Single-mode-laser phase dynamics

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We study the phase dynamics of a single-mode ring laser described by the complex Maxwell-Bloch equations. We identify three reference-frame frequencies and determine the properties of the field dynamics observed in these frames. In one of these reference frames, the phase jumps are always equal to π, irrespective of the detuning, while in another reference frame quasiperiodic field portraits reduce to periodic field portraits. We also apply the recent theory of Ning and Haken [Phys. Rev. Lett. **68**, 2109 (1992)] to prove that the laser phase can be decomposed into a geometrical component that is frame invariant and a dynamical component that is frame dependent.

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I. INTRODUCTION

The Lorenz equations [1] are one of the generic models for the study of dynamical systems. As shown by Haken [2], there is a complete equivalence between the Lorenz equations and the Maxwell-Bloch equations that describe a very simple model of a tuned laser. This analogy was extended by Fowler *et al.* [3], who established the equivalence between the detuned laser model and the complex Lorenz equations. Interest in these equivalencies was greatly piqued when Weiss *et al.* [4] showed experimentally that the simple laser model that is equivalent to the Lorenz equations describes the complex dynamics of some lasers. Until recently, single-mode laser fields were characterized only by their amplitude or photon average. Few papers mentioned the possibility of a phase dynamics [5–7], and never included a systematic study. This situation changed drastically when Weiss *et al.* [8–10] started to analyze the complex field experimentally. This led to the observation of either the real and imaginary parts of the field, or its amplitude and phase. These results were soon followed by a theoretical analysis [11,12]. This analysis was generalized to cover Doppler broadening [13], Raman lasers [14], and four-level optically pumped lasers [15]. The bidirectional ring laser was also analyzed experimentally and theoretically along the same lines in the visible and infrared domains [16–19]. The semiclassical phase concept was extended to the laser below threshold in Ref. [20], and phase diffusion was studied in chaotic lasers in Refs. [21] and [22]. The phase evolution in intracavity two-photon process was studied in Ref. [23]. A possible analogy with Berry's geometrical phase [24] was suggested for single-mode lasers [12,25] and for intracavity two-photon processes [23]. This idea was amplified by Ning and Haken [26,27], who recently proposed a generalization of the Berry phase to dissipative systems [27].

As the single-mode laser phase was studied more attentively, a problem arose, mainly in the theoretical literature. The Maxwell-Bloch equations are usually studied in a rotating frame of reference. There is, however, a complete arbitrariness in the choice of this reference frame and its rotation frequency. In the two-level homogeneously broadened ring laser, we deal with three "natural" frequencies: the empty-cavity frequency, the atomic frequency, and the lasing frequency. Any of these three frequencies could be chosen as a reference frequency. In fact, any frequency can be used as a reference. The question is, therefore, to determine which properties of the laser phase are invariant with respect to a change of reference frame, and which properties depend on the reference frame. In the latter case, we should also know what the frequency dependence is. These problems are not mere theoretical considerations; they have their counterpart in the experimental procedure used to study the laser phase. Indeed, the frequency is determined by heterodyning the laser output with some reference frequency [8]. Here again, the heterodyne reference is arbitrary and thus the experimental result is also affected by an arbitrary constant. The purpose of this paper is to focus on the phase dynamics and its relation to the reference frequency. It is organized as follows. In Sec. II, we formulate the problem and define our notation. In Sec. III, we analyze and compare three rotating reference frames. Finally, Sec. IV is devoted to a general discussion of the phase dynamics.

II. FORMULATION

Our analysis is centered on the Maxwell-Bloch equations for a single-mode ring laser filled with a collection of identical two-level atoms:

$$\frac{\partial E}{\partial t}=-\kappa[(1+i\bar{\delta})E+AP]\ ,$$
$$\frac{\partial P}{\partial t}=-\gamma_\perp[(1+i\bar{\Delta})P+EF]\ , \qquad (1)$$
$$\frac{\partial F}{\partial t}=\gamma_\parallel[-F+1+\tfrac{1}{2}(EP^*+PE^*)]\ ,$$

where E is the cavity electric field with decay rate κ; P and F are the atomic polarization and population inversion with decay rates $\gamma_\perp$ and $\gamma_\parallel$, respectively. The detuning parameters are defined through

$$\bar{\delta}=(\omega_c-\omega_r)/\kappa\ ,\quad \bar{\Delta}=(\omega_a-\omega_r)/\gamma_\perp\ , \qquad (2)$$

where ω_a is the atomic frequency, ω_c is the cavity frequency nearest the atomic frequency, and ω_r is the frequency of the reference frame. This frequency is fundamentally arbitrary, and we shall study the consequences of this degeneracy on the solutions of Eq. (1). The "natural" choice for the reference-frame frequency is $\omega_r=\Omega$, where Ω is the field frequency under steady-state operation; it is the solution of $\bar{\delta}=-\bar{\Delta}$. This natural choice is the only one for which Eqs. (1) have a steady solution. For any other choice of the reference-frame frequency, the simplest nontrivial solution of Eqs. (1) is of the form $E(t)=\mathcal{E}\exp(-i\varphi)$, where $\varphi(t)=(\Omega-\omega_r)t+\varphi(0)$. These solutions correspond to a constant intensity $I=\mathcal{E}^2$. Thus the reference-frame frequency appears as a fairly irrelevant quantity. This is true only for the steady-state solutions. When the dynamical system undergoes bifurcations to time-periodic or to chaotic states, the intensity is no longer constant and both phase and amplitude are functions of time. In the time-periodic states only the field intensity and the population inversion show a periodic evolution; the other variables can be periodic with the same or another period, or can be quasiperiodic. Furthermore, since in these domains the steady state is unstable, it is difficult (and often impossible) to determine the frequency Ω experimentally. Although intrinsic properties of the dynamical system should be independent of the reference frame, we shall see that some reference frames reveal more properties than others, and that a bad choice of the reference frame may lead to erroneous conclusions.

It is common, in optics, to characterize optical fields by their frequency and intensity (i.e., mean photon number). For the monomode fields, which are the only ones we consider in this paper, these two quantities are related to the polar decomposition of the complex field into a real amplitude $\mathcal{E}$ and a real phase φ:

$$E(t)=\mathcal{E}(t)\exp[-i\varphi(t)]\ . \qquad (3)$$

The instantaneous frequency is, by definition, the time derivative of the phase φ. This transformation is singular if the real amplitude $\mathcal{E}(t)$ vanishes or becomes arbitrarily small. To appreciate fully the difficulty associated with this singularity, let us consider the evolution of the field $E(t)$ in a typical situation of periodic motion, as shown in Fig. 1. Inspection of Figs. 1(a) and 1(b) shows the coincidence in time between the peaks in the frequency $\dot{\varphi}\equiv\partial\varphi/\partial t$ and the minima in the intensity I. As resonance is approached ($\omega_c\rightarrow\omega_a$), the intensity maintains a well-behaved evolution (with its minima approaching zero), while the peaks in $\dot{\varphi}$ diverge and the corresponding frequency loses its meaning. This structure in Fig. 1(b) is, in fact, an invariant since a change of the reference-frame frequency ω_r will only shift the frequency $\dot{\varphi}$ by a constant amount, leaving $I(t)$ unmodified. Thus, the dynamical frequency $\dot{\varphi}$ has little to do with our intuitive notion of the field frequency. A more regular decomposition of the field is expressed in terms of its Cartesian coordinates:

$$E(t)=\mathrm{Re}[E(t)]+i\ \mathrm{Im}[E(t)]\equiv\mathcal{R}(t)+i\ \mathcal{J}(t)\ . \qquad (4)$$

The plot of $E(t)$ in the $(\mathcal{R},\mathcal{J})$ plane is the field portrait of $E(t)$; an example is displayed in Fig. 1(c). It should be stressed that it is precisely $\mathcal{R}$ and $\mathcal{J}$ that are first determined in the experiments of Weiss *et al.* [8–10]. The phase φ and frequency $\dot{\varphi}$ are extracted from these functions numerically. It was shown by Zeghlache *et al.* [11] that, for any periodic solution of Eqs. (1), there exists a reference-frame frequency such that the phase and the field portrait are periodic and have the same periodicity as all other dynamical variables. Vilaseca *et al.* [12] later showed that there is a countable infinite set of reference-frame frequencies verifying the same property for the phase portrait and *the other variables,* but not for the phase (special attention was paid to one of these reference frequencies). These frequencies will be identified later in this paper. In agreement with those results, Ning and Haken [25] proved that, in the reference frame rotating at the steady lasting frequency Ω, the field portrait near the onset of periodic motion is necessarily quasiperiodic.

III. REFERENCE FRAMES

As can be easily deduced from Eqs. (1), the reference-frame frequency influences the phase evolution $\varphi(t)$ in any dynamic regime according to

$$\varphi_\Omega(t)+\Omega t=\varphi_{\omega_r}(t)+\omega_r t\ , \qquad (5)$$

where the subscripts Ω and ω_r denote the reference-frame frequency utilized to compute the phase. Thus the complex field $E(t)$ is affected by ω_r. A natural way to obtain ω_r-independent information on the field evolution is by using its normalized power spectrum $S(\omega)$, defined through

$$S(\omega)\equiv\frac{|\mathcal{F}(E^*(t)\exp(i\omega_r t))|^2}{\int_0^\infty|\mathcal{F}(E^*(t)\exp(i\omega_r t))|^2\,d\omega}\ , \qquad (6)$$

where $\mathcal{F}(E)$ is the Fourier transform of E. Clearly, $S(\omega)$ is an invariant since it is independent of the value assigned to the reference-frame frequency ω_r. It is worth noting, however, that, when solving Eqs. (1) numerically, $S(\omega)$ is determined by assigning an arbitrary value to ω_r, which must be large enough to avoid overlapping be-

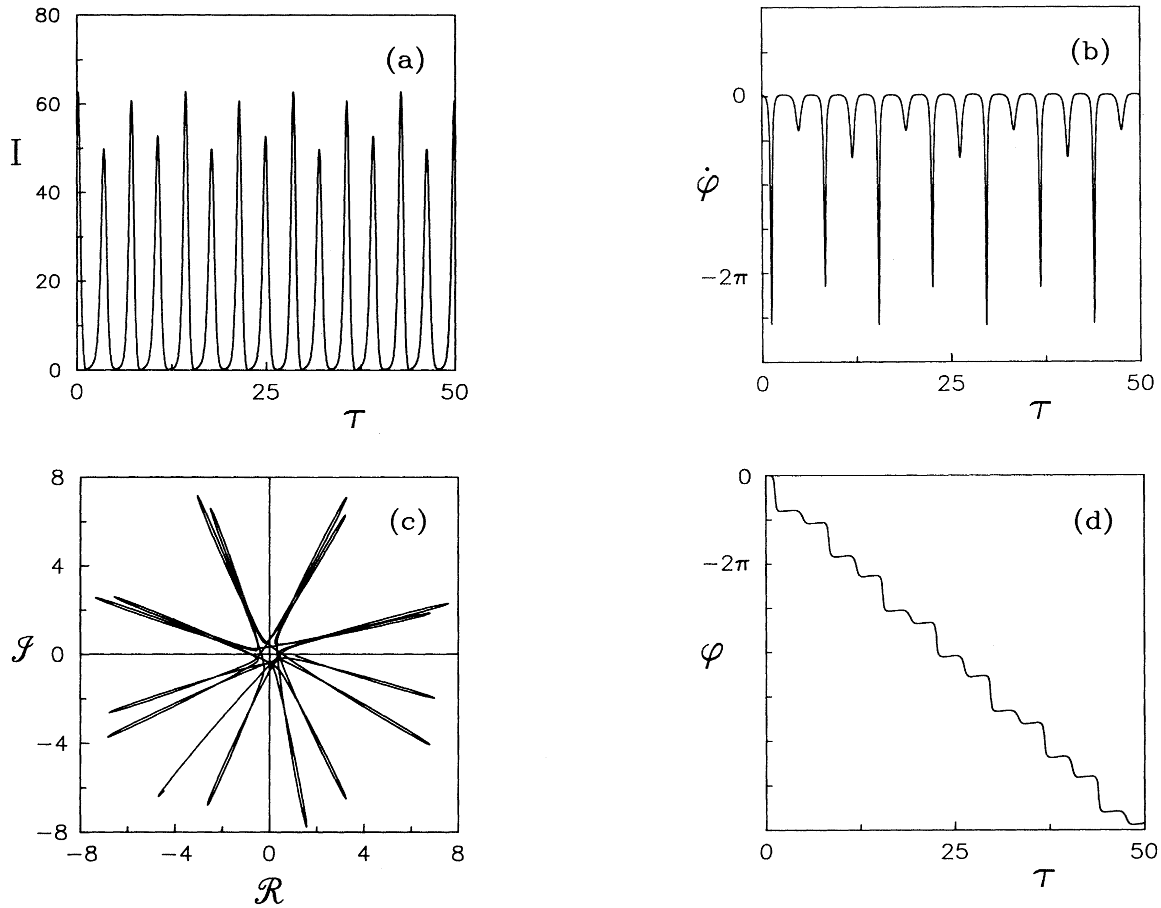

FIG. 1. (a) Intensity vs time; (b) instantaneous frequency $\dot{\varphi}$ vs time; (c) field portrait; and (d) phase φ vs time. Parameters are $\kappa/\gamma_\perp=2$, $\gamma_\parallel/\gamma_\perp=0.25$, $\omega_r=\Omega$, $\delta_{ca}=(\omega_c-\omega_a)/\gamma_\perp=0.69$, and $A=16$. All frequencies are given in units of $\gamma_\perp$.

tween the positive and negative parts of the spectrum, but small enough to deal with manageable time series (in our calculations, we took $\omega_r=50\gamma_\perp$). The relative positions of the peaks of $S(\omega)$ with respect to the value assigned to ω_r is the only relevant quantity. To fix the notation, we have plotted in Fig. 2 an example of a power spectrum for a periodic solution of Eqs. (1). Three new frequencies can be defined quite simply with this spectrum. The first is ω_1, the frequency of the highest peak. The second is ω_0, the algebraic mean of the peak frequencies, i.e., the middle point or center of $S(\omega)$ (there will or will not be a peak at this point, depending on whether the periodic solution is asymmetry or symmetric). The third frequency is the mean frequency, defined as

$$\bar{\omega}=\int_0^\infty \omega S(\omega)d\omega \ . \tag{7}$$

These three new frequencies can be used as reference-frame frequencies leading to specific advantages in the analysis of the field evolution. We shall now compare the field portrait, phase evolution, and attractor projection in these reference frames.

A. Highest peak frequency

When the reference frame rotates at the frequency ω_1, the field portrait, the phase, and the attractor projection in the (E,P) plane are shown in Figs. 3(a), 3(b), and 3(c), respectively, for the same parameters as in Fig. 1. The main feature is that in this reference frame the field portrait is no longer quasiperiodic, but has become periodic.

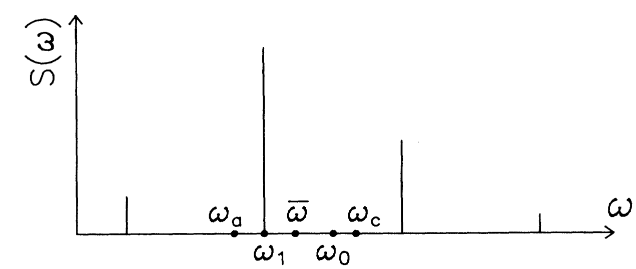

FIG. 2. Power spectrum with the position of the maximum side-mode frequency ω_1, the algebraic mean ω_0, and the mean frequency $\bar{\omega}$ computed according to Eq. (7).

In fact, this reference frame is precisely the reference frame that was found numerically by Zeghlache *et al.* [11]. It was called the "irreducible representation" of the periodic attractor because, in this reference frame, all five dynamical variables—$\mathcal{R}(t)$, $\mathcal{I}(t)$, $\mathrm{Re}[P(t)]$, $\mathrm{Im}[P(t)]$, and $F(t)$—are periodic with the same periodicity. Note, however, that this frame is not the only one that freezes (i.e., makes periodic) the field portrait. As discussed in Ref. [12], there exist many other reference-frame frequencies (which we can identify here as those corresponding

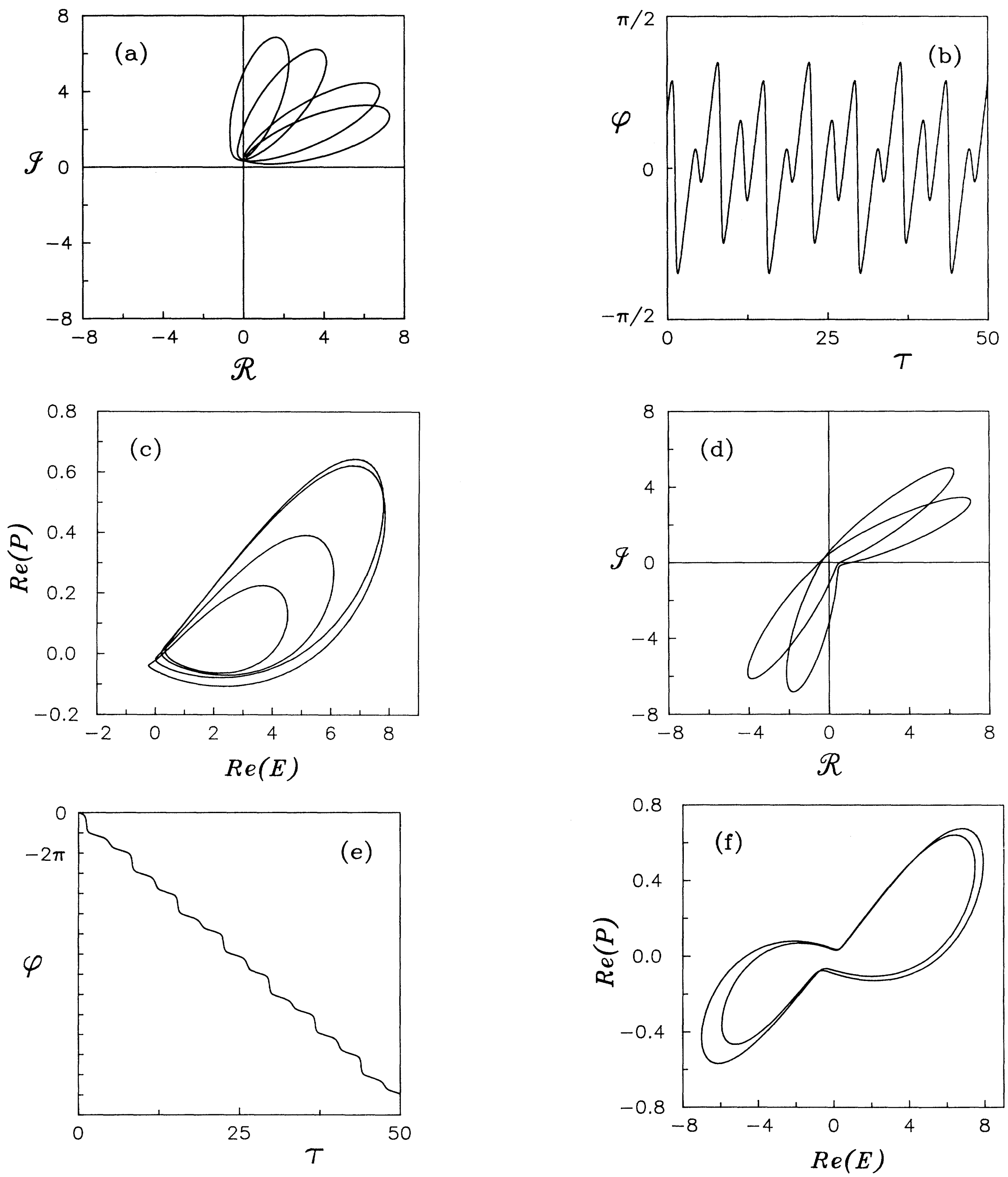

FIG. 3. (a) Field portrait, (b) phase evolution, and (c) attractor projection in the reference frame rotating at the frequency of the highest peak, $\omega_1=\Omega-0.4999$. (d) Field portrait, (e) phase evolution, and (f) attractor projection in the reference frame rotating at the algebraic mean frequency $\omega_0=\Omega+0.3829$. The other parameters are as in Fig. 1. Note the high precision required in the determination of ω_1 and ω_0 to avoid noticeable spurious rotations (and destruction of the periodicity) in panels (a), (c), (d), and (f).

to the other peaks in the power spectrum) with this property, providing field-portrait representations with the same periodicity.

B. Algebraic mean frequency

An obvious problem with the choice of ω_1 as a reference-frame frequency is that it is discontinuous as the atom-field detuning changes from positive to negative values (since there is a change in the sign of the phase jumps). Therefore, we may consider another choice, namely the algebraic mean frequency ω_0. The field portrait, phase, and attractor projection in this reference frame are displayed in Figs. 3(d), 3(e), and 3(f), respectively. It is quite clear that here, again, we have a field portrait that is periodic (with the same periodic as before) instead of quasiperiodic, in spite of the fact that the φ displayed in Fig. 3(e) is no longer periodic. Furthermore, the shape of the field portrait is similar to the shape of the portrait on resonance. For instance, the periodic attractors retain the butterfly shape that is characteristic of the real Lorenz attractor, as shown in Fig. 3(f). This frame was proposed in Ref. [12]. Contrary to the highest-peak-frequency case, the phase displays an accumulation rather than a periodic motion. However, the important feature of this reference frame is that no matter what the detuning is, the phase jumps are always equal to π in average. This means that the total phase accumulation along one period is exactly $n\pi$, where n is the number of phase jumps in the period. This property was used in Ref. [12] as an operative definition for ω_0. Thus, even the phase-jump amplitude cannot be used as a signature of some topological feature since it also depends on the reference-frame frequency.

C. Mean frequency

The surprising result we have obtained is that the mean frequency defined by Eq. (7) coincides with the frequency Ω, which is the solution of the steady-state dispersion equation $\bar{\Delta}=-\bar{\delta}$. This result holds for periodic and chaotic solutions. It is a rather general property of the Lorenz equations, though we have been unable to produce a convincing analytical proof of that result. Therefore, the representation in that reference frame is identical with that of Figs. 1(c) and 1(d).

The equality between ϖ and Ω has interesting consequences. It indicates the existence of some internal consistency in the shape of $S(\omega)$. This is all the more surprising since it is known that the mean intensity has an angular point when the system enters the chaotic domain [28]. Another consequence is that the mean refraction index, defined as

$$\bar{n}=\int_0^\infty \frac{\omega_c}{\omega} S(\omega)d\omega \;, \tag{8}$$

is practically equal to $n(\varpi)$, which is the mean refraction index of the steady solution, stable or not. The approximation $\bar{n}\cong n(\varpi)$ holds provided $S(\omega)$ has finite support, i.e., $S(\omega)=0$ for $|\omega-\omega_c|/\omega_c>\varepsilon$ with $0<\varepsilon\ll 1$.

IV. DISCUSSION

We have seen that the choice of the reference-frame frequency is not "innocent" and that it can significantly modify the interpretation of the model. Figure 4 displays the variation of the three frequencies discussed in the preceding section as a function of the detuning, for three different values of the gain parameter A. In the first case, Fig. 4(a), the gain is large and the emission regimes are simple [29,30]: the steady-state (cw) solution for large cavity detuning ($10<|\omega_c-\omega_a|\le 15$ in our case) and the periodic solution P_1 (single-round-trip orbit in the phase space) for small detuning ($|\omega_c-\omega_a|\le 10$). The algebraic mean frequency ω_0 is continuous at resonance, whereas the highest peak frequency ω_1 is discontinuous. Strictly speaking, what occurs at exact resonance is that ω_1 is not uniquely defined: any value ranging within the vertical interval joining the upper and lower branches in Fig. 4(a) is permitted. This is so because at resonance the phase jumps are exactly equal to π and their sign is not defined [the complex field amplitude $E(t)$ goes exactly through zero]. This means that the temporal evolution of the phase admits different combinations of positive and negative phase jumps, so that each one is characterized by a different average slope (and, as a consequence, by a different value of ω_1). Another feature in Fig. 4(a) is that both ω_0 and ω_1 vary almost linearly with detuning. The difference $|\omega_1-\omega_0|$ is equal to π/T, where T is the period associated with the regime P_1. Since T varies slightly with detuning, the difference $|\omega_1-\omega_0|$ remains almost constant. Finally, the mean frequency ϖ varies linearly across the two regions P_1 and cw (without discontinuity at the frontier), since it coincides with the frequency of the stationary solution Ω.

In Fig. 4(b), which corresponds to a smaller gain, the dynamic regimes are more complex since period P_1 doubles to P_2 for small detunings. For simplicity, only positive detunings have been displayed in this case. The cw solution occurs for larger detunings, $\omega_c-\omega_a\gtrsim 8$, not shown in the figure. In the region around the bifurcation point between P_1 and P_2, the frequencies ω_0 and ω_1 vary smoothly and ϖ remains linear. The fact that ω_c coincides with ϖ and, as a consequence, with Ω at resonance [in both cases of Figs. 4(a) and 4(b)] is one of the reasons that the shape of the field portraits obtained with the reference frame ω_0 are similar to the shape of the portraits on resonance (i.e., similar to the typical Lorenz-model portraits), as expressed above [Fig. 3(f)].

For still smaller gain, Fig. 4(c), there is a complete Feigenbaum scenerio when detuning is decreased. ω_0 and ω_1 are defined only in the periodic regimes, and vary smoothly across the region where the period-doubling bifurcations occur. In Figs. 4(b) and 4(c) the frequencies ω_1 and ω_0 are not shown in the vicinity of the Hopf bifurcation, where the periodic solution P_1 emerges and the steady-state solution becomes unstable. This is because the Hopf bifurcation is subcritical for these parameters and there is a small bistable domain around the lasing second threshold. In this domain there is critical slowing down, and the precise determination of ω_1 is some-

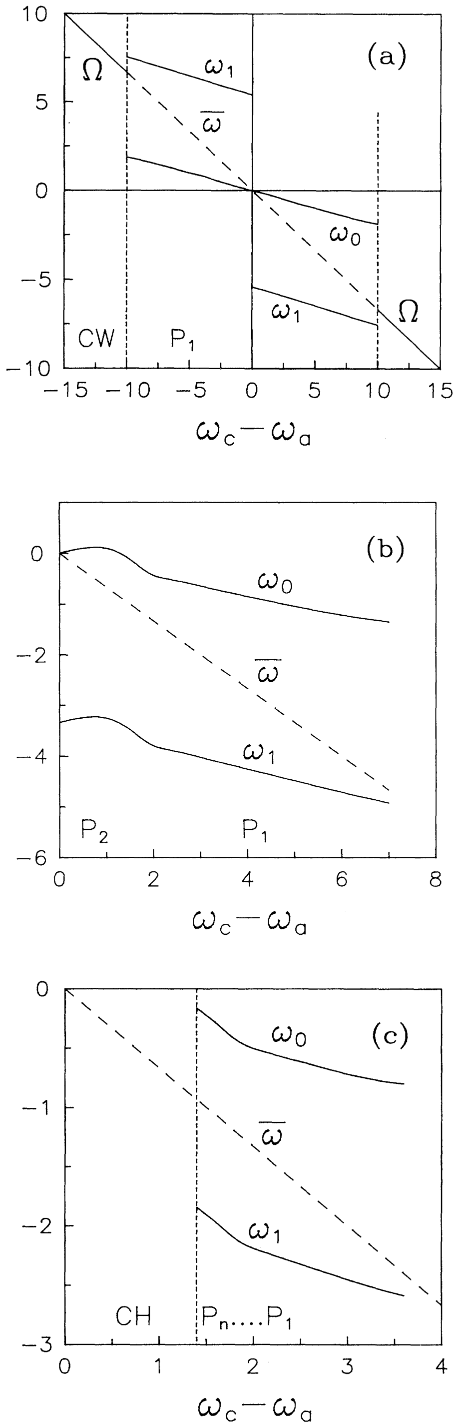

FIG. 4. Maximum peak frequency ω_1, algebraic mean frequency ω_0, and mean frequency $\overline{\omega}$ as a function of cavity-atom detuning. (a) $A=500$; only period-1 and cw solutions occur in the domain represented by the figure. (b) $A=180$; period-1 and -2 solutions are present (the cw solution occurs for larger values of $\omega_c-\omega_a$). (c) $A=50$; a complete cascade from period 1 to chaos occurs. $\kappa/\gamma_\perp$ and $\gamma_\parallel/\gamma_\perp$ as in Fig. 1. The three frequencies ω_1, ω_0, and $\overline{\omega}$ have been expressed with respect to ω_a. The chaotic domain is marked CH.

what delicate and time consuming, and does not elicit any relevant information.

The second problem that we shall discuss concerns the phase jumps. It is commonly admitted that the phase jumps of π are associated with a zero value for the field amplitude $\mathcal{E}(t)$. This misconception has its origin in the fact that, on resonance, it is indeed true that phase jumps of π are necessarily associated with $\mathcal{E}(t)=0$ because, as mentioned above, the resonant field portrait is a straight line that goes through the origin of the $(\mathcal{R},\mathcal{J})$plane. Any amount of detuning will lead to a field portrait that is a closed or an open curve, corresponding to periodic or quasiperiodic motion, respectively. Depending on the detuning, the field portrait will either pass close to the origin or remain far from it, but it will not cross the origin—otherwise the field frequency $\dot{\varphi}$ would diverge. To illustrate this point, a sequence of phase portraits in the same reference frame—but for increasing detuning—is displayed in Fig. 5. They clearly show that the minimum distance of the field portrait to the origin increases with detuning. Thus a phase jump of π indicates either that the laser is operating on resonance and that the reference-frame frequency is Ω, or that the laser is detuned and that the reference frame rotates at the frequency ω_0.

Another problem that can be clarified by our reference-frame analysis is the relation between the phase-accumulation behavior and the geometric phase introduced by Berry [24]. This connection has been suggested by Vilaseca *et al.*, [12] and by Ning and Haken [25], and has been strongly advocated by the latter authors [25–27] in the context of the complex Lorenz model. It has also been discussed in detail by Mandel *et al.* [23] in the context of two-photon–cavity optics, making a comparison with the case of the detuned-laser model. Ning and Haken have recently proposed a generalization of the Berry phase theory adapted to dissipative systems [27]. In that context, they were able to show that, for periodic solutions of the complex Lorenz equations, the phase of the complex variables can be decomposed into the sum of a dynamical phase and a geometrical phase. However, the transformation from Eqs. (1) into the complex Lorenz equations introduces in the coefficient of the Lorenz equations a dependence on the reference frame. Therefore, we apply their method directly to the physical laser equations, Eqs. (1). We first define a dynamical frequency ω_d through

$$\omega_d = -\mathrm{Im}\left[\left\langle\Psi\left|\frac{\partial\Psi}{\partial t}\right.\right\rangle \Big/ \langle\Psi|\Lambda|\Psi\rangle\right], \tag{9}$$

where $|\Psi\rangle \equiv \mathrm{col}(\Psi_1,\Psi_2,\Psi_3)=\mathrm{col}(E,P,F)$ and Λ is a 3×3 diagonal matrix with real elements $\alpha_1,\alpha_2,\alpha_3$ chosen in such a way that the periodic solutions of Eqs. (1) verify the relation $|\Psi(t+T)\rangle=\mathcal{T}(-\delta\phi)|\Psi(t)\rangle$, where $\delta\phi$ is real and $\mathcal{T}$ is a diagonal matrix of elements $\exp(i\alpha_j\delta\phi)$ with $j=1,2,3$. For Eqs. (1), the obvious property $\Psi_{1,2}(t+T)=\exp(-i\delta\phi)\Psi_{1,2}(t)$ and $\Psi_3(t+T)=\Psi_3(t)$, where $\delta\phi$ is the phase accumulation over one period, implies the choice $\alpha_1=\alpha_2=1$ and $\alpha_3=0$. From these properties, it is easy to verify that the dynamical frequency (9)

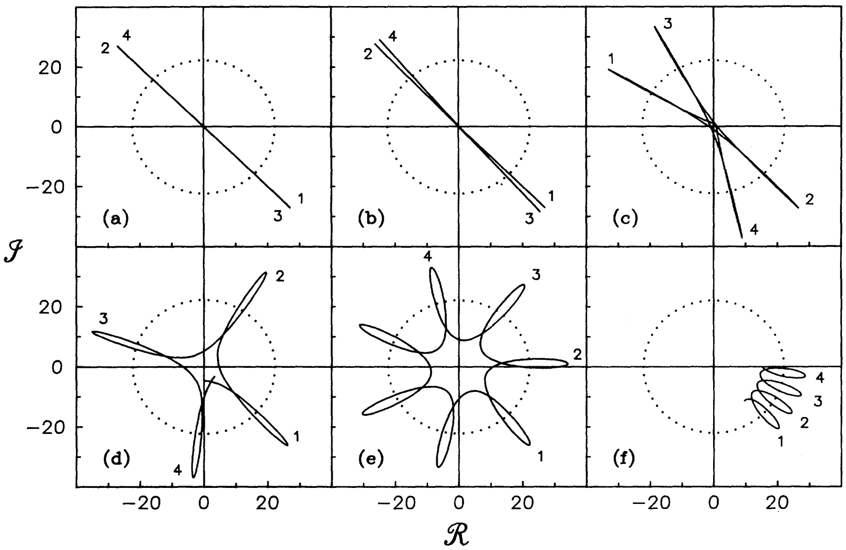

FIG. 5. Field portrait of the laser equations (1) as a function of the detuning in the reference frame rotating at the frequency Ω of the steady-state solution. The dots arranged along a circle are unstable steady states. (a) $\delta_{ca}=0$; (b) $\delta_{ca}=0.1$; (c) $\delta_{ca}=1$; (d) $\delta_{ca}=5$; (e) $\delta_{ca}=8.43$; (f) $\delta_{ca}=11$. Only portraits (a) and (e) correspond to closed trajectories. All other portraits describe quasiperiodic motion. $A=500$, while $\kappa/\gamma_\perp$ and $\gamma_\parallel/\gamma_\perp$ are as in Fig. 1.

can be written as

$$\omega_d = -\operatorname{Im}\left[E^*\frac{\partial E}{\partial t}+P^*\frac{\partial P}{\partial t}\right]\Big/(E^*E+P^*P) \tag{10a}$$

$$= -\omega_r+\omega_c+[(\omega_a-\omega_c)|P|^2+(\kappa A-\gamma_\perp F)\operatorname{Im}(EP^*)]/(|E|^2+|P|^2)$$

$$\equiv -\omega_r-\Omega_d(t)\ , \tag{10b}$$

where $\Omega_d(t)$ is manifestly frame invariant. The geometrical phase φ_g is defined as the difference between the total phase and the dynamical phase

$$\varphi_g \equiv \varphi-\varphi_d=\varphi+\omega_r t+\int \Omega_d(t)dt\ . \tag{11}$$

From this definition and Eq. (5), it is simple to prove that the frame invariance of $\Omega_d(t)$ implies the frame invariance of the geometrical phase $\varphi_g(t)$. Although the Ning and Haken method identifies a fraction of the total phase that is frame invariant, there seems to be no way to measure it. This results from the fact that the geometrical phase is not associated with an independent physical phenomenon in the laser case. This aspect of the measurement was stressed in Ref. [23], where cavities driven by an external field were considered. In such a case, the external field provides the reference.

So far, we have mainly considered the cases of periodic dynamic regimes. For chaotic regimes the frequencies ω_1 and ω_0 cannot be defined and indeed there is no reference-frame frequency for which the field portrait and the attractor projections can be frozen. These representations are affected by random rotations as discussed in Refs. [10], [21], and [22], since the phase jumps appear at random times and with random sizes (diffusivelike phase evolution). As explained in Sec. III, the only feature that remains in the case of chaotic evolution is that $\bar{\omega}=\Omega$, i.e., the mean frequency coincides with that of the stationary solution.

Finally, it should be borne in mind that a separate problem is that of the dependence on the initial phase $\varphi(0)$. Different choices of that initial phase correspond to rotations in the five-dimensional space of the solutions of Eqs. (1). This rotation factors out into a rotation in the field plane $(\mathcal{R},\mathcal{I})$ and a rotation in the atomic-polarization plane. Therefore the shape of the field portrait will also depend on the initial phase.

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