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Classical entropy of quantum states of light

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Classical-like, Wehrl's entropies of various quantum states of light are calculated and investigated as functions of the average value of the photon-number operator. Qualitatively similar behavior of the Wehrl entropies, corresponding to quantum states with quite different properties, is observed. A type of damping of the Wehrl entropies for states with high coherent components is pointed out and discussed.

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I. INTRODUCTION

Since the first mathematically consistent formulation of the Heisenberg uncertainty relations by Robertson [1], the standard deviation of an observable is commonly considered to be the most natural measure of the fundamental uncertainty connected with quantum fluctuations. In quantum optics such important effects as squeezing and antibunching are also defined using standard deviations of quadrature and photon-number operators [2]. Using standard deviations, we can also build various complex measures of the quantum uncertainty such as total noise [3] or uncertainty radius, surface, and volume [4]. Other moments are also useful, e.g., in the description of higher-order squeezing [5]. However in many cases standard deviations are not appropriate measures of the uncertainty, and some serious problems connected with using this quantity have been pointed out and discussed [6]. It is no wonder that alternative approaches based, e.g., on the concept of entropy have been intensively developed. It is worthwhile noticing in this context the entropic uncertainty relations [7]. Recently, statistical properties of light emitted from a two-level atom (in the framework of the Jaynes-Cummings model) have been studied using the Shannon and the von Neumann entropies [8].

It is known that the standard von Neumann definition of the quantum-mechanical entropy [9] gives zero for all pure states. Thus this entropy does not differentiate between various pure states (it is, rather, the measure of the "purity" of states). A different definition of "classical" entropy associated with a quantum state of the system under consideration has been proposed by Wehrl [10]. The definition exploits the notion of the Glauber coherent states and the Q representation [11] of the density operator. This entropy has many interesting properties and has stimulated very deep theoretical investigations. However, quantum optics applications of this concept seem to be very restricted. Marginal Wehrl's entropies of two-photon coherent states are considered in Ref. [12], but the Wehrl entropy itself is not calculated for these states therein. Very recently the Wehrl entropy of squeezed states has also been calculated in the context of entropic uncertainty relations [13]. In fact, we know a great deal about general properties of the Wehrl entropy, but we know its explicit value only for coherent states (for which it is equal to 1) and for squeezed states.

The main purpose of the present paper is to investigate whether the Wehrl entropy (which clearly displays the unique character of the coherent states) can provide a reasonable classification of states with respect to their nonclassical behavior. To check such a possibility we calculate analytically (once numerically) the explicit values of the Wehrl entropies for various quantum states of light. Let us note that the existence of a proper classification is a very important and nontrivial problem of modern quantum optics [14]. It is remarkable that in spite of evident differences between the considered states, their Wehrl entropies exhibit qualitatively very similar behavior if written as functions of the average value of the photon number operator $\langle \hat{N} \rangle$. Nevertheless, we observe that the Wehrl entropy is a good measure of the strength of the coherent component. In other words, it measures how "close" a given state is to the coherent states.

This paper is organized as follows. Section II contains the definition and the basic properties of the Wehrl entropy, including the Wehrl conjecture. In Sec. III, examples of the Wehrl entropies for two-photon coherent states (squeezed states), photon-number states, ideal laser light, chaotic (thermal) radiation, and displaced photon-number states (semicoherent states) are given. We finish with some comments and concluding remarks.

II. WEHRL'S ENTROPY

The standard von Neumann definition of the quantum-mechanical entropy reads [9]

$$S = -\mathrm{Tr}(\hat{\rho} \ln \hat{\rho}) , \tag{1}$$

where $\hat{\rho}$ is the density operator describing a given quantum state and the Boltzmann constant is taken to be $k=1$ for simplicity. Let us note that the above definition gives zero for all pure states $\hat{\rho}=\hat{\rho}^2$. Being the measure of the "purity" of states, this entropy does not differentiate between pure states. A different definition, which clearly exhibits the unique character of coherent states, has been proposed by Wehrl [10]. It reads

$$S = -\frac{1}{\pi}\int d^2\alpha\, Q(\alpha)\ln Q(\alpha) \ , \tag{2}$$

where

$$Q(\alpha) = \langle \alpha|\hat{\rho}|\alpha\rangle \ , \tag{3}$$

is the so-called Q representation of the density operator that satisfies the normalization condition

$$\frac{1}{\pi}\int d^2\alpha\, Q(\alpha) = 1 \ . \tag{4}$$

In contrast to other quasiprobability functions (such as the Wigner function or the P representation), this quantity is always positive and uniquely definite. However, it does not possess correct marginal properties, so it cannot be considered as a true probability distribution over the quantum-mechanical phase space (it is still only a quasidistribution). By $|\alpha\rangle$ we denote Glauber's coherent states as usual,

$$|\alpha\rangle = D(\alpha)|0\rangle = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})|0\rangle \ . \tag{5}$$

It was conjectured by Wehrl [10], and indeed proved by Lieb [15], that we always have

$$S \geq 1 \ , \tag{6}$$

and equality holds only for coherent states, i.e.,

$$\hat{\rho} = |\beta\rangle\langle\beta| = D(\beta)|0\rangle\langle 0|D^\dagger(\beta) \ . \tag{7}$$

It is easy to calculate that for coherent states we have $S=1$, but the proof that it is really a global minimum is very complicated [15].

III. EXAMPLES

In this section we calculate the Wehrl entropy for various quantum states of interest in quantum optics and its applications. To provide a sound basis for comparisons we rewrite (where necessary) obtained results as functions of the average number of photons $\langle\hat{N}\rangle$ in a given state.

A. Two-photon coherent states

Two-photon coherent states form the most important class of squeezed states. Introduced by Stoler [16] as generalized coherent states, they were studied in details by Yuen [17] as two-photon coherent states (see also [2] for equivalent definitions, applications, and references). Let us consider the Bogoliubov transformation [18] of the annihilation and creation operators $\hat{b} = \mu\hat{a} + \nu\hat{a}^\dagger$, where $|\mu|^2 - |\nu|^2 = 1$. Following Yuen we define these states as eigenstates of $\hat{b}$ with a complex eigenvalue β,

$$\hat{b}|\mu,\nu;\beta\rangle = \beta|\mu,\nu;\beta\rangle \ . \tag{8}$$

We have $\langle\hat{N}\rangle = |\nu|^2 + |\beta|^2$ in these states. Their Q representation reads

$$Q(\alpha) = \frac{1}{|\mu|}\exp(-|\alpha|^2 - |\beta|^2) \times \exp\left[-\frac{\nu}{2\mu}\alpha^{*2} + \frac{\nu^*}{2\mu}\beta^2 + \frac{1}{\mu}\alpha^*\beta + \text{c.c.}\right] \ . \tag{9}$$

After straightforward but strenuous calculations we obtain

$$S = \ln|\mu| + 1 \ . \tag{10}$$

Let us note that S does not depend on the coherent component β. Written as a function of the average number of photons $\langle\hat{N}\rangle$ it takes the form

$$S = 1 + \tfrac{1}{2}\ln(1 + \langle\hat{N}\rangle - |\beta|^2) \ . \tag{11}$$

Thus the Wehrl entropy of squeezed states with nonzero coherent component is smaller than the value obtained for the squeezed vacuum with the same average photon number (see Fig. 1). Moreover, in the limiting case of a coherent state ($|\mu|=1$), we immediately obtain the correct value $S=1$. Let us note that the independence from the coherent component is not accidental, and its origin lies in an invariance property of the Wehrl entropy, which will be briefly discussed in Sec. III E, which is devoted to displaced photon-number states.

B. Photon-number states

For these states, the Q representation is given by the Poisson distribution

$$Q(\alpha) = \frac{|\alpha|^{2n}}{n!}\exp(-|\alpha|^2) \ . \tag{12}$$

As expected, it does not depend on the phase of α. Direct integration in polar coordinates immediately gives

$$S = 1 + n + \ln n! - n\psi(n+1) \ , \tag{13}$$

where we have $\psi(n+1) = -\gamma + \sum_{k=1}^{n} 1/k$ and $\gamma \approx 0.577\,215\,664\,9$ is the well-known Euler constant. Also in this case we have the correct limiting behavior of the Wehrl entropy for $n=0$. It is seen from Fig. 2 that the qualitative behavior of S as a function of n is very similar to that of squeezed states but values of the Wehrl

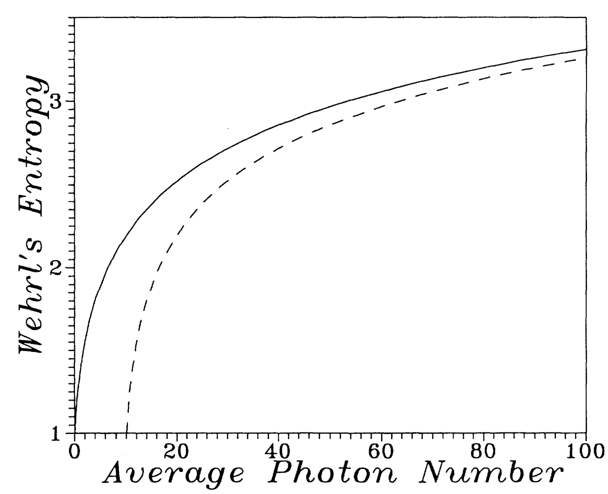

FIG. 1. Wehrl's entropy of the squeezed vacuum (solid line) and the squeezed state with the coherent component $|\beta|^2=10$ (dashed line).

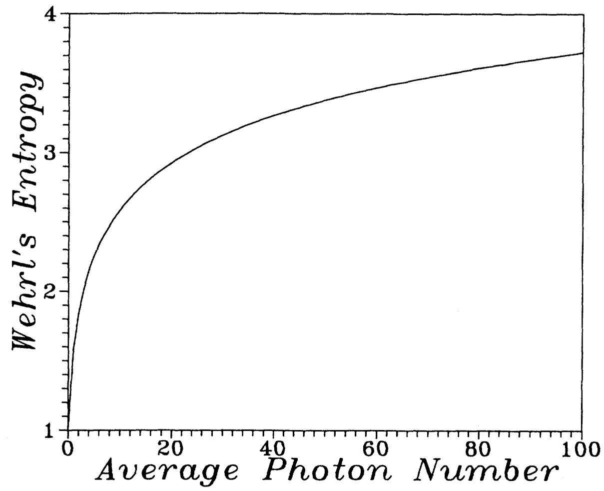

FIG. 2. Wehrl's entropy of photon-number states.

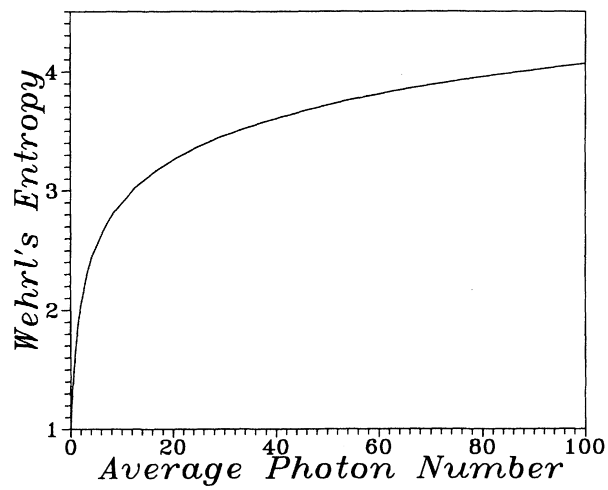

FIG. 3. Wehrl's entropy of the ideal laser light.

entropy for photon-number states are greater than S of the squeezed states corresponding to the same average number of photons.

C. Ideal laser light

It is commonly assumed that as the output of the model laser we obtain coherent states. But it is true only if we know precisely the absolute phase of the oscillating polarization generating the laser radiation. When we lack the knowledge of the polarization phase we should perform an average of coherent states over phase to get the proper density operator describing the laser light [19],

$$\hat{\rho}=\frac{1}{2\pi}\int_0^{2\pi}d\phi|\exp\iota\phi|\beta|\rangle\langle|\beta|\exp\iota\phi| \ . \tag{14}$$

As a result of this integration, we obtain a mixed state,

$$\hat{\rho}=\sum_{n=0}^{\infty}p(n)|n\rangle\langle n| \ , \tag{15}$$

where the probability of finding of n photons in the state is still given by the Poisson distribution

$$p(n)=\frac{|\beta|^{2n}}{n!}\exp(-|\beta|^2) \ , \tag{16}$$

and $|\beta|^2$ is the average number of photons. This kind of state is also known as a "random-phase" coherent state [20]. The Q representation of these states is given by

$$Q(\alpha)=\exp(-|\alpha|^2-|\beta|^2)\sum_{n=0}^{\infty}\frac{(|\alpha||\beta|)^{2n}}{(n!)^2} \ . \tag{17}$$

It can be written in a compact form using modified Bessel functions [21]:

$$Q(\alpha)=\exp(|\alpha|^2-|\beta|^2)I_0(2|\alpha||\beta|) \ . \tag{18}$$

In this case the Wehrl entropy cannot be calculated analytically but can be easily evaluated numerically. The results of numerical calculations are given in Fig. 3.

D. Thermal radiation

For light emitted by a source in thermal equilibrium at temperature T we have the Q representation [22]

$$Q(\alpha)=(1-\xi)\exp[-(1-\xi)|\alpha|^2] \ , \tag{19}$$

where $\xi=\exp(-\hbar\omega/kT)=\langle\hat{N}\rangle/(\langle\hat{N}\rangle+1)$. Also in this case it is convenient to perform the integration in polar coordinates. The Wehrl entropy is then equal to

$$S=1+\ln(1+\langle\hat{N}\rangle) \ . \tag{20}$$

For $\langle\hat{N}\rangle=0$, which corresponds to the zero temperature T, we have again $S=1$, as expected. Graphical presentation is given in Fig. 4. Making comparisons among investigated states, we see that for a given average number of photons, the Wehrl entropy takes the maximum value just for thermal radiation.

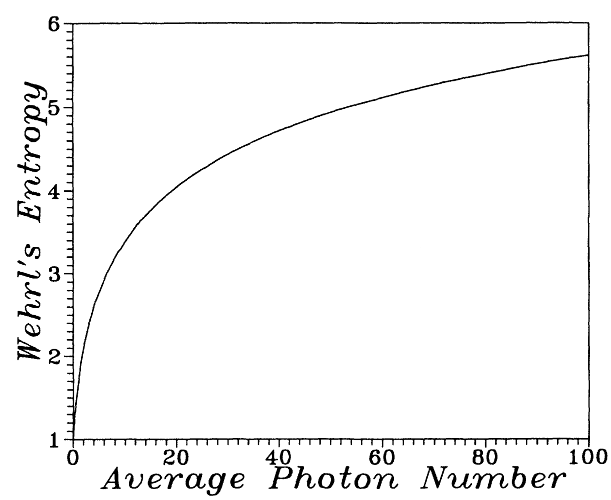

FIG. 4. Wehrl's entropy of the thermal radiation.

E. Displaced photon-number states

These states are defined as a result of action of the displacement operator $D(\beta)$ on a photon-number state different from the vacuum,

$$|\beta,n\rangle=\exp(\beta\hat{a}^{\dagger}-\beta^{*}\hat{a})|n\rangle\ . \tag{21}$$

The idea of such states can be found in early work of Cahill and Glauber [23]. They are also known as semicoherent states [24] or generalized coherent states [25]. Recently, various properties of these states have been extensively studied [26] in the context of nonclassical behavior. It can be easily shown that their Wehrl entropy is the same as that for the corresponding photon-number states, but with a different value of the average photon number. Indeed, we have

$$\langle\beta,n|\alpha\rangle=\exp[\iota\,\mathrm{Im}(\alpha\beta^{*})]\langle n|D(\alpha-\beta)|0\rangle\ . \tag{22}$$

Thus for the Q representation we get the formula

$$Q(\alpha)=\frac{|\alpha-\beta|^{2n}}{n!}\exp(-|\alpha-\beta|^{2})\ . \tag{23}$$

In this case, the integral defining the Wehrl entropy has the form of a convolution on the complex plane and does not depend on the parameter β. So the Wehrl entropy is the same as calculated above for photon-number states but the average number of photons is now equal to $n+|\beta|^2$. Of course, this observation is generally valid. For arbitrary states of the form

$$D(\beta)\hat{\rho}D^{-1}(\beta)\ , \tag{24}$$

the Wehrl entropy does not depend on the coherent component β. It is connected with the above-mentioned invariance property of the Wehrl entropy under the above transformation of states. However, it should be noted that it is not true for a general unitary transformation but only for the displacement operators [10].

IV. REMARKS

We have calculated analytically the Wehrl entropies of various quantum states of light. We have seen that despite the apparent differences in the quantum properties of the considered states, the Wehrl entropies of them exhibit qualitatively very similar behavior. But we have also observed that Wehrl entropies of coherently displaced states are smaller than Wehrl entropies of original states if written as functions of $\langle\hat{N}\rangle$. Having the same shape, they are displaced with respect to the $\langle\hat{N}\rangle$ axis. Thus the Wehrl entropy can be considered as a good measure of the strength of the coherent component and, in this specific sense, also the classical properties of states. Similar qualitative behavior of the Wehrl entropies of states under consideration is obviously connected with the similar shape of their Q representations (Gaussian or modified Gaussian shapes). But why do such different states have so similar Q representations? The answer to this question is beyond the scope of this paper.

Finally, let us remark about possible connections of the Wehrl entropy with another measure of nonclassical properties, namely, the total noise [3]. There are some interesting similarities. First, the total noise is also invariant under the state transformation (24). Second, it also takes the minimum value for the coherent states. However, for a given average photon number, the total noise has the same value for the squeezed vacuum, the photon-number state, and all mixed states, including our examples of the thermal radiation and the ideal laser light. Thus the Wehrl entropy is the more sensitive measure, in the sense that it can distinguish between these states.

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