integrable systems and their relation to classical integrable systems have recently been attracting much attention [1–4]. Superintegrable systems in N dimensions have more than N integrals of motion, while maximally superintegrable systems have $2N-1$ integrals. The classical superintegrable systems in two dimensions have been reviewed in [5], while several examples of classical superintegrable systems in three dimensions are given in [6]. Two examples of quantum superintegrable systems, the isotropic harmonic oscillator and the Kepler problem in a space with constant curvature, have been studied in [7,8].

In the present work we are going to demonstrate how quantum-algebraic techniques can be used for the study of quantum superintegrable systems. It is known that q-deformed oscillators [9,10] are necessary for constructing boson realizations of quantum algebras (also called quantum groups) [11], which are nonlinear algebras reducing to the corresponding Lie algebras when the deformation parameter q is set equal to 1. We are going to show that the study of quantum superintegrable systems can be greatly simplified through the use of an appropriate generalized deformed oscillator [12].

II. CLASSICAL SUPERINTEGRABLE SYSTEMS

Let us first consider a classical superintegrable system in two dimensions, described by the Hamiltonian

$$H=H(x,y,p_x,p_y)\ . \tag{1}$$

If the system is superintegrable there are two independent additional integrals of motion I and C, such that

$$\{H,I\}_{\rm PB}=\{H,C\}_{\rm PB}=0\ ,\quad \{I,C\}_{\rm PB}=F(H,I,C)\ , \tag{2}$$

where $\{\ ,\ \}_{\rm PB}$ denotes the Poisson bracket and $F=F(H,I,C)$ is a constant of motion which depends on the three independent constants of motion H,I,C.

Superintegrable systems in two dimensions are necessarily maximally superintegrable, i.e., they possess the maximum number of independent classical invariants. Therefore any other integral can be expressed as a function of the basic integrals H,I,C. As a result we can in general choose two new integrals of motion,

$$L=L(H,I,C)\ ,\quad A=A(H,I,C)\ ,$$

such that

$$\{L,A\}_{\rm PB}=B\ ,\quad \{L,B\}_{\rm PB}=-A\ . \tag{3}$$

One can then prove that

$$B^2+A^2=G(H,L)\ ,$$

where $G(H,L)$ is some function depending only on the integrals of motion H,L, and

$$\{A,B\}_{\rm PB}=\Phi(H,L)=-\frac{1}{2}\frac{\partial G}{\partial L}\ . \tag{4}$$

The structure of the algebra defined by Eqs. (3) and (4) is in many ways similar to the algebraic structure of the deformed oscillator given in Refs. [12–15], where L is some kind of number operator, while A and B are like the creation and annihilation operators. Therefore it is quite natural to attempt to study the quantum superintegrable systems in terms of suitable generalized deformed oscillators, allowing for the determination of the energy spectrum through purely algebraic manipulations.

III. QUANTUM SUPERINTEGRABLE SYSTEMS

Let us now consider a two-dimensional quantum system described by a Hamiltonian H. H and all relevant operators are generated by nonlinear combinations of the basic algebra of generators x,p_x,y,p_y satisfying the usual commutation relations

$$[x,p_x]=[y,p_y]=i\ ,\quad \text{other commutators}=0\ .$$

The system is called *superintegrable,* by analogy to the classical definitions, if there are two operators, I and C, linearly independent of H and of each other, which commute with H but *not* with each other,

$$[H,I]=0\ ,\quad [H,C]=0\ ,\quad [I,C]\neq 0\ .$$

We propose the following working hypothesis: Consider the superintegrable systems for which one can construct an associative algebra,

$$\begin{aligned}
&\mathcal{N}=\mathcal{N}(H,I,C)\ ,\\
&\mathcal{N}^\dagger=\mathcal{N}\ ,\\
&\mathcal{A}=\mathcal{A}(H,I,C)\ ,\\
&[\mathcal{N},\mathcal{A}]=-\mathcal{A}\ ,\\
&\mathcal{A}^\dagger\mathcal{A}=\Phi(H,\mathcal{N})\ ,\\
&[\mathcal{A}^\dagger\mathcal{A},\mathcal{A}\mathcal{A}^\dagger]=0\ ,
\end{aligned} \tag{5}$$

where $\Phi(E,x)$ is a real positive function definite for $x\geq 0$ and

$$\Phi(E,0)=0\ . \tag{6}$$

From the above equations one can then prove that

$$[\mathcal{N},\mathcal{A}^\dagger]=\mathcal{A}^\dagger\ ,\quad \mathcal{A}\mathcal{A}^\dagger=\Phi(H,\mathcal{N}+1)\ .$$

If this construction is possible one can then define the Fock space for each energy eigenvalue,

$$\begin{aligned}
&H|E,n\rangle=E|E,n\rangle\ ,\\
&\mathcal{N}|E,n\rangle=n|E,n\rangle,\quad n=0,1\ldots\ ,\\
&\mathcal{A}|E,0\rangle=0\ ,\\
&|E,n\rangle=\left[\frac{1}{\sqrt{[n]!}}\right](\mathcal{A}^\dagger)^n|E,0\rangle\ ,
\end{aligned}$$

where

$$[0]!=1\ ,\quad [n]!=\Phi(E,n)[n-1]!\ .$$

In the case of a system with discrete energy eigenvalues, for every energy eigenvalue E there is some degeneracy of dimension N_d+1. Therefore the dimensionality of the Fock space corresponding to that energy eigenfunction should be equal to N_d+1. This is equivalent to the condition

$$\Phi(E,N_d+1)=0\ . \tag{7}$$

The two conditions (6) and (7), and the positiveness of the structure function $\Phi(E,x)$ suffice to determine the energy spectrum of the quantum maximally superintegrable systems. We are going to verify this fact below, using the two-dimensional isotropic harmonic oscillator in a space with constant curvature [7] as an example.

IV. EXAMPLE: HARMONIC OSCILLATOR IN A SPACE WITH CONSTANT CURVATURE

The curved space is geometrically described by the metric

$$ds^2=\frac{dx^2+dy^2+\lambda(xdy-ydx)^2}{[1+\lambda(x^2+y^2)]^2}\ ,$$

the flat space corresponding to $\lambda=0$. The harmonic oscillator in this space is defined in Ref. [7] by the Hamiltonian

$$H=\tfrac{1}{2}(\pi_x^2+\pi_y^2+\lambda L^2)+\frac{\omega^2}{2}(x^2+y^2)\ , \tag{8}$$

where

$$\begin{aligned}
&L=xp_y-yp_x\ ,\\
&\pi_x=p_x+\frac{\lambda}{2}[x(xp_x+yp_y)+(xp_x+yp_y)x]\ ,\\
&\pi_y=p_y+\frac{\lambda}{2}[y(xp_x+yp_y)+(xp_x+yp_y)y]\ .
\end{aligned} \tag{9}$$

As in [7] one can define the Fradkin [16] operators

$$\begin{aligned}
&B=S_{xx}-S_{yy}=(\pi_x^2+\omega^2x^2)-(\pi_y^2+\omega^2y^2)\ ,\\
&S_{xy}=\tfrac{1}{2}(\pi_x\pi_y+\pi_y\pi_x)+\omega^2xy\ .
\end{aligned} \tag{10}$$

These operators satisfy the following commutation relations:

$$[H,L]=[H,B]=0\ , \tag{11}$$

$$[L,B]=4iS_{xy}\ ,\quad [L,S_{xy}]=-iB\ , \tag{12}$$

i.e., the operators L,B do not commute, but they form a closed algebra with the operator S_{xy}.

The above relations suggest the possibility of expressing the two-dimensional harmonic oscillator algebra by using the deformed oscillator formulation:

$$\mathcal{N}=\frac{L}{2}-u\,1\ ,\quad \mathcal{A}^{\dagger}=\frac{B}{2}+iS_{xy}\ ,\quad \mathcal{A}=\frac{B}{2}-iS_{xy}\ , \tag{13}$$

where u is a constant to be determined and

$$\begin{aligned}
&[\mathcal{N},\mathcal{A}^{\dagger}]=\mathcal{A}^{\dagger}\ ,\\
&[\mathcal{N},\mathcal{A}]=-\mathcal{A}\ ,\\
&\mathcal{A}^{\dagger}\mathcal{A}=H^2-\left[\omega^2+\frac{\lambda^2}{4}+\lambda H\right](L-1)^2+\frac{\lambda^2}{4}(L-1)^4\\
&\quad=H^2-\left[\omega^2+\frac{\lambda^2}{4}+\lambda H\right](2\mathcal{N}+2u-1)^2\\
&\qquad+\frac{\lambda^2}{4}(2\mathcal{N}+2u-1)^4\\
&\quad=\Phi(H,\mathcal{N})\ ,
\end{aligned} \tag{14}$$

where the function $\Phi(E,x)$ is given by

$$\Phi(E,x)=E^2-\left[\omega^2+\frac{\lambda^2}{4}+\lambda E\right](2x+2u-1)^2+\frac{\lambda^2}{4}(2x+2u-1)^4\ , \tag{15}$$

and we can see that

$$\mathcal{A}\mathcal{A}^{\dagger}=\Phi(H,\mathcal{N}+1)\ .$$

The existence of a finite-dimensional representation of the oscillator algebra is equivalent to the existence of a maximum number $N+1$, which is a root of the structure function, with N being the dimensionality of the algebra representation, coinciding with the dimensionality of the appropriate Fock space. This restriction, combined with the annihilation of the structure function for $x=0$, is written as

$$\Phi(E,0)=0\ ,\quad \Phi(E,N+1)=0\ . \tag{16}$$

Solving this system of two equations with two unknowns, E and u, one obtains the eigenvalues of the harmonic oscillator in a space with constant curvature

$$E=E_N=\left[\omega^2+\frac{\lambda^2}{4}\right]^{1/2}(N+1)+\frac{\lambda}{2}(N+1)^2\ , \tag{17}$$

which coincide with the findings of [7], while the value of the constant

$$u=-\frac{N}{2}$$

determines through the first of Eq. (13) the angular momentum eigenvalues allowed for each N,

$$L=-N,-N+2,\ldots,N-2,N\ ,$$

in agreement with [7].

We have therefore proven that the quantum isotropic harmonic oscillator in a two-dimensional space with constant curvature can be described in terms of a generalized deformed oscillator, the eigenvalues of the energy, as well as the angular momentum values allowed for each energy eigenvalue, being directly obtainable from the properties of the relevant structure function, which can be put in the form

$$\begin{aligned}\Phi(E_N,x)=4x(N+1-x)[\lambda(N+1-x)+\sqrt{\omega^2+\lambda^2/4}]\\ \times(\lambda x+\sqrt{\omega^2+\lambda^2/4})\ .\end{aligned}$$

V. APPLICATION: THE HOLT POTENTIAL

Having verified in the previous section that the method works in a known [7] case, we now apply it in order to construct the quantum analogue of a classical superintegrable system, the Holt potential [17].

The classical superintegrable Holt system corresponds to the Hamiltonian

$$H=\tfrac{1}{2}(p_x^2+p_y^2)+(x^2+4y^2)+\frac{\delta}{x^2}\ . \tag{18}$$

This system has two additional classical invariants of motion,

$$T=p_y^2+8y^2\ ,\quad C=p_x^2p_y+8xyp_x-2x^2p_y+\frac{2\delta}{x^2}p_y\ , \tag{19}$$

the second of them (C) being a cubic function of the momenta. The quantum version of the Hamiltonian (18) corresponds to a quantum superintegrable system with two additional integrals:

$$T=p_y^2+8y^2\ ,\quad B=p_x^2p_y+4\{xy,p_x\}-2x^2p_y+\frac{2\delta}{x^2}p_y\ .$$

It is clear that the quantum integral B is the symmetrized version of the classical integral C. From the above definitions we can verify that

$$[H,T]=0\ ,\quad [H,B]=0\ ,$$

and

$$\begin{aligned}
&[T,B]=R,\quad [T,R]=32B\ ,\\
&[R,B]=-96+256\delta-64H^2+128HT-48T^2\ ,\\
&R^2-32B^2=1024H-704T+512\delta T-128TH^2\\
&\qquad+128T^2H-32T^3\ .
\end{aligned}$$

From the above closed nonlinear algebra we can define

$$\mathcal{N}=\frac{T}{\sqrt{32}}-u,\quad \mathcal{A}^{\dagger}=8B+\sqrt{2}R\ ,\quad \mathcal{A}=8B-\sqrt{2}R\ ,$$

where u is a constant to be determined. These operators correspond to a deformed oscillator algebra:

$$\begin{aligned}
&[\mathcal{N},\mathcal{A}^{\dagger}]=\mathcal{A}^{\dagger}\ ,\quad [\mathcal{N},\mathcal{A}]=-\mathcal{A}\ ,\\
&\mathcal{A}^{\dagger}\mathcal{A}=2^6(T-2\sqrt{2})\left[H-\frac{T}{2}+\sqrt{2}+\left[\frac{1+8\delta}{2}\right]^{1/2}\right]\\
&\qquad\times\left[H-\frac{T}{2}+\sqrt{2}-\left[\frac{1+8\delta}{2}\right]^{1/2}\right]\\
&\quad=\Phi(H,\mathcal{N})\ ,\\
&\mathcal{A}\mathcal{A}^{\dagger}=\Phi(H,\mathcal{N}+1)\ .
\end{aligned} \tag{20}$$

The corresponding structure function is defined by

$$\Phi(E,x)=2^{23/2}[(x+u)-\tfrac{1}{2}]\times\left[\frac{E}{\sqrt{8}}-(x+u)+\frac{1}{2}+\left[\frac{\sqrt{1+8\delta}}{4}\right]^{1/2}\right]\times\left[\frac{E}{\sqrt{8}}-(x+u)+\frac{1}{2}-\left[\frac{\sqrt{1+8\delta}}{4}\right]^{1/2}\right].$$

The existence of a finite representation of the algebra for each energy eigenvalue implies that the structure function satisfies Eq. (16). Therefore we can find the possible energy eigenvalues having degeneracy equal to $N+1$:

$$u=\tfrac{1}{2},\quad E_N=\sqrt{8}\left[N+1+\frac{\sqrt{1+8\delta}}{4}\right],$$

where $(1+8\delta)\geq 0$. The corresponding structure function is

$$\Phi(E_N,x)=2^{23/2}x(N+1-x)\left[N+1-x+\frac{\sqrt{1+8\delta}}{2}\right].$$

In the special case where $-\frac{1}{8}\leq\delta\leq\frac{3}{8}$ there are energy eigenvalues given by

$$u=\tfrac{1}{2},\quad E_N=\sqrt{8}\left[N+1-\frac{\sqrt{1+8\delta}}{4}\right],$$

and the structure function is

$$\Phi(E_N,x)=2^{23/2}x(N+1-x)\left[N+1-x-\frac{\sqrt{1+8\delta}}{2}\right],$$

which is positive for $0<x\leq N$ if $-\frac{1}{8}\leq\delta\leq\frac{3}{8}$.

In both cases the degeneracy of the levels is determined by $T=\sqrt{32}(\mathcal{N}+u)$. Since $\mathcal{N}$ obtains the $N+1$ values $0,1,\ldots,N$, as a result T also obtains $N+1$ values.

VI. DISCUSSION

The method presented here is of general applicability. Such a construction can be carried out for the Kepler system in a two-dimensional space with constant curvature, a quantum superintegrable system also studied in [7]. The method can also be used for constructing the quantum superintegrable versions of well-known classical superintegrable systems in two dimensions, such as the Fokas-Lagerstrom potential [18], the Smorodinsky-Winternitz potential ([19] and references therein), and the Hartmann potential ([20] and references therein). It can also be extended to quantum superintegrable systems in three dimensions [6]. Work in these directions is in progress.

In Ref. [21] the deformed oscillator theory has been connected to $N=2$ supersymmetric quantum mechanics, by proving that the deformed oscillator corresponds to an example of $N=2$ SUSY QM (supersymmetric quantum mechanics), where the operators $A^\dagger A$ and $AA^\dagger$ commute. Furthermore, in Ref. [22] a connection between $N=2$ SUSY QM and the representation of the Korteweg–deVries (KdV) equation has been provided. The connection of the deformed oscillator theory through $N=2$ SUSY QM to the representations of the KdV equations is an interesting open problem.

ACKNOWLEDGMENT

One of the authors (D.B.) is grateful to the Greek Ministry of Research and Technology for support.

[1] S. Weigert, Physica D **56**, 107 (1992).
[2] M. Robnik, J. Phys. A **19**, L841 (1986).
[3] H. J. Korsch, Phys. Lett. A **90**, 113 (1982).
[4] J. Hietarinta, J. Math. Phys. **25**, 1833 (1984).
[5] J. Hietarinta, Phys. Rep. **147**, 87 (1987).
[6] N. W. Evans, Phys. Rev. A **41**, 5666 (1990).
[7] P. W. Higgs, J. Phys. A **12**, 309 (1979).
[8] H. I. Leemon, J. Phys. A **12**, 489 (1979).
[9] L. C. Biedenharn, J. Phys. A **22**, L873 (1989).
[10] A. J. Macfarlane, J. Phys. A **22**, 4581 (1989).
[11] V. G. Drinfeld, in *Proceedings of the International Congress of Mathematicians,* edited by A. M. Gleason (American Mathematical Society, Providence, RI, 1987), p. 798.
[12] C. Daskaloyannis, J. Phys. A **24**, L789 (1991).
[13] C. Daskaloyannis, J. Phys. A **25**, 2261 (1992).
[14] C. Daskaloyannis and K. Ypsilantis, J. Phys. A **25**, 4147 (1992).
[15] D. Bonatsos and C. Daskaloyannis, Chem. Phys. Lett. **203**, 150 (1993).
[16] D. M. Fradkin, Am. J. Phys. **33**, 207 (1965).
[17] C. R. Holt, J. Math. Phys. **23**, 1037 (1982).
[18] A. S. Fokas and P. A. Lagerstrom, J. Math. Anal. Appl. **74**, 325 (1980).
[19] N. W. Evans, Phys. Lett. A **147**, 483 (1990).
[20] M. Kibler and P. Winternitz, J. Phys. A **20**, 4097 (1987).
[21] D. Bonatsos and C. Daskaloyannis, Phys. Lett. **307B**, 100 (1993).
[22] P. K. Roy, A. Lahiri, and B. Bagghi, Mod. Phys. Lett. A **5**, 1339 (1990).