Quantum optical master equations: The one-atom laser

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We present a detailed numerical study of the one-atom laser, that is, a single two-level atom interacting with one lasing mode, whereby both the atom and the photon field are coupled to reservoirs. The stationary as well as the dynamical properties of the model are calculated directly from the quantum master equation with the help of two numerical methods. These numerical methods do not need any quasi-probability representation and they do not require approximations. We find that the one-atom laser exhibits most of the typical features of a normal laser. In the region far below threshold some aspects, among them the linewidth, are changed due to eigenvalues of the master equation with imaginary parts. In this regime the complexity of the eigenvalues prominently enters the dynamical behavior.

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I. INTRODUCTION

With the successful realization of the one-atom maser [1,2] and advances in cavity-QED technology [3] an experimental investigation of a one-atom laser seems to be feasible in the near future, for it now appears possible to provide what is needed: a large single-mode coupling constant, a low field-dissipation rate, and a strong coupling between the atom and the pump. In such an experiment one would be able to study the laser process in detail. For example, it would be possible to investigate if the standard way of modeling atomic reservoirs correctly describes the pump process or if this approximation is too crude and leads to wrong results. In preparation of such an experiment we present a detailed numerical analysis of the standard laser model in the special situation where the lasing medium consists of a single atom. By the use of two methods we are able to solve the corresponding quantum master equation directly, without resorting to partial differential equations for some quasiprobability distribution. These approaches are advantageous because one does not have to make any structural approximations, such as neglecting higher-order derivatives to obtain a proper Fokker-Planck equation for the P function, as in one standard approach to the problem [4]. The first of our numerical methods uses the damping basis formalism that is presented in a recently published companion paper [5]. The essence of the second method is the direct integration of the time evolution of a slightly generalized density operator.

The paper is organized as follows. In Sec. II we present the model and introduce the objects we are going to calculate. Section III briefly describes the main ideas on how to tackle the problem: the use of the damping basis on the one hand and direct integration on the other hand. In Sec. IV the numerical solutions, concerning stationary as well as dynamical properties of the model, are presented. In Sec. V we summarize the results and point out the essential advantages of our approaches over the usual ones.

II. THE STANDARD MODEL

The standard laser model has N two-level atoms interact with one mode of a resonator. The pump process is modeled by N inverted reservoirs which are coupled directly to the atoms. The field mode interacts with the modes of free space, which also serve as a reservoir. Normally, this model is studied in the limit of a very large number of atoms. In this paper we shall concentrate on the other extreme case, that of one single atom. The corresponding quantum master equation is

$$\frac{\partial}{\partial t}P=\frac{1}{i\hbar}[H,P]+L_aP+L_sP\equiv LP \tag{1}$$

in which P denotes the density operator of the total system. The total Liouville operator L on the right-hand side consists of a unitary part and a nonunitary one. The unitary part involves, in an interaction picture, the usual dipole Hamiltonian (in the rotating wave approximation)

$$H=-\tfrac{1}{2}\hbar g(a^{\dagger}\sigma_{-}+a\sigma_{+})-\tfrac{1}{2}\hbar\Delta\sigma_z\ . \tag{2}$$

It describes the interaction between the photon mode and the atom. The dynamical variables are the ladder operators $a^{\dagger}$ and a for the field and $\sigma_{\pm}=\sigma_x\pm i\sigma_y$ for the atom, where σ_x, σ_y, and σ_z are Pauli's spin operators. The constants g and $\Delta=\omega_0-\Omega$ denote the coupling constant

and the detuning between the atom (level spacing $\hbar\Omega$) and the mode (frequency ω_0).

The nonunitary part accounts for losses to and gains from reservoirs. The first term,

$$L_a P = -\frac{A}{2}(\nu+1)[a^\dagger a P + P a^\dagger a - 2aPa^\dagger] - \frac{A}{2}\nu[aa^\dagger P + Paa^\dagger - 2a^\dagger P a] , \tag{3}$$

couples the field to a thermal reservoir at a temperature which corresponds to a mean number ν of thermal photons in the cavity. The second term,

$$L_s P = -\frac{B}{8}(1-s)[\sigma_+\sigma_- P + P\sigma_+\sigma_- - 2\sigma_- P\sigma_+] - \frac{B}{8}s[\sigma_-\sigma_+ P + P\sigma_-\sigma_+ - 2\sigma_+ P\sigma_-] - \frac{2C-B}{4}[P - \sigma_z P \sigma_z] , \tag{4}$$

describes incoherent pumping and decay processes of the atom. In (3) and (4) the constants A, B, and C are the relaxation rates of the mean photon number, the atomic inversion, and the atomic polarization, respectively, under the sole action of their reservoirs. The parameter s, ranging from 0 to 1, characterizes the nature of the atomic reservoir. If $s=0$, the terms model processes such as incoherent transitions from the upper to the lower level due to spontaneous emission of photons into modes other than the privileged cavity mode. The last term accounts for atomic dephasing; it does not change the level population. When $s>0$, there are also transitions from the lower to the upper level. In particular, when $s>\frac{1}{2}$, the atomic reservoir acts like a pump which inverts the atomic population. In the sequel we call s the pump parameter of the model. The steady-state behavior of a similar model is discussed in [6] and a treatment of this problem, with the help of the positive P representation, is given in [7].

To find the stationary as well as the dynamical properties of this model, it is expedient to know the Green's operator $G(t+\tau|t)$ of Eq. (1). This operator $G(t+\tau|t)$ is formally given by

$$G(t+\tau|t) = e^{L\tau} . \tag{5}$$

Once G is known explicitly, all quantities of interest can be computed. For instance, important stationary aspects of the model can be calculated as exemplified by the mean photon number $\langle n \rangle$:

$$\langle n \rangle = \lim_{t\to\infty} \mathrm{Tr}\{a^\dagger a G(t|t=0)P(t=0)\} = \mathrm{Tr}\{a^\dagger a P_{\mathrm{SS}}\} , \tag{6}$$

where $P(t=0)$ is an arbitrary initial state of the system, and P_{SS} is its stationary state.

More interesting than stationary aspects are dynamical properties of the system, such as the linewidth. The power spectrum, for instance, is the Fourier transform of the field correlation function

$$\gamma(t) = \frac{\langle a^\dagger(t) a(t=0)\rangle}{\langle a^\dagger(0) a(0)\rangle} ,$$

which can be calculated with the aid of the Green's operator G according to

$$\gamma(t) = \frac{\mathrm{Tr}\{a^\dagger G(t|t=0) a P_{\mathrm{SS}}\}}{\mathrm{Tr}\{a^\dagger a P_{\mathrm{SS}}\}} . \tag{7}$$

To find the correlation function we shall use two different strategies. In the first approach we get the Green's operator explicitly in terms of the eigenvalues and eigenstates (left and right) of the Liouville operator L. In the second strategy we calculate $\gamma(t)$ by integrating Eq. (1) directly with a generalized state P.

III. NUMERICAL METHODS

A. The damping basis

For the explicit construction of G it is expedient to solve the corresponding eigenvalue equations for the left and right eigenstates:

$$LP_n = \lambda_n P_n \tag{8}$$

and

$$\check{P}_n L = \check{L}\check{P}_n = \lambda_n \check{P}_n . \tag{9}$$

Here $\check{L}$ is the dual conjugate operator to L, whose defining property is

$$\mathrm{Tr}\{O(LP)\} = \mathrm{Tr}\{(\check{L}O)P\} , \tag{10}$$

for all states P and all observables O. As already anticipated in (9), the $\check{P}_n$ can be regarded as left eigenstates of L, dual to the right eigenstates P_n. Suppose we have solved (8) and (9) and found all eigenvalues λ_n and the corresponding left and right eigenstates $\check{P}_n$ and P_n, then the Green's operator $G(t|t=0)$ is known:

$$G(t|t=0)(\cdot) = \sum_n e^{\lambda_n t} P_n \mathrm{Tr}\{\check{P}_n \cdot\} . \tag{11}$$

The time development of the combined system, subject to an arbitrary initial condition, reads

$$P(t) = G(t|t=0)P(t=0) = \sum_n e^{\lambda_n t} P_n \mathrm{Tr}\{\check{P}_n P(t=0)\} , \tag{12}$$

and all relevant correlation functions, such as

$$\langle a^\dagger(t) a(t=0)\rangle = \mathrm{Tr}\left[a^\dagger \sum_n e^{\lambda_n t} P_n \mathrm{Tr}\{\check{P}_n a P_{\mathrm{SS}}\}\right] , \tag{13}$$

are available, too. To obtain the Green's operator by this method requires, in principle, the knowledge of infinitely many eigenvalues and eigenstates, but, since the convergence of the series in Eq. (13) is very rapid, the knowledge of a few eigenstates suffices for an extremely good approximation in most practical cases.

In order to solve the eigenvalue problem (8), we first

expand P into eigenstates of the nonunitary parts of the master equation:

$$P(t)=\sum_{\mu=1}^{4}\sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty}\breve{\xi}_{nk;\mu}(t)\rho_n^{(k)}\sigma_\mu\ , \tag{14}$$

with the spin operators

$$\begin{aligned}\sigma_1&=\tfrac{1}{2}[1+(2s-1)\sigma_z]\ ,\\ \sigma_2&=\sigma_z\ ,\\ \sigma_3&=\sigma_-\ ,\\ \sigma_4&=\sigma_+\ ,\end{aligned} \tag{15}$$

which are the eigenstates of L_σ, and the photon operators ($k\geq 0$)

$$\begin{aligned}\rho_n^{(k)}&=\rho_n^{(-k)\dagger}\\ &=\frac{(-1)^n}{(1+\nu)^{k+1}}a^{\dagger k}{:}L_n^{(k)}\left[\frac{a^\dagger a}{1+\nu}\right]\exp\left[-\frac{a^\dagger a}{1+\nu}\right]{:}\ ,\end{aligned} \tag{16}$$

which are the eigenstates of L_a; for details [5] should be consulted.

Expansion (14) transforms the master equation (1) into a system of differential equations for the coefficients $\breve{\xi}_{nk;\mu}$. When properly arranged, the corresponding eigenvalue equation (8) acquires the form of a three-term recurrence relation

$$\lambda\breve{X}_{n,k}=M_n^{(k)}\breve{X}_{n,k}+G_n^{(k)}\breve{X}_{n+1,k}+F_n\breve{X}_{n-1,k}\ , \tag{17}$$

which is Eq. (3.8) in [5]. Here, coefficients with different indices μ for the atomic degree of freedom make up the four-entry columns

$$\breve{X}_{n,k}=\begin{pmatrix}\breve{\xi}_{n+1,k;1}\\ \breve{\xi}_{n,k;2}\\ \frac{1}{i}\breve{\xi}_{n,k+1;3}\\ \frac{1}{i}\breve{\xi}_{n+1,k-1;4}\end{pmatrix} \tag{18}$$

(for $k>0$ and similarly for $k=0$ [8]), and $M_n^{(k)}$, $G_n^{(k)}$ as well as F_n are 4×4 matrices.

Note that in (17) only columns with the same index k are coupled. So, with this expansion, Eq. (8) has decayed into decoupled subsets of equations with different indices k. This simplifies the further numerical treatment substantially. The stationary state, for instance, belongs to the subset with $k=0$, whereas the subset with the index $k=1$ is needed for the calculation of the correlation function $\gamma(t)$.

The left eigenstates of Eq. (9) are found by employing a similar expansion as in (14), now using the left eigenstates of L_σ and L_a as a basis for $\breve{P}$. This leads to a recursion

$$\lambda X_{n,k}=X_{n,k}M_n^{(k)}+X_{n-1,k}G_{n-1}^{(k)}+X_{n+1,k}F_{n+1}\ , \tag{19}$$

where four-entry rows $X_{n,k}$ comprise the corresponding expansion coefficients and are acted upon by the same matrices that appear in (17).

In the special case where $\nu=0$ and $s=0$, (17) and (19) reduce to a two-term recurrence relation and can therefore be solved analytically [5]. In the general situation, $s\geq 0$ and $\nu\geq 0$, one is left with a truly three-term recursion which can be solved numerically with the method of matrix continued fraction [9]. To see this, let us introduce matrices $R_n^{(k)}$ which couple columns $\breve{X}_{n,k}$ with different indices n,

$$\breve{X}_{n+1,k}=R_n^{(k)}\breve{X}_{n,k}\ . \tag{20}$$

Then (17) is solved if

$$(M_n^{(k)}-\lambda)R_{n-1}^{(k)}+G_n^{(k)}R_n^{(k)}R_{n-1}^{(k)}+F_n=0 \tag{21}$$

or

$$R_{n-1}^{(k)}=\frac{1}{(\lambda-M_n^{(k)})-G_n^{(k)}R_n^{(k)}}F_n \tag{22}$$

holds, which gives a continued-fraction expansion for the matrices $R_n^{(k)}$. To find the eigenvalues we insert (20) into (17) for $n=0$ and $n=-1$, respectively. This leads to the quantization conditions

$$\det[\lambda-M_0^0-K_0^0(\lambda)]=0 \tag{23}$$

and

$$\det[\lambda-M_{-1}^{(k)}-K_{-1}^{(k)}(\lambda)]=0\quad\text{for } k=1,2,3,\ldots\ , \tag{24}$$

with

$$K_n^{(k)}(\lambda)=G_n^{(k)}\frac{1}{\lambda-M_{n+1}^{(k)}-K_{n+1}^{(k)}}F_{n+1}\ , \tag{25}$$

where $n=0,1,2,\ldots$ if $k=0$, and $n=-1,0,1,2,\ldots$ if $k=1,2,\ldots$. Equations (23) and (24) together with the steady-state value $\lambda=0$ provide us with all eigenvalues of the master equation (1) in the form of a continued-fraction expansion. The corresponding eigenstates are then obtained from (22) and (20). This supplies the general solution of the master equation and gives the explicit form of the Green's propagator G according to (11).

The eigenvalues and eigenstates for $k=0$ determine the dynamics of the field intensity and the atomic inversion. It is worth mentioning that the corresponding $k=0$ recursions (17) and (19) for the columns $\breve{X}_{n,0}$ and rows $X_{n,0}$ can be reduced to three-term recursions for one coefficient only. These can be solved with the aid of ordinary continued fractions.

B. Direct integration

In contrast to the preceding section where we found the Green's operator explicitly we are now trying to calculate the properties of the system without an explicit form of the Green's operator. For stationary properties, such as the mean photon number $\langle n\rangle$, it is easy to see how this works. From Eq. (6) one can see that it is sufficient to take an arbitrary initial state of the system, propagate this state for some time—that means just integrate Eq. (1) with this arbitrary state as a starting point—until the steady state is reached, and then per-

form the trace with the number operator $a^\dagger a$.

To perform the integration one has to use a basis in the product space of the mode and the atom. A convenient choice is given by the following basis states:

$$|n,\pm\rangle\ ,\quad n=0,1,2,\ldots\ , \tag{26}$$

where n labels the nth Fock state of the mode and $|\pm\rangle$ stand for the upper and lower atomic levels. The matrix elements of the state P then carry four indices:

$$P^{i,j}_{n,m}:=\langle n,i|P|m,j\rangle\ ,$$
$$n,m=0,1,2,\ldots\ ;i,j=+,-\ . \tag{27}$$

For the sake of simplicity, we choose $\Delta=0$ in this section, because that allows us to do all the calculations with real numbers only, provided that we also use a slightly different, but unitarily equivalent, form for the interaction Hamiltonian,

$$H=\frac{i}{2}\hbar g(a^\dagger\sigma_- - a\sigma_+)\ . \tag{28}$$

The equations of motion thus obtained are

$$\dot{P}^{-,-}_{n,m}=g(\sqrt{n}\,P^{+,-}_{n-1,m}+\sqrt{m}\,P^{-,+}_{n,m-1})+B(1-s)P^{+,+}_{n,m}$$
$$-\left[Bs+\frac{A}{2}[n+m+2(n+m+1)\nu]\right]P^{-,-}_{n,m}$$
$$+A(\nu+1)\sqrt{n+1}\sqrt{m+1}P^{-,-}_{n+1,m+1}$$
$$+A\nu\sqrt{nm}\,P^{-,-}_{n-1,m-1}\ , \tag{29}$$

$$\dot{P}^{-,+}_{n+1,m}=g(\sqrt{n+1}P^{+,+}_{n,m}-\sqrt{m+1}P^{-,-}_{n+1,m+1})$$
$$-\left[C+\frac{A}{2}[n+1+m+2(n+m+2)\nu]\right]P^{-,+}_{n+1,m}$$
$$+A(\nu+1)\sqrt{n+2}\sqrt{m+1}P^{-,+}_{n+2,m+1}$$
$$+A\nu\sqrt{(n+1)m}\,P^{-,+}_{n,m-1}\ , \tag{30}$$

$$\dot{P}^{+,-}_{n,m+1}=g(-\sqrt{n+1}P^{-,-}_{n+1,m+1}+\sqrt{m+1}P^{+,+}_{n,m})$$
$$-\left[C+\frac{A}{2}[n+m+1+2(n+m+2)\nu]\right]P^{+,-}_{n,m+1}$$
$$+A(\nu+1)\sqrt{n+1}\sqrt{m+2}P^{+,-}_{n+1,m+2}$$
$$+A\nu\sqrt{n(m+1)}P^{+,-}_{n-1,m}\ , \tag{31}$$

$$\dot{P}^{+,+}_{n,m}=g(-\sqrt{n+1}P^{-,+}_{n+1,m}-\sqrt{m+1}P^{+,-}_{n,m+1})+BsP^{-,-}_{n,m}$$
$$-\left[B(1-s)+\frac{A}{2}[n+m+2(n+m+1)\nu]\right]P^{+,+}_{n,m}$$
$$+A(\nu+1)\sqrt{n+1}\sqrt{m+1}P^{+,+}_{n+1,m+1}$$
$$+A\nu\sqrt{nm}\,P^{+,+}_{n-1,m-1}\ . \tag{32}$$

Similar sets of equations have been used by Savage and Carmichael [10] in their treatment of single-atom bistability. The most serious problem encountered with the integration of these equations is their enormous number. If one follows the method suggested in this paper, a large number of variables have to be handled even if Eqs. (29)–(32) are truncated for moderate photon numbers N (typically 20 000 variables for $N=100$).

But this is not necessary. In our situation we need to deal with not more than 400 variables, which is only four times the photon number $N=100$ at which we typically truncate our Eqs. (29)–(32). This can be seen as follows. The decoupling that is mentioned at Eq. (17) occurs here, too. Consider the four variables $P^{-,-}_{n,m}$, $P^{-,+}_{n+1,m}$, $P^{+,-}_{n,m+1}$, $P^{+,+}_{n,m}$ as a set $\tilde{P}_{n,m}$. Then one immediately observes that Eqs. (29)–(32) involve only $\tilde{P}_{n,m}$'s with a common difference $m-n=k$. This number k is identical with the k in Eqs. (17) and (19).

In the steady state all $\tilde{P}_{n,m}$ with $n\neq m$ (or $k\neq 0$) vanish, and this reduces the number of relevant equations, which superficially appears to be proportional to N^2, to a number that is proportional to N. Even if the master equation is supplemented by additional terms that lead to the phenomenon of bistability and introduce a coupling of $\tilde{P}_{n,m}$ with differing k values, most of the $k\neq 0$ variables will be very close to zero, the more so the larger $|k|$. Therefore neglecting these variables does not change the results. This is a straightforward way to reduce the complexity of the problem. And, in addition, it enables one to control the quality of the calculation rather easily by taking more and more off-diagonal elements into account. More about this is planned to be reported in a forthcoming paper [11].

To calculate the field correlation function a similar idea can be applied. As one can see from Eq. (7) all one has to change is the state P that has to be propagated in time. In contrast to the stationary case where P is an arbitrary density operator (of unit trace, positive semidefinite, and therefore Hermitian), now one has to propagate the operator $aP_{\rm SS}$ (traceless and not Hermitian anymore). Again this propagation is done by integrating Eqs. (29)–(32) with the starting point $P(t=0)=aP_{\rm SS}$, where $P_{\rm SS}$ is the stationary state of the system. Thus, at the initial time, only those $\tilde{P}_{n,m}$ are nonzero for which $m-n=k=1$ holds, and this remains so in the course of time. Consequently, again the number of the relevant variables is proportional to N, and not to N^2. Obviously this method can be extended to compute more complicated correlation functions.

IV. NUMERICAL RESULTS

When computing the basic results by these two independent methods we are in a position to test the accuracy of the calculations, and find that the agreement is always to five digits, at least. Both methods are therefore very reliable and the results are actual properties of the investigated system and not artifacts of the numerical treatment.

In Figs. 1 and 2 we concentrate on the stationary aspects of the one-atom laser. We have taken g as the natural unit of frequency and $1/g$ as the unit of time. The values of the other parameters A, B, C, ν, Δ ($A/g=1/20$, $B/g=2$, $C/g=1$, $\nu=0$, $\Delta=0$) are chosen such that for $s=1$ the model exhibits laser behavior. In Figs. 1(a)–1(d), the stationary reduced state $\rho=\mathrm{Tr}_{\rm atom}\{P_{\rm SS}\}$ for the mode is plotted for various pump

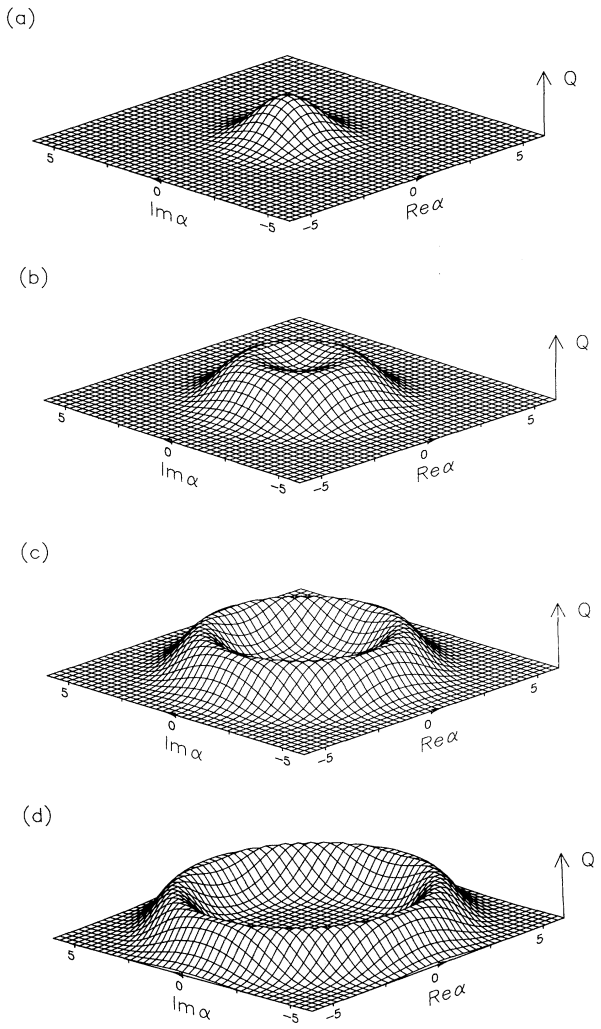

FIG. 1. Stationary Q function $Q(\alpha,\alpha^*)$ of the mode for different pump parameters s, (a) $s=0.4$, (b) $s=0.6$, (c) $s=0.8$, (d) $s=1.0$.

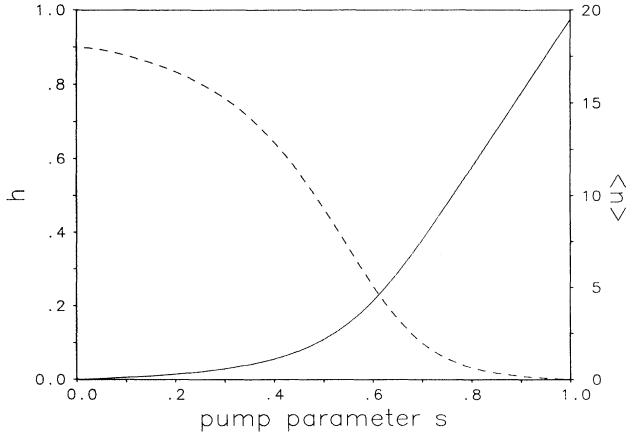

FIG. 2. Mean photon number (—) and h parameter (— — —) as a function of the pump parameter s.

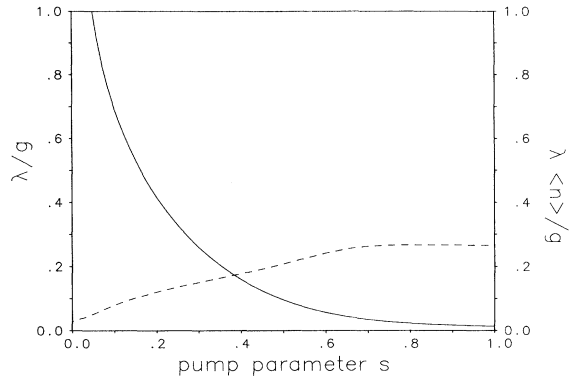

FIG. 3. Smallest real eigenvalue λ (—) and the linewidth factor $\lambda\langle n\rangle$ (— — —) as a function of the pump parameter s.

parameters s. The density operator ρ is here represented with the help of the corresponding Q function. Please note that this Q function serves pedagogical purposes only; it is calculated after we already know ρ in terms of the coefficients $\xi_{n,k;\mu}$ or in terms of the matrix elements $P^{--}_{n,m}$ and $P^{++}_{n,m}$. The plots show that even the one-atom laser exhibits typical laser behavior, which justifies the use of this name. Figure 2 shows the mean photon number and the parameter h (defined by $\langle\Delta n\rangle = \langle n\rangle + h\langle n\rangle^2$). Here the familiar aspects of the laser transition are confirmed as well. For small values of s the laser behaves like a thermal light source ($h\approx 1$) but for maximum pumping the emitted light follows a Poissonian distribution ($h\approx 0$). This result is in contrast to what Smith and Gardiner [7] found in their treatment of the one-atom laser for a similar set of parameters. Indeed, they have in effect $A/B=0.0001$ which should favor lasing more than our value $A/B=1/40$ does. Inasmuch as we get our result by two independent methods, we are quite confident that the one-atom laser does actually show laser behavior.

In Figs. 3–6 we are concerned with dynamical aspects of the system. In Fig. 3 the s dependence of the smallest real eigenvalue of the Liouville operator and the so-called linewidth factor $\lambda\langle n\rangle$ are plotted. As one can see in Fig. 4, the correlation function for large s is given by a single exponential. This means that only one single pair of left

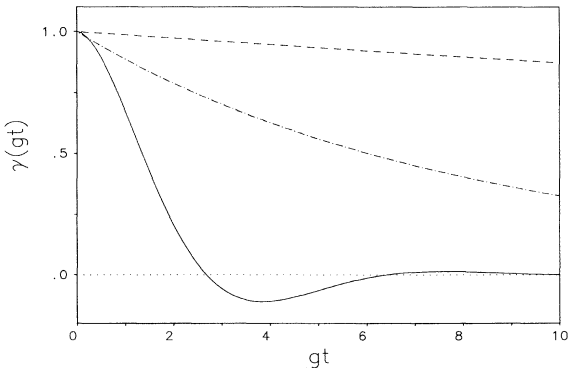

FIG. 4. Correlation function γ for $s=1$ (— — —), $s=0.5$ (·–·–), and $s=0.05$ (—).

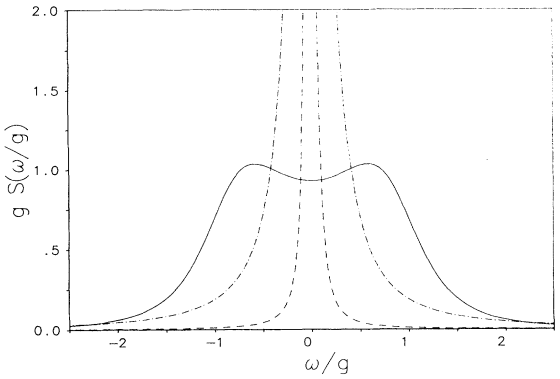

FIG. 5. Power spectrum S for $s=1$ (— — —), $s=0.5$ (·–·–), and $s=0.05$ (—).

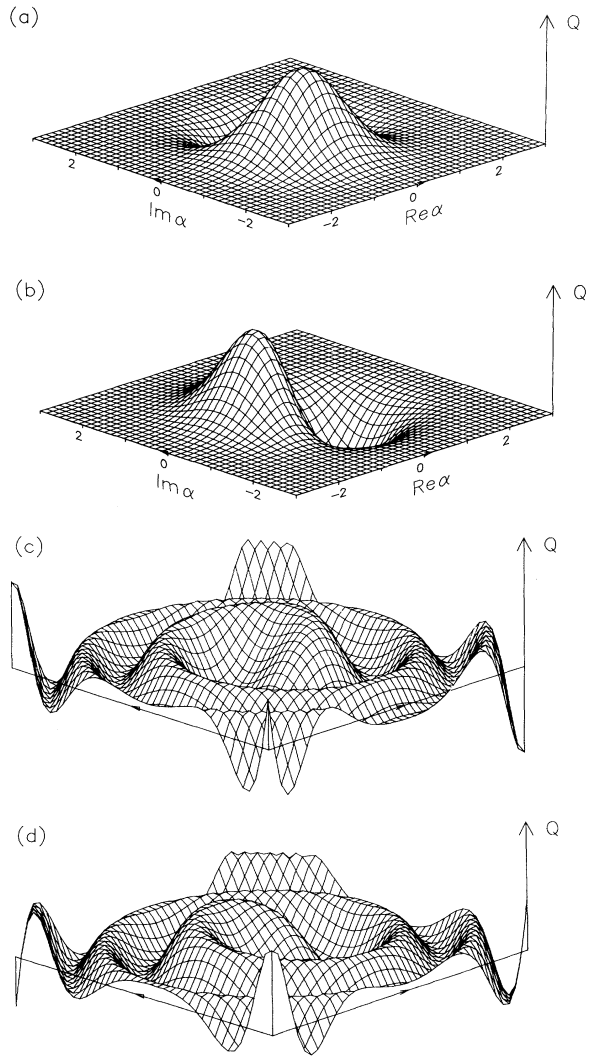

FIG. 6. Relevant complex eigenstate of the quantum master equation with the eigenvalue $\lambda/g=0.56-i0.89$. (a) Real part of the right eigenstate, (b) imaginary part of the right eigenstate, (c) real part of the left eigenstate, (d) imaginary part of the left eigenstate [(c) and (d) are plotted over the same domain as (a) and (b)].

and right eigenstates determines the Green's operator. The Fourier transform of this exponentially decaying function is a Lorentz curve (cf. Fig. 5) with the width given by the first eigenvalue λ. Figure 3 shows the s dependence of this smallest real eigenvalue and therefore, at least for large s, the s dependence of the linewidth. The linewidth factor $\lambda\langle n\rangle$ now shows that the linewidth of the one-atom laser is proportional to the inverse of the mean photon number for large pumping, a familiar aspect of laser theory. But Figs. 4 and 5 also demonstrate that for small values of s the field correlation function is not a pure exponential anymore. More eigenstates and eigenvalues must be taken into account and these eigenvalues may have nonvanishing imaginary parts, which give rise to the oscillations in the correlation function. As a consequence the power spectrum $S(\omega)$ is not a Lorentz curve. This is clearly visible in Fig. 5. The imaginary parts of the eigenvalues lead to sidebands that broaden the spectrum. These imaginary parts are remnants of the familiar Rabi splitting that survive the effects of damping and pumping for the parameters given. Note that ω is the frequency in the interaction picture; the physical frequency is $\omega_0+\omega$.

Figure 6 presents the pair of complex left and right eigenfunctions that determines the Green's operator for $s=0.05$. Please note, in particular, the somewhat funny shape of the left eigenstates in Figs. 6(c) and 6(d), reminiscent of a snail's shell. We emphasize that the knowledge of the left eigenstate is inevitable, if one calculates the correlation function with the aid of an eigenvalue approach. If one tries to obtain the correlation function with the help of a quasiprobability distribution it is far from easy to get the corresponding left eigenfunction of the partial differential operator that governs the time evolution [12]. For our second method this problem does not arise. One automatically obtains the effect of all eigenstates and therefore this method is a good check whether one has taken enough eigenstates.

The last figures (Figs. 7 and 8) demonstrate the effects of the detuning. In Fig. 7 the mean photon number is plotted as a function of the detuning for two different values of the parameter C ($C=B$ and $B/2$ for the solid and the dashed curve, respectively). Please note that the

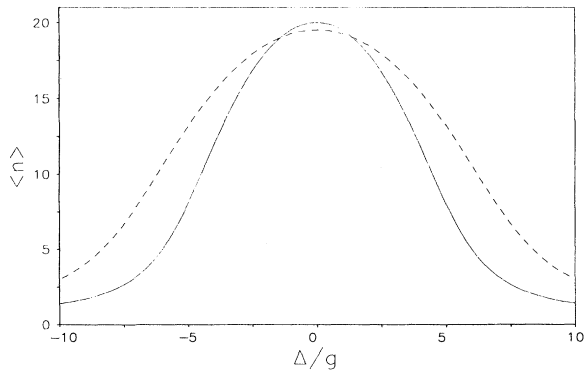

FIG. 7. Mean number of photons as a function of the detuning Δ for $C=B/2$ (—) and $C=B$ (— — —).

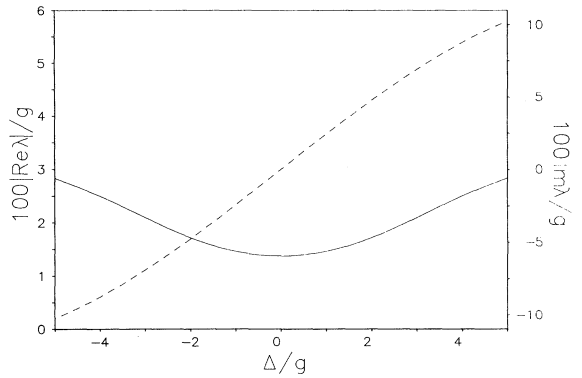

FIG. 8. Real (—) and imaginary (— — —) parts of the smallest eigenvalue as a function of Δ.

decrease in the photon number brought about by an increase of the detuning Δ can be partly compensated for by an increase in the parameter C, the decay constant for the atomic polarization. Figure 8 shows the real and imaginary parts of the smallest eigenvalue. The power spectrum $S(\omega)$ has its maximum for $\omega=\mathrm{Im}\lambda$. The linear dependence of Imλ on Δ near $\Delta=0$ is thus consistent with the well-known prediction of semiclassical laser theory ("mode pulling"), according to which

$$\mathrm{Im}\lambda=\frac{A}{A+B}\Delta\ , \tag{33}$$

wherein the ratio $A/(A+B)$ equals 1/41 for the chosen values. Both figures refer to $\nu=0.5$ and $s=1.0$. The other parameters g, A, B are as before.

V. CONCLUSION

We have demonstrated that the use of the two numerical methods presented here enables us to solve a fundamental physical problem: the one-atom laser, an open quantum system with two degrees of freedom (one of spin-$\frac{1}{2}$ type, the other a harmonic oscillator) coupled to two reservoirs, one active, the other passive. Although both methods are independent and follow completely different strategies, their results are in excellent agreement. Both are able to treat the quantum master equation directly. So most of the problems that are connected with a quasiprobability representation, such as the truncation of higher-order derivatives or negative diffusion, are avoided right from the start. In particular, with the help of the damping basis the left- and the right-eigenvalue problems can be treated on equal footing. In contrast, in the usual way of handling such problems with a quasiprobability distribution, the left-eigenvalue problem cannot be solved without serious additional mathematical difficulties. As a consequence the Green's operator of the problem cannot be calculated and therefore correlation functions cannot be computed systematically. For this reason the field correlation function is usually approximated by an exponentially decaying function and the power spectrum is then given by a Lorentz curve. But, as shown in Fig. 5, this is not the case in general.

The method of integrating the quantum master equation is a very direct way to get the correlation functions. It has the advantage that one automatically takes into account the effects of all the important eigenfunctions. With the observation that in most situations only a small fraction of the density matrix elements are different from zero, even problems that involve fairly large photon numbers can be handled with reasonable effort.

In most of our calculations we have used a fixed set of values for the parameters g, A, B, C, ν, Δ. These values are not chosen judiciously to produce particular features, such as lasing, but are typical numbers; other sets of parameters are handled just as easily. Note that in the model all the parameters have a definite microscopic meaning, in contrast to the standard approach. We have shown that the one-atom laser exhibits most of the typical features of a normal laser. In contrast to what has been found by Smith and Gardiner [7] we do believe that even the one-atom laser shows laser action for a set of parameters that are within the experimental reach. In the region far below threshold some aspects, among them the linewidth, are changed due to eigenvalues with nonvanishing imaginary parts. In this regime the complexity of the eigenvalues of the master equation prominently enters the dynamical behavior.

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