Paraxial wave optics and harmonic oscillators

G. Nienhuis and L. Allen*
Huygens Laboratorium, Rijksuniversiteit Leiden, Postbus 9504, 2300 RA Leiden, The Netherlands
(Received 23 November 1992)

The operator algebra of the quantum harmonic oscillator is applied to the description of Gaussian modes of a laser beam. Higher-order modes of the Hermite-Gaussian or the Laguerre-Gaussian form are generated from the fundamental mode by ladder operators. This approach allows the description of both free propagation and refraction by ideal astigmatic lenses. The paraxial optics analog of a coherent state is shown to be a light beam with a displaced beam axis which is refracted by lenses according to geometric optics. The expectation value of the orbital angular momentum of a paraxial beam of light is found to be expressible in terms of a contribution analogous to the angular momentum of the oscillator plus contributions which arise from the ellipticity of the wave fronts and of the light spot. This clarifies the process by which a transfer of orbital angular momentum between a light beam and astigmatic lenses or diaphragms occurs.

PACS number(s): 42.50.Vk, 42.25.Bs

I. INTRODUCTION

It is well known that the analytical form of Gaussian modes of a laser beam resembles the wave functions of the stationary states of a two-dimensional quantum-mechanical harmonic oscillator [1,2]. This suggests that algebraic treatments of the harmonic oscillator may be fruitfully applied to wave optics. Such a treatment can give insight into the structure and the properties of paraxial modes, and the connection between modes of different order. Moreover, algebraic methods often greatly simplify explicit calculations of physical quantities. Recently it has been pointed out that a Laguerre-Gaussian beam carries an orbital angular momentum along its propagation direction [3]. This angular momentum arises from the transverse momentum density of the field. It might be expected that the orbital angular momentum of a beam of light be analogous to the angular momentum of the harmonic oscillator.

In this paper, operator algebra is applied to describe Gaussian modes of a laser beam in the presence of ideal, but possibly, astigmatic lenses. Raising and lowering operators are introduced, which generate all higher-order modes from the fundamental one. These operators depend in a simple way on the coordinate in the propagation direction, and they can be expressed as a unitary transformation of the real harmonic-oscillator ladder operators. The ladder operators are characterized by three z-dependent beam parameter for each transverse dimension. These parameters are the radius of curvature, the spot size, and the phase. The algebraic connection between modes with different mode indices is the same as that for the number states of the harmonic oscillator. The fundamental mode is the eigenvector of the lowering operator with eigenvalue zero. The connection between modes of different order, and between Laguerre-Gaussian and Hermite-Gaussian modes, follows directly. Eigenvectors with different eigenvalues, which are analogous to the coherent states, represent beams with a displaced axis. The axis of the beam simply obeys the rules of geometric optics while leaving unaffected the three beam parameters, which reflect the wave-optical nature of the beam. This separation between geometric-optical and wave-optical aspects of the beam during its propagating through lens systems follows in a direct fashion from the algebraic treatment. Finally, we apply the operator algebra to the description of the orbital angular momentum of a Gaussian beam, the expression for which is found to separate into a term that is analogous to the orbital angular momentum of the isotropic two-dimensional harmonic oscillator, a contribution expressing the ellipticity of the spot size, and a term arising from the ellipticity of the wave fronts. Only this final term is modified when the beam passes an astigmatic lens. The torque exerted on the lens vanishes when the intensity distribution of the beam is symmetric with respect to one of the axes of the lens. This explains why a nonastigmatic Laguerre-Gaussian beam cannot transfer angular momentum to an astigmatic lens [4]. However, we demonstrate that such a beam transfers a significant amount of angular momentum to the lens if one applies angular aperturing of the beam just in front of the lens. Such a transfer depends only on the radial mode index.

II. OPERATOR ALGEBRA FOR FREE PARAXIAL MODE

As is well known, a complete set of solutions of the wave equation of a beam of light in the paraxial approximation consists of the products of a Gaussian with a Hermite polynomial H_n [1,2]. For a light beam propagating in the z direction, and a single transverse dimension x, these Hermite-Gaussian modes can be expressed as the normalized functions [2]

$$u_n(x,z)=\exp\left[\frac{ikx^2}{2s(z)}-i\chi(z)(n+\tfrac{1}{2})\right]\times\frac{1}{\sqrt{\gamma(z)}}\psi_n\left[\frac{x}{\gamma(z)}\right]. \quad (2.1)$$

Here the functions ψ_n represent the real normalized eigenfunctions of the harmonic oscillator, defined by the Hamiltonian

$$H=\frac{1}{2}\left[-\frac{\partial^2}{\partial\xi^2}+\xi^2\right] . \tag{2.2}$$

Their explicit expressions are

$$\psi_n(\xi)=[2^n n!\sqrt{\pi}]^{-1/2}\exp(-\xi^2/2)H_n(\xi) . \tag{2.3}$$

The formal analogy between the Hermite-Gaussian modes and the harmonic-oscillator eigenstates offers the possibility of applying the operator algebra of the harmonic oscillator to paraxial beam optics.

The Hermite-Gaussian mode functions (2.1) depend on three z-dependent mode parameters γ, s, and χ. The expressions for the spot size γ and radius of curvature of the wave front s can be combined in the single complex equality

$$\frac{1}{\gamma^2}-\frac{ik}{s}=\frac{k}{b+iz} , \tag{2.4}$$

with k the wave number of the light. The parameter b is the Rayleigh range of the beam, which determines the size of the focal region. Finally, the phase factor χ is given by

$$\tan\chi=z/b . \tag{2.5}$$

In the present case of Gaussian beams, this factor describes the phase jump of π that occurs over the focal region of any spherical converging wave, which was first recognized by Gouy [5,2]. The basis set (2.1) depends on the value of the Rayleigh range b and on the location of the focal plane, which we have chosen at $z=0$. The spot size at focus is equal to $\sqrt{b/k}$, and it increases as $z/\sqrt{bk}$ for $z \gg b$. Hence a small spot size at focus implies a large divergence angle of the beam.

Equation (2.1) displays both the analogy and the difference between the propagation of a Hermite-Gaussian mode of light and the evolution of a harmonic oscillator. The diffraction of the beam is expressed by a variation of the spot size γ and the radius of curvature s with z. This is absent for the oscillator. As may be noted in (2.1), the variation of the Gouy phase χ during propagation, which multiplies the level energy $n+\frac{1}{2}$, is analogous to the phase Ωt of the oscillator with frequency Ω during its evolution. In this sense, the free propagation of a Gaussian beam from $-\infty$ to ∞ corresponds to half a cycle of the oscillator.

The Hermite-Gaussian modes (2.1) are solutions of the paraxial wave equation [1,2,6]

$$\frac{\partial^2}{\partial x^2}u(x,z)=-2ik\frac{\partial}{\partial z}u(x,z) . \tag{2.6}$$

When the variations of u in the transverse direction x are small over a wavelength, a solution u of (2.6) determines, to a good approximation, the electric field $\mathbf{E}$ of a monochromatic light wave with frequency $\omega=ck$, of the form

$$\mathbf{E}(\mathbf{r},t)=\mathrm{Re}\mathbf{E}_0 u(\mathbf{r})e^{ikz-i\omega t} . \tag{2.7}$$

The constant transverse vector $\mathbf{E}_0$ determines the polarization and the amplitude of the paraxial beam. Equation (2.6) has the same form as Schrödinger's equation for a free particle in one dimension, with the z coordinate replacing the time variable [7]. The formalism of quantum mechanics in terms of a Hilbert state space, and with linear operators representing observables, can be carried over directly to a classical light beam in the paraxial approximation, as has been done by Stoler [8]. In the same spirit, we introduce state vectors $|u(z)\rangle$, which have $u(x,z)$ for the mode function. Furthermore, we define the coordinate operator X, and the momentum operator P, according to the equalities

$$Xu(x,z)=xu(x,z), \quad Pu(x,z)=\frac{1}{i}\frac{\partial}{\partial x}u(x,z) . \tag{2.8}$$

(Operators acting on states vectors are indicated by roman capitals throughout this paper.) Then the propagation equation (2.6) can be put in the form

$$\frac{d}{dz}|u(z)\rangle=-\frac{i}{2k}P^2|u(z)\rangle . \tag{2.9}$$

Equation (2.9) has the formal solution

$$|u(z)\rangle=U(z)|u(0)\rangle , \tag{2.10}$$

with the propagation operator U defined by

$$U(z)=\exp\left[-\frac{i}{2k}P^2 z\right] . \tag{2.11}$$

To complete the analogy with quantum mechanics, we assume normalized state vectors obeying the identity

$$\langle u(z)|u(z)\rangle \equiv \int dx\, u^*(x,z)u(x,z)=1 . \tag{2.12}$$

When $|u\rangle$ is normalized for one position z, the normalization for all other z values is automatic. For an inner product defined as in (2.12), the operators X and P are Hermitian. The expectation value of operators is defined in the standard way as

$$\langle P(z)\rangle \equiv \langle u(z)|P|u(z)\rangle = \int dx\, u^*(x,z)Pu(x,z) . \tag{2.13}$$

It may be shown that with this normalization (2.12) $\hbar\langle P(z)\rangle$ is equal to the transverse momentum per photon in the beam [4].

A. Ladder operators

The expressions (2.1), which are solutions of the paraxial wave equation (2.6), can be obtained by operator algebra techniques. Our starting point is the introduction of z-dependent field operators, which allow a series of solutions to be created from one. For $z=0$, we define the operators

$$A(0)=\frac{1}{\sqrt{2bk}}[kX+ibP], \quad A^\dagger(0)=\frac{1}{\sqrt{2bk}}[kX-ibP] , \tag{2.14}$$

where b enters as a free parameter. These operators are real: they transform a real function into another real function. The z dependence of these operators is defined by the requirement that, for any solution $|u(z)\rangle$ of (2.9), $A^\dagger(z)|u(z)\rangle$ is also a solution. This implies that

$$A(z)=U(z)A(0)U^\dagger(z), \quad A^\dagger(z)=U(z)A^\dagger(0)U^\dagger(z) . \tag{2.15}$$

For free propagation, explicit expressions for these z-dependent operators follow from the operator identity

$$\exp\left[-\frac{i}{2k}P^2 z\right]X\exp\left[\frac{i}{2k}P^2 z\right]=X-\frac{z}{k}P . \tag{2.16}$$

Combining (2.15) and (2.16) gives the simple z dependence

$$A(z)=\frac{1}{\sqrt{2bk}}[kX+i(b+iz)P] , \quad A^\dagger(z)=\frac{1}{\sqrt{2bk}}[kX-i(b-iz)P] . \tag{2.17}$$

These operators obey the standard commutation rule

$$[A(z),A^\dagger(z)]=1 \tag{2.18}$$

for boson raising and lowering operators and, as shown in every quantum mechanics textbook, this commutation rule is sufficient to prove that the number operator

$$N(z)=A^\dagger(z)A(z) \tag{2.19}$$

has the natural numbers as its eigenvalue spectrum. If we indicate the eigenstates for $z=0$ as $|u_n(0)\rangle$, and define their z dependence by

$$|u_n(z)\rangle=U(z)|u_n(0)\rangle , \tag{2.20}$$

then these states $|u_n(z)\rangle$ are solutions of (2.9), obeying the eigenvalue relation

$$N(z)|u_n(z)\rangle=n|u_n(z)\rangle , \tag{2.21}$$

for $n=0,1,2,\ldots$. Moreover, the phases of these states can be chosen so that

$$A^\dagger(z)|u_n(z)\rangle=\sqrt{n+1}|u_{n+1}(z)\rangle , \quad A(z)|u_n(z)\rangle=\sqrt{n}\,|u_{n-1}(z)\rangle . \tag{2.22}$$

Hence, as for the harmonic oscillator, the higher-order modes can be obtained from the fundamental mode by applying the raising operator, according to

$$|u_n(z)\rangle=\frac{1}{\sqrt{n!}}[A^\dagger(z)]^n|u_0(z)\rangle . \tag{2.23}$$

We have thus proved the existence of a set of normalized solutions $|u_n(z)\rangle$ of (2.9), which are coupled by the operators $A^\dagger(z)$ and $A(z)$ according to the usual relations for the ladder operators of the harmonic oscillator.

These results allow the analytic form of the Hermite-Gaussian modes (2.1) to be explained in terms of the harmonic-oscillator eigenfunctions. In order to show this, we rewrite the operators (2.17) as a transformation of real operators, by using the transformation

$$\exp\left[\frac{ik}{2s}X^2\right]P\exp\left[-\frac{ik}{2s}X^2\right]=P-\frac{k}{s}X , \tag{2.24}$$

which is fully analogous to (2.16). This allows us to write (2.17) in the alternative form

$$A(z)=\exp\left[\frac{ik}{2s}X^2\right]B(z)\exp\left[-\frac{ik}{2s}X^2\right]\exp(i\chi) , \tag{2.25}$$

with $B(z)$ the real operator

$$B(z)=\frac{1}{\sqrt{2}}\left[\frac{X}{\gamma}+i\gamma P\right] . \tag{2.26}$$

The conjugate expressions hold for $A^\dagger$ and $B^\dagger$. As is obvious from (2.1), when A operates on a mode $u_n(x,z)$, the real operator B acts on the harmonic-oscillator state ψ_n. The z-dependent quantities s, γ, and χ are given in (2.4)–(2.6). As the higher-order modes $|u_n(z)\rangle$ can be generated from the fundamental one by repeated action of $A^\dagger$, it is sufficient to find the analytical form of the mode function $u_0(x,z)$. This is not difficult if one realizes that the lowering operators $A(z)$ must give zero when operating on the fundamental mode. For the normalized mode function, this gives as a solution of (2.6)

$$u_0(x,z)=\left[\frac{bk}{\pi}\right]^{1/4}\frac{1}{\sqrt{b+iz}}\exp\left[-\frac{kx^2}{2(b+iz)}\right] , \tag{2.27}$$

which may be rewritten in terms of the ground state of the harmonic oscillator as

$$u_0(x,z)=\exp\left[\frac{ik}{2s}x^2\right]\exp\left[-\frac{i\chi}{2}\right]\frac{1}{\sqrt{\gamma}}\psi_0\left[\frac{x}{\gamma}\right] . \tag{2.28}$$

The analytical form (2.1) of the higher-order modes follows directly after applying Eq. (2.23) and the recognition that the operator $B^\dagger$ is the raising operator for the harmonic-oscillator eigenfunctions with Gaussian width γ.

B. Coherent states and the Gouy phase

In view of the analogy of the complete set of Hermite-Gaussian modes with a harmonic oscillator, or equivalently with a single quantized field mode, it is natural to consider the analog of a coherent state [9]. Such a state arises from the ground state after the application of a displacement in phase space. The displacement operator

$$D(0)=\exp[iq_0X-ia_0P] \tag{2.29}$$

displaces position over a_0, and momentum over q_0, according to the relations [10]

$$D^\dagger(0)XD(0)=X+a_0, \quad D^\dagger(0)PD(0)=P+q_0 . \tag{2.30}$$

We consider the mode which, for $z=0$, is equivalent to the displaced ground state

$$|u(0)\rangle=D(0)|u_0(0)\rangle . \tag{2.31}$$

This state is an eigenstate of the lowering operator $A(0)$ with complex eigenvalue

$$\alpha_0=\frac{1}{\sqrt{2bk}}[ka_0+ibq_0] . \tag{2.32}$$

The solution of (2.9), which is given by the coherent state (2.31) for $z=0$, can be expressed as

$$|u(z)\rangle=D(z)|u_0(z)\rangle , \tag{2.33}$$

with

$$D(z)=U(z)D(0)U^{\dagger}(z) . \tag{2.34}$$

Obviously, $|u(z)\rangle$ is an eigenvector of $A(z)$ with eigenvalue α_0. Application of (2.16) shows that $D(z)$ is again a displacement operator, and takes the form

$$D(z)=\exp[iq(z)X-ia(z)P] , \tag{2.35}$$

with the z-dependent displacements of position and momentum

$$a(z)=a_0+\frac{z}{k}q_0,\quad q(z)=q_0 . \tag{2.36}$$

Hence the state $|u(z)\rangle$, which is the analog of a coherent state, is simply the fundamental mode $|u_0(z)\rangle$, with the transverse momentum displaced by the constant amount q_0, and the position by $a(z)$, as given by (2.36). This equation simply describes a ray of light, tilted with respect to the z axis at an angle q_0/k.

In view of the analogy between the Gouy phase χ and the phase of the oscillator, we may expect that the variation of the displacements (2.36) with z corresponds to a variation in time of the displacement of a coherent state of the oscillator during its evolution. This may be illustrated by rewriting the displacement operator in the form

$$D(z)=\exp[\alpha_0A^{\dagger}(z)-\alpha_0^{*}A(z)] , \tag{2.37}$$

with α_0 given in (2.32). If we substitute the alternative form (2.25) of A and $A^{\dagger}$ into (2.37), we obtain an expression for D in terms of the real field operators

$$D(z)=\exp\left[\frac{ik}{2s}X^2\right]\exp[\alpha(z)B^{\dagger}(z)-\alpha^{*}(z)B(z)]\times\exp\left[-\frac{ik}{2s}X^2\right] , \tag{2.38}$$

with

$$\alpha(z)=\alpha_0e^{-i\chi} . \tag{2.39}$$

In Eq. (2.39), the z-dependent displacement is described by the dimensionless quantity α, which indicates the relative displacement in units of γ. In contrast, the displacements in position and momentum in Eq. (2.35) are indicated by a and q on an absolute scale. Hence the Gouy phase jump over π near a focus, which corresponds to an oscillation over half a cycle of the relative displacement, corresponds to the rectilinear motion of the light on an absolute scale.

III. IDEAL LENSES

For the description of experiment, it is necessary to extend the field-operator description by including optical elements, such as lenses and diaphragms. An ideal lens is sufficiently thin so that no propagation occurs in the lens. Its only effect is to add an x-dependent phase factor to the field. When a lens with focal length f with its center on the z axis is located at the position z_1, the relation between the field incident on the lens and the outgoing field is given [8] by

$$u(x,z_+)=\exp\left[-\frac{ik}{2f}x^2\right]u(x,z_-) , \tag{3.1}$$

with z_+ (z_-) a position immediately behind (before) the lens. This local phase jump can be described by writing for the propagation operator

$$U(z_+)=\exp\left[-\frac{ik}{2f}X^2\right]U(z_-) . \tag{3.2}$$

When the lens at position z_1 is the only one in the interval $[0,z]$, the propagation operator is equal to

$$U(z)=\exp\left[-\frac{i}{2k}P^2(z-z_1)\right]\exp\left[-\frac{ik}{2f}X^2\right]\times\exp\left[-\frac{i}{2k}P^2z_1\right] , \tag{3.3}$$

for $z\geq z_1$. In this way, we can compose the propagation operator for an optical axis with an arbitrary set of lenses, with free propagation in between lenses.

A. Ladder operators and fundamental mode

If we start with a definition of a lowering operator $A(0)$, as in Eq. (2.14), then the propagation operator U defines the field operators for all values of z, according to (2.15). As the commutation relation (2.18) is unaffected by the presence of lenses, we still have the fundamental mode as an eigenstate of $A(z)$ with eigenvalue zero, together with the higher-order modes, which are related by (2.22). The analytical expressions are modified to account for the presence of lenses, but the operator algebra remains the same.

It is important to recognize that the explicit analytical expressions for all Hermite-Gaussian modes are fully determined by the radius of curvature s, the spot size γ, and the phase χ. The z dependence of these three parameters, which fully describe the fundamental mode for a given configuration of lenses, defines the framework of the operator algebra. To understand this, it is sufficient to note that a field operator A, defined as an arbitrary linear combination of the operators X and P for a single z value, keeps a similar form for all positions. This is obvious from the transformation rules (2.16) and (2.24). Physically, this reflects the fact that a Gaussian beam remains Gaussian during free propagation and in passage through (ideal) lenses.

If an optical axis with an arbitrary number of lenses of arbitrary focal length is considered, we can define the lowering operator A at one position as a linear combination of X and P, and define the z dependence of $A(z)$ in terms of the propagation operator, as in (2.15). Then A retains a similar form for all z, and we may write

$$A(z)=\frac{1}{\sqrt{2}}[\kappa(z)X+i\beta(z)P] , \quad (3.4)$$

with κ and β complex-valued functions of z. In order that A and $A^\dagger$ obey the commutation rule (2.18), κ and β must be related by the equality

$$\mathrm{Re}\kappa\beta^*=1 . \quad (3.5)$$

Upon inversion, Eq. (3.4) takes the form

$$X=\frac{1}{\sqrt{2}}[\beta^* A+\beta A^\dagger], \quad P=\frac{1}{i\sqrt{2}}[\kappa^* A-\kappa A^\dagger] . \quad (3.6)$$

In an interval without lenses, where free propagation occurs, the z dependence of κ and β may be determined using the transformation (2.16). We find that κ is constant in such an interval, whereas β varies linearly with z. The derivatives are

$$\frac{d\beta}{dz}=\frac{i\kappa}{k}, \quad \frac{d\kappa}{dz}=0 . \quad (3.7)$$

On the other hand, application of (2.24) shows that a lens with focal length f does not change β, but it does cause a jump in κ. The result is

$$\beta(z_+)=\beta(z_-), \quad \kappa(z_+)=\kappa(z_-)+\frac{ik\beta}{f} . \quad (3.8)$$

Hence, β is a continuous function that varies linearly between lenses with a slope proportional to κ. Conversely, κ makes jumps at the lens positions with a jump size that depends on the local value of β. In the absence of any lens, we again find the ladder operators to be given by (2.17).

The z-dependent lowering operator A has an eigenvector $|u_0(z)\rangle$ with eigenvalue zero, which is a solution of (2.7) with a propagation operator that is modified by the lenses. This light beam is the fundamental Gaussian mode for this arbitrary lens configuration, and its normalized wave function is equal to

$$u_0(x,z)=[\beta\sqrt{\pi}]^{-1/2}\exp\left[-\frac{\kappa x^2}{2\beta}\right] . \quad (3.9)$$

It is important to note that the intensity distribution has width $|\beta|$, whereas the momentum distribution, which is determined by the Fourier transform of (3.9), has width $|\kappa|$. When κ and β vary according to the rules (3.6) and (3.7), this expression obeys the evolution equation (2.6) between two lens positions and makes the phase jump (3.1) at the lens positions. This result generalizes that for which no lenses are present given in (2.27).

B. Higher-order modes

In order to obtain the analytic form for the higher-order modes, it is convenient to express the lowering operator (3.4) and the fundamental mode function (3.9) in terms of the radius of curvature s, the spot size γ, and the phase χ, as in (2.25) and (2.28). The quantities β and κ, with the normalization (3.5), uniquely determine the values of the fundamental mode parameters s, γ, and χ, by the relations

$$s=-k\left[\mathrm{Im}\frac{\kappa}{\beta}\right]^{-1}, \quad \gamma=|\beta|, \quad \chi=\arg\beta . \quad (3.10)$$

These results generalize (2.4) and (2.5). In the intervals between lenses, the variation of s, γ, and χ is determined by the linear variation of β with z, as expressed in (3.7). In the region of negative values of s, the beam converges and γ decreases, while the beam diverges where s is positive. A focus occurs where κ has the same argument as β, so that $s=0$. The phase angle χ can only increase, and it varies most rapidly at a focus. Since a lens does not change β, both γ and χ remain unmodified by the lens. According to (3.8), κ/β changes by ik/f, and the only effect of the lens is a change of the radius of curvature,

$$\frac{1}{s(z_+)}=\frac{1}{s(z_-)}-\frac{1}{f} , \quad (3.11)$$

in accordance with (3.1). This is just the lens formula of geometric optics. Since the expressions (2.25)–(2.27) remain valid, we can apply the raising operator to the fundamental mode repeatedly to obtain the higher-order modes. We conclude that they are given by the same expression (2.1) as in the case of free propagation, but with the behavior of the fundamental mode parameters as determined in this section. In the presence of lenses, the spot size γ can pass through several foci. Correspondingly, the Gouy phase χ can execute a series of π jumps. This is equivalent to the harmonic oscillator, which is the analog of the light beam in the presence of lenses, performing a number of cycles. The algebraic connection between the modes of various orders is completely unaffected by these changes in the three fundamental mode parameters.

C. Coherent states and ray optics

The discussion of Sec. II B on displaced fundamental modes in analogy to coherent states remains largely valid in the presence of lenses. The displacement of the spot at $z=0$, as described by $D(0)$ defined in (2.29), leads to a displacement for arbitrary z, where the operator $D(z)$ is still defined in (2.34). The only difference is that now the propagation operator $U(z)$ is changed. Likewise, the expression (2.37) for $D(z)$ in terms of the z-dependent field operators A and $A^\dagger$ remains valid, and, moreover, these operators still obey Eqs. (2.25) and (2.26), and (2.38) and (2.39) still hold. Hence the expression for the displacement on the relative scale in units of γ remains the same. The effect of the lenses is completely hidden in the modified behavior of the fundamental mode parameters s, γ, and χ.

The refraction by the lenses becomes directly obvious when we express the displacement on the absolute scale, by rewriting $D(z)$ in the form (2.35) in terms of a momentum displacement $q(z)$ and a position displacement $a(z)$. In an interval between two lenses, where free propagation occurs, the z dependence of q and a is determined by the transformation (2.16). One finds that a varies linearly with z, whereas q is constant. The derivatives are

$$\frac{da}{dz}=\frac{q}{k}, \quad \frac{dq}{dz}=0 . \tag{3.12}$$

Applying (3.11) shows that the effect of a lens with focal length f is described by the transformation

$$a(z_+)=a(z_-), \quad q(z_+)=q(z_-)-\frac{ka}{f} . \tag{3.13}$$

One should recall that q/k has the significance of the angle of a light ray with the z axis, whereas a is the distance of the ray from the axis at the position z. Hence Eq. (3.12) describes the rectilinear motion of a ray of light in free space, while (3.13) describes the refraction of the ray by a lens with focal length f.

Hence, the behavior of the axis of a beam in a displaced fundamental mode (the analog of a coherent state) simply obeys the laws of geometric optics. This behavior is completely independent of the variation of the fundamental beam parameters s, γ, and χ, which are a reflection of the wave-optical nature of the beam. The variation of these beam parameters is governed by the transformation (3.7) and (3.8) for β and κ, which strongly resemble the geometric-optical rules (3.12) and (3.13). This is no accident, as both pairs of rules follow from the transformations (2.16) and (2.24). In summary, a Gaussian beam is completely specified by its first and second moments. The first moments give the average position and momentum: namely a and q. They determine the geometric-optical properties of the beam. The second moments, which are specified by the complex quantities β and κ, determine the position and momentum widths of the fundamental mode, and describe the wave aspect of the beam.

IV. ORBITAL ANGULAR MOMENTUM

In this section, we apply the operator formalism to describe the orbital angular momentum of a paraxial beam of light. This requires the explicit consideration of the two transverse dimensions. Then we have two momentum operators P_x and P_y for the x and the y direction, and two position operators X and Y. We introduce the operator

$$L=XP_y-YP_x , \tag{4.1}$$

in analogy to the quantum operator of orbital angular momentum. In cylindrical coordinates r and ϕ, with

$$x=r\cos\phi, \quad y=r\sin\phi , \tag{4.2}$$

L takes the well-known form

$$L=\frac{1}{i}\frac{\partial}{\partial\phi} . \tag{4.3}$$

The total angular momentum of a radiation field is given by Maxwell's theory in the form [11]

$$\mathbf{J}=\varepsilon_0\int d\mathbf{r}\{\mathbf{r}\times[\mathbf{E}(\mathbf{r})\times\mathbf{B}(\mathbf{r})]\} . \tag{4.4}$$

This angular momentum can be separated into a term resulting from the derivatives of the amplitudes, and a term that derives from the polarization of the field. With some caution, this may be viewed as a separation into orbital angular momentum and spin of the field [12]. For a monochromatic field in the paraxial approximation (2.7), the z component J_z is separated as [4]

$$J_z=\frac{\varepsilon_0}{2\omega}\int dx\,dy\,dz\left[\mathbf{E}_0^*\cdot\mathbf{E}_0u^*Lu+\frac{1}{i}u^*u(E_{0x}^*E_{0y}-E_{0y}^*E_{0x})\right] . \tag{4.5}$$

This implies that for an arbitrary normalized mode $|u(z)\rangle$, the expectation value $\hbar\langle L\rangle$ is equal to that of the orbital angular momentum in the z direction per photon [4].

For each transverse dimension, we have independent values for the fundamental mode parameters s, γ, and χ, which we distinguish with suffices x and y. The corresponding lowering operators A_x and A_y are determined by these parameters, as in (2.25), in terms of the real operators B_x and B_y. In the presence of cylindrical lenses, the operators for the two dimensions can be treated independently only when the lens axes coincide with the x and the y axis, which is what will be assumed from now on. Hence the spot size and the wave front will, in general, be elliptical and with coinciding axes. We exclude for the moment the general astigmatism that arises by using cylindrical lenses oriented at oblique angles to each other [13]. Then the Hermite-Gaussian modes $|u_{nn'}(z)\rangle$ have mode functions $u_{nn'}(\mathbf{r})=u_n(x,z)u_{n'}(y,z)$, which are products of one-dimensional modes. All higher-order modes can then be generated by repeated action of ladder operators on the fundamental mode $|u_{00}(z)\rangle$.

Laguerre-Gaussian modes are the laser mode analog of the angular-momentum eigenstates of the isotropic two-dimensional harmonic oscillator. It has been shown that a higher-order Hermite-Gaussian beam can be converted into a Laguerre-Gaussian beam by using astigmatic lenses [14–16]. We wish to demonstrate that this and similar conversions are conveniently understood by operator algebra. First, we give the connection between these modes in an operator form.

A. Hermite-Gaussian, Laguerre-Gaussian, and elliptical Gaussian modes

Generalizing the algebraic treatment of the two-dimensional harmonic oscillator [17], we introduce mixed ladder operators by the definition

$$A_\pm=\frac{1}{\sqrt{2}}[A_x\mp iA_ye^{i\theta}] , \tag{4.6}$$

in terms of a phase θ that is independent of z. These operators obey the commutation rules

$$[A_\pm(z),A_\pm^\dagger(z)]=1, \quad [A_\pm(z),A_\mp^\dagger(z)]=0 . \tag{4.7}$$

The operators for the sum and the difference of the number operators are

$$N(z)=N_+(z)+N_-(z),\quad M(z)=N_+(z)-N_-(z)\ , \tag{4.8}$$

with

$$N_\pm(z)=A_\pm^\dagger(z)A_\pm(z)\ . \tag{4.9}$$

In terms of the field operators A_x and A_y, the operators N and M are

$$N=A_x^\dagger A_x+A_y^\dagger A_y,\quad M=iA_y^\dagger A_x e^{-i\theta}-iA_x^\dagger A_y e^{i\theta}\ . \tag{4.10}$$

Because N and M commute, they have a common basis of eigenstates $|u_{nm}(z)\rangle$, with eigenvalues n and m. As A_+ or A_- decrease the eigenvalue of N_+ and N_- by one unit, the lowering operators $A_\pm$ both decrease the total excitation number n, whereas A_+ decreases m and A_- increases m by one unit. This is apparent from the commutation rules

$$[N(z),A_\pm(z)]=-A_\pm(z),\quad [M(z),A_\pm(z)]=\mp A_\pm(z)\ . \tag{4.11}$$

With the proper phase convention of the eigenfunctions, the explicit action of the ladder operators may be given by

$$A_\pm(z)|u_{nm}(z)\rangle=\left[\frac{n\pm m}{2}\right]^{1/2}|u_{n-1,m\mp1}(z)\rangle\ , \tag{4.12}$$

$$A_\pm^\dagger(z)|u_{nm}(z)\rangle=\left[\frac{n\pm m+2}{2}\right]^{1/2}|u_{n+1,m\pm1}(z)\rangle\ .$$

The total mode number n can take all natural values $0,1,2,\ldots$, and for each value of n, m can take the $n+1$ values $-n,-n+2,\ldots,n-2,n$. The mode $|u_{nm}\rangle$ is created from the fundamental mode by applying the raising operators $A_+^\dagger$ and $A_-^\dagger$, and one finds

$$|u_{nm}(z)\rangle=\left[\frac{1}{p!q!}\right]^{1/2}[A_-^\dagger(z)]^q[A_+^\dagger(z)]^p|u_{00}(z)\rangle\ , \tag{4.13}$$

where we introduced the integers p and q, defined by

$$p=\frac{n+m}{2},\quad q=\frac{n-m}{2} \tag{4.14}$$

in terms of n and m. This result allows the expansion of the eigenfunctions $u_{nm}(\mathbf{r})$ of N and M in terms of the Hermite-Gaussian modes. If we substitute the definitions (4.6) into (4.13), the brackets can be worked out in terms of the coefficients g_s, which are defined by the expansion

$$(1-t)^q(1+t)^p=\sum_{s=0}^{n}g_st^s\ . \tag{4.15}$$

As the fundamental mode u_{00} is the product of two one-dimensional fundamental modes, we may directly apply (2.23) for each transverse dimension, with the result

$$u_{nm}(\mathbf{r})=\sum_{s=0}^{p+q}g_si^se^{-is\theta}\left[\frac{(p+q-s)!s!}{2^{p+q}p!q!}\right]^{1/2}\times u_{p+q-s}(x,z)u_s(y,z)\ . \tag{4.16}$$

The equalities (4.10)–(4.13) and (4.16) hold for any beam specified by its fundamental mode parameters. In the special case of a nonastigmatic region of the beam, where $\gamma_x=\gamma_y=\gamma$ and $s_x=s_y=s$, the phases χ_x and χ_y can only differ by a constant. Then the expression for $A_\pm$ in terms of the real ladder operators B_x and B_y takes the form

$$A_\pm=e^{i\chi_x}\exp\left[\frac{ik}{2s}(X^2+Y^2)\right]B_\pm\exp\left[-\frac{ik}{2s}(X^2+Y^2)\right]\ . \tag{4.17}$$

Here we denote

$$B_\pm=\frac{1}{\sqrt{2}}(B_x\mp ie^{i\zeta}B_y)\ , \tag{4.18}$$

where B_x and B_y are determined by the spot sizes $\gamma_x=\gamma_y=\gamma$, in analogy to (2.26), and where

$$\zeta=\chi_y-\chi_x+\theta\ . \tag{4.19}$$

As the fundamental mode function takes the form

$$u_{00}(\mathbf{r})=\exp\left[\frac{ikr^2}{2s}\right]\exp\left[-\frac{i}{2}(\chi_x+\chi_y)\right]\times\frac{1}{\gamma}\psi_0\left[\frac{x}{\gamma}\right]\psi_0\left[\frac{y}{\gamma}\right]\ , \tag{4.20}$$

operating on u_{00} with $A_\pm$ is equivalent to operating on the harmonic-oscillator ground state with $B_\pm$. This demonstrates that it is the value of ζ that determines the nature of the mode $|u_{nm}\rangle$ in a nonastigmatic region.

When $\zeta=\pi/2$, the operators $B_\pm$ are equal to

$$B_\pm=\frac{1}{\sqrt{2}}[B_x\pm B_y]\ . \tag{4.21}$$

Substituting (4.17) and (4.20) in (4.13) then gives for the higher-order mode functions

$$u_{nm}(\mathbf{r})=\exp\left[\frac{ik}{2s}r^2-\frac{i}{2}(\chi_x+\chi_y)-i\chi_x(p+q)\right]\times\frac{1}{\gamma}\psi_p\left[\frac{x+y}{\gamma\sqrt{2}}\right]\psi_q\left[\frac{x-y}{\gamma\sqrt{2}}\right]\ , \tag{4.22}$$

with p and q defined by (4.14). The functions (4.22) are normalized Hermite-Gaussian modes with their symmetry axis rotated over 45°. If we combine (4.22) with (4.16), we obtain the transformation of these rotated Hermite-Gaussian modes in the nonrotated Hermite-Gaussian modes. In the special case that $\chi_x=\chi_y$, so that $\theta=\zeta=\pi/2$, this is equivalent to an analytical expansion of products of Hermite polynomials in $(x\pm y)/\sqrt{2}$, as given by Abramochkin and Volostnikov [14].

When $\zeta=0$, we find that

$$B_{\pm}=\frac{1}{\sqrt{2}}[B_x \mp iB_y] , \quad (4.23)$$

which are circular ladder operators. It is easy to verify that, in this case, the operator M is equal to the angular momentum L, and the quantum number $m=l$ is the corresponding eigenvalue. The beam carries an orbital angular momentum $\hbar l$ per photon [3]. For the mode functions, we obtain, after using (4.15),

$$u_{nl}(\mathbf{r})=\exp\left[\frac{ik}{2s}r^2-i\chi(n+1)\right]\frac{1}{\gamma}\psi_{nl}\left[\frac{x}{\gamma},\frac{y}{\gamma}\right] , \quad (4.24)$$

with ψ_{nl} the wave functions of the isotropic harmonic oscillator that are eigenfunctions of both energy and angular momentum, with quantum numbers n and l. The azimuthal dependence of ψ_{nl} is given by $\exp(il\phi)$, and its radial part has the form [17,2]

$$\exp(-r^2/2\gamma^2)r^{|l|}L_p^{|l|}(r^2/\gamma^2) ,$$

with $p=(n-|l|)/2$, and $L_p^{|l|}$ the generalized Laguerre polynomial. As both the Laguerre-Gaussian and the Hermite-Gaussian modes form a complete set, a basis transformation must exist. The analytical derivation of this transformation is painstaking and not very transparent [15]. The transformation, obtained by operator algebra, is given by (4.16) in the special case that $\theta=0$. A similar result has recently been found by group-theoretic methods by Danakas and Aravind [18].

It is natural to compare the Laguerre-Gaussian modes to circular polarization and the Hermite-Gaussian modes to linear polarization. We have seen that they arise as special cases of the modes $|u_{nm}\rangle$ defined by (4.13) with (4.17) and (4.18), for $\zeta=0$ and $\zeta=\pi/2$, respectively. For intermediate values of ζ, the modes $|u_{nm}\rangle$ have an elliptical nature. This is obvious when we rewrite (4.18) in the form

$$B_{\pm}=\left[\frac{1}{\sqrt{2}}(B_x \pm iB_y)\cos\left[\frac{\zeta}{2}-\frac{\pi}{4}\right]-\frac{i}{\sqrt{2}}(B_x \mp iB_y)\sin\left[\frac{\zeta}{2}-\frac{\pi}{4}\right]\right]\times\exp\left[i\left[\frac{\zeta}{2}-\frac{\pi}{4}\right]\right] . \quad (4.25)$$

The corresponding creation operators $A_{\pm}^{\dagger}$ acting on the fundamental mode generate elliptical patterns of the transverse momentum density, with the axes oriented in the xy plane along the lines $x=\pm y$. This ellipticity should not be confused with the elliptic intensity distribution in the case of an astigmatic beam.

B. Mode conversion

In general, a mode converter is defined as a configuration of lenses that transforms a nonastigmatic input beam into a nonastigmatic output beam with an astigmatic region between them. Let us consider a configuration of astigmatic lenses with their elliptical axes oriented along the x and the y axis. The lenses are located between z_1 and z_2. The incoming beam for $z\le z_1$ is supposed to be nonastigmatic, so that $\gamma_x=\gamma_y=\gamma$ and $s_x=s_y=s$. The lens configuration determines the values of the fundamental mode parameters s_x, γ_x, χ_x and s_y, γ_y, and χ_y for all values of z. The output beam for $z\ge z_2$ is also nonastigmatic, provided that the last lens is located at a position where $\gamma_x=\gamma_y$, and that it has different focal lengths f_x and f_y along the two axes that make up for a possible difference in s_x and s_y, according to

$$\frac{1}{s_x(z_-)}-\frac{1}{f_x}=\frac{1}{s_y(z_-)}-\frac{1}{f_y} . \quad (4.26)$$

As the characteristics of the mode $|u_{nm}\rangle$ in a nonastigmatic region are determined by the value of the parameter ζ, the properties of a converter are fully specified by the difference between the ζ values in the output and the input beam, which is equal to

$$\Delta\zeta=(\zeta)_{\text{out}}-(\zeta)_{\text{in}}=(\chi_x-\chi_y)_{\text{out}}-(\chi_x-\chi_y)_{\text{in}} . \quad (4.27)$$

Consider a Hermite-Gaussian input beam with its axes oriented at 45° with respect to the axes of the lenses, so that

$$u_{\text{in}}(z)=u_p\left[\frac{x+y}{\sqrt{2}},z\right]u_q\left[\frac{x-y}{\sqrt{2}},z\right] , \quad (4.28)$$

with u_p and u_q the one-dimensional Hermite-Gaussian mode functions (2.1). According to (4.22), this input mode is identical to the mode $|u_{nm}\rangle$ with $\zeta=\pi/2$, and with $n=p+q$, and $m=p-q$. In order that the output beam be in the Laguerre-Gaussian mode (4.24), we must have $(\zeta)_{\text{out}}=0$, or $\Delta\zeta=-\pi/2$. The angular-momentum quantum number is $l=p-q$. Such a $\pi/2$ converter has recently been realized with a system consisting of two identical cylindrical lenses [16]. Then the Rayleigh range is the same at input and output. As the Gouy phases χ_x and χ_y cannot increase by more than π in a region of free propagation, the value of $\Delta\zeta$ for a system of only two lenses must obey the inequality $-\pi<\Delta\zeta<\pi$. For a system consisting of more lenses, $\Delta\zeta$ can be outside this range and the Rayleigh range can be different at input and output. Inspection of (4.17) and (4.18) shows that for $\zeta=\pi$, the operator M is equal to $-L$. Hence a lens system with $\Delta\zeta=\pi$ inverts the azimuthal mode index l of a Laguerre-Gaussian beam. It cannot, however, be realized with less than three lenses [4].

Our treatment demonstrates that the conversion properties of an arbitrary astigmatic lens system with a single axis with a nonastigmatic input and output is determined by the single parameter $\Delta\zeta$, which is the difference of the change in Gouy phase for the two axes of the lens system. As indicated in (4.18), this phase determines the change in the ladder operator $B_{\pm}$ from the input to the output region. An elliptical Gaussian beam specified by a value of ζ is realized by a converter with an appropriately selected value of $\Delta\zeta$ when the input beam is Hermite-Gaussian or Laguerre-Gaussian. The action of the converter results from the modification of the eigenoperator M, while the eigenvalues n and m remain unchanged. The propagation of the beam through the lens system is fully analogous to the evolution of a two-dimensional harmonic oscillator, with a modification of the oscillation frequencies Ω_x and Ω_y. The change in the phase χ over a propagation distance plays the role of the oscillator phase $\int dt\, \Omega(t)$ over a time interval. In each case, it is the total phase difference between the two directions that determines the state change for an arbitrary initial state. When the mode function is simply the product of one-dimensional functions for the x and the y direction, this phase difference does not affect the nature of the mode. It only plays a role for an entangled state, which is a linear combination of different products of one-dimensional eigenstates for the two axes. This explains why an input Hermite-Gaussian mode must have axes that do not coincide with the symmetry axes of the lenses, in order for mode conversion to take place.

C. Angular-momentum transfer

It is informative to derive expressions for the transfer of orbital angular momentum when a cylindrical lens is traversed by a possibly astigmatic Gaussian beam that is an eigenfunction of N. Again, clearly, it is necessary to consider both transverse directions. The transverse field amplitude for a given position z of the beam is generally described by the function $u(x,y)$, that can be separated as

$$u(\mathbf{r}) = \exp\left[\frac{ik}{2}\left[\frac{x^2}{s_x}+\frac{y^2}{s_y}\right]\right]\frac{1}{\sqrt{\gamma_x\gamma_y}}\Psi(\xi,\eta) . \qquad (4.29)$$

We introduced the scaled transverse coordinates and momenta

$$\xi = x/\gamma_x,\quad \eta = y/\gamma_y,\quad P_\xi = \frac{1}{i}\frac{\partial}{\partial \xi},\quad P_\eta = \frac{1}{i}\frac{\partial}{\partial \eta}, \qquad (4.30)$$

so that Ψ is the normalized wave function of an energy eigenstate of a two-dimensional isotropic harmonic oscillator. As we are interested only in the distribution at a given position z, we suppressed the z dependence in (4.29). The Gouy phases χ_x and χ_y are absorbed in Ψ. When γ_x and γ_y are different, the Gaussian part of u has unequal widths along the two axes, whereas the function Ψ is unsqueezed. If we evaluate the orbital angular momentum per photon $\hbar\langle L\rangle = \hbar\langle u|L|u\rangle$ for the beam (4.29) while using (2.24), we can express $\langle L\rangle$ as an expectation value over the wave function Ψ in terms of the scaled coordinates, with the result

$$\begin{aligned}\langle L\rangle = {} & k\gamma_x\gamma_y\left[\frac{1}{s_y}-\frac{1}{s_x}\right]\langle\Psi|\xi\eta|\Psi\rangle \\ & +\frac{1}{2}\left[\frac{\gamma_x}{\gamma_y}-\frac{\gamma_y}{\gamma_x}\right]\langle\Psi|(\xi P_\eta+\eta P_\xi)|\Psi\rangle \\ & \times\frac{1}{2}\left[\frac{\gamma_x}{\gamma_y}+\frac{\gamma_y}{\gamma_x}\right]\langle\Psi|(\xi P_\eta-\eta P_\xi)|\Psi\rangle . \end{aligned} \qquad (4.31)$$

The first term on the right-hand side of (4.31) results from the difference in the two radii of curvature. This reflects the ellipticity of the wave front. The second term is due to the difference in spot size and results from the elliptical shape of the light spot. The last term contains the angular momentum in scaled coordinates. It is the only surviving term in a nonastigmatic beam. Note that for a freely propagating astigmatic beam, all three terms in (4.31) vary with z, whereas their sum remains constant.

If the beam described by u passes a cylindrical lens, with focal length f in the x direction, the field leaving the lens is given by $u(\mathbf{r})\exp(-ikx^2/2f)$. Hence the last two terms in (4.31) are unaffected and only the first one changes, due to the change in s_x. The net increase of angular momentum per photon passing the lens is

$$\hbar\langle\delta L\rangle = \frac{\hbar k}{f}\gamma_x\gamma_y\langle\Psi|\xi\eta|\Psi\rangle = \frac{\hbar k}{f}\langle u|xy|u\rangle . \qquad (4.32)$$

This term determines the torque that the field exerts on the lens. It vanishes when the intensity distribution $|u|^2$ has the x or the y axis as symmetry axis. In particular, the torque on the lens vanishes when the input beam is an astigmatic Hermite-Gaussian mode $u_n(x,z)u_{n'}(y,z)$. This is understandable, as the lens orientation is at an extremum of energy. Furthermore, the angular-momentum transfer (4.32) vanishes for a Laguerre-Gaussian input mode, and for symmetry reasons [4] it will also vanish when the beam leaving the lens is Laguerre-Gaussian. Notice that this angular-momentum transfer is proportional to the wave number k. Therefore, when this matrix element is nonvanishing, it can be as large as many units $\hbar$, even for modes of moderate (nonzero) order.

D. Angular aperturing and angular momentum

In Sec. IV C, it was argued that a Laguerre-Gaussian beam cannot transfer angular momentum to an astigmatic lens. However, when angular aperturing is applied just in front of the lens, an appreciable transfer of angular momentum can occur. Suppose that the Laguerre-Gaussian mode (4.24) passes a diaphragm with a transmittance that depends only on the azimuthal angle ϕ, and not on the radial coordinate r. Then the mode function leaving the diaphragm can be given in polar coordinates as

$$\begin{aligned} u(\mathbf{r}) = {} & \exp\left[\frac{ik}{2s}r^2 - i\chi(n+1)\right]\frac{1}{\gamma\sqrt{2\pi}}R_{nl}(r/\gamma) \\ & \times e^{il\phi}A(\phi) , \end{aligned} \qquad (4.33)$$

with R_{nl} the normalized radial-wave function of the angular-momentum eigenstates ψ_{nl} of the two-dimensional harmonic oscillator. For an ideal diaphragm, the transmittance function $A(\phi)$ attains only the value zero or 1. When a cylindrical lens with focal length f in the x direction is placed immediately after the diaphragm, so that no propagation occurs between them, the transfer of angular momentum to the lens is found after substitution of (4.33) in (4.32). The result is the product of a radial and an angular average. By using the well-known fact that the average potential and kinetic energy are equal in an energy eigenstate of the harmonic oscillator, we obtain for the transfer of angular momentum per incident photon the expression

$$\hbar\langle\delta L\rangle=\frac{\hbar k}{2f}\gamma^2(n+1)\frac{1}{2\pi}\int_0^{2\pi}d\phi|A(\phi)|^2\sin(2\phi)\ . \quad (4.34)$$

This result is independent of the azimuthal mode index l, and it is proportional to a Fourier coefficient of the ϕ-dependent transmission function $|A(\phi)|^2$. The angular momentum transferred to the lens can be much larger than the angular momentum per photon $\hbar l$ in the incident beam. The transfer is maximal when $|A(\phi)|^2$ is 1 for $0<\phi<\pi/2$, and $\pi<\phi<3\pi/2$, and zero elsewhere.

V. CONCLUSIONS

The propagation of Gaussian laser beams in free space, and their refraction by ideal astigmatic lenses, are described by harmonic-oscillator operator algebra. A mode is characterized by z-dependent eigenoperators and constant eigenvalues. This explains why the Hermite-Gaussian form of a beam is conserved during its free propagation and at refraction by lens systems. Moreover, it shows that the displacement of the axis of a slightly misaligned beam follows the rules of geometric optics, which does not affect the beam diffraction. Such a displaced beam is the analog of a coherent state of the harmonic oscillator. The relationship between Laguerre-Gaussian and Hermite-Gaussian modes takes a simple algebraic form. Conversion from one mode into another by astigmatic lenses is directly characterized by a single phase difference. The orbital angular momentum of a light beam in its propagation direction is separated into a contribution that resembles the angular momentum of a harmonic oscillator and terms that originate from the astigmatism of the beam. These latter terms can be considerably larger than the first. The torque of a beam exerted on an astigmatic lens is expressed in terms of a single expectation value. When a Laguerre-Gaussian beam, with angular momentum $\hbar l$ per photon, traverses an astigmatic lens after passing an angular aperture, it may transfer more angular momentum to the lens than $\hbar l$ per photon. Apart from the physical interest of these results, they demonstrate the advantage of the algebraic method in terms of ladder operators for paraxial optics.

ACKNOWLEDGMENT

This work is part of the research program of the Stichting voor Fundamenteel Onderzoek der Materie (FOM), which is financially supported by the Nederlandse Organisatie voor Wetenschappelijk Onderzoek (NWO).

*Present address: Physics Department, University of Essex, England.

[1] H. A. Haus, *Waves and Fields in Optoelectronics* (Prentice-Hall, Englewood Cliffs, NJ, 1984).

[2] A. E. Siegman, *Lasers* (University Science Books, Mill Valley, CA, 1986).

[3] L. Allen, M. W. Beijersbergen, R. J. C. Spreeuw, and J. P. Woerdman, Phys. Rev. A **45**, 8185 (1992).

[4] S. J. van Enk and G. Nienhuis, Opt. Commun. **94**, 147 (1992).

[5] L. G. Gouy, C. R. Acad. Sci. **110**, 1251 (1890).

[6] M. Lax, W. H. Louisell, and W. B. McKnight, Phys. Rev. A **11**, 1365 (1975).

[7] D. Gloge and D. Marcuse, J. Opt. Soc. Am. **59**, 1629 (1969).

[8] D. Stoler, J. Opt. Soc. Am. **71**, 334 (1981).

[9] R. Loudon, *The Quantum Theory of Light* (Clarendon, Oxford, 1981).

[10] P. Meystre and M. Sargent III, *Elements of Quantum Optics* (Springer, Berlin, 1990).

[11] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962).

[12] J. W. Simmons and M. J. Guttmann, *States, Waves and Photons* (Addison-Wesley, Reading, MA, 1970).

[13] J. A. Arnaud and H. Kogelnik, Appl. Opt. **8**, 1687 (1969).

[14] E. Abramochkin and V. Volostnikov, Opt. Commun. **83**, 123 (1991).

[15] C. Tamm and C. O. Weiss, J. Opt. Soc. Am. B **7**, 1034 (1990).

[16] M. W. Beijersbergen, L. Allen, H. E. L. O. van der Veen, and J. P. Woerdman, Opt. Commun. **96**, 123 (1993).

[17] A. Messiah, *Mécanique Quantique* (Dunod, Paris, 1964).

[18] S. Danakas and P. K. Aravind, Phys. Rev. A **45**, 1973 (1992).