Mode entanglement in nondegenerate down-conversion with quantized pump

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(Received 11 January 1993)

We study entanglement between field modes in the process of nondegenerate two-photon down-conversion with quantized pump. We show that due to the quantum nature of the dynamics, strong entanglement between the pump and the signal-idler subsystems can be observed. We find that the higher the initial intensity of the pump mode the stronger the entanglement between the pump and the signal-idler subsystem is established during the first instants of the time evolution. We also show that the signal and the idler modes are strongly entangled (correlated). This entanglement is much stronger than the entanglement between the pump and the signal-idler subsystem. Correlation between the signal and the idler modes leads to a high degree of two-mode squeezing, which can be observed during the first instants of the time evolution when the pump mode is still approximately in a pure state. On the other hand, the back action of the signal-idler subsystem on the pump mode leads to a strong single-mode squeezing of the pump mode. At the time interval during which squeezing of the pump mode can be observed the pump mode is far from being in the minimum uncertainty state. We also analyze the long-time behavior of the quantum-optical system under consideration and we show that the interesting collapse-revival effect in the time evolution of the mean photon number and of the purity parameters of field modes can be observed. Finally, we show that the degree of entanglement between modes in the nondegenerate quantum-optical down-conversion strongly depends on the initial state of the system.

PACS number(s): 42.50.Dv, 03.65.Bz

I. INTRODUCTION

Nonlinear quantum-optical processes are generally considered as candidates for the production of highly nonclassical states of light [1]. In particular, a great deal of attention has been paid to quadratic processes, both degenerate as well as nondegenerate, described by the following Hamiltonians [2]:

$$\hat{H}_D=\omega_a\hat{a}^\dagger\hat{a}+\lambda_D\gamma[(\hat{a}^\dagger)^2\exp(-i\omega_c t)+\hat{a}^2\exp(i\omega_c t)]\ , \tag{1.1}$$

$$\hat{H}_N=\omega_a\hat{a}^\dagger\hat{a}+\omega_b\hat{b}^\dagger\hat{b}+\lambda_N\gamma[\hat{a}^\dagger\hat{b}^\dagger\exp(-i\omega_c t)+\hat{a}\hat{b}\exp(i\omega_c t)]\ , \tag{1.2}$$

where $\hat{a}$ and $\hat{a}^\dagger$ ($\hat{b}$ and $\hat{b}^\dagger$) are the annihilation and the creation operators of the signal (idler) mode, respectively. These operators obey the usual bosonic commutation relations $[\hat{a},\hat{a}^\dagger]=1$, $[\hat{b},\hat{b}^\dagger]=1$. The coupling constants λ_D and λ_N are proportional to the second-order nonlinear polarizability of the medium. The pump mode is assumed to be a classical field with amplitude γ and frequency ω_c.

The degenerate two-photon down-conversion process with the classical pump described by the Hamiltonian (1.1) has served as a prototype of the quantum-optical process in which a single-mode squeezed-light field can be produced [3]. In fact, Kimble and co-workers [4] using the degenerate down-conversion process described by the Hamiltonian (1.1) have experimentally generated squeezed light exhibiting a high degree of quadrature noise reduction. On the other hand, the nondegenerate Hamiltonian (1.2) describes a process in which a pair of highly correlated signal and idler photons is created. This correlation results in a high degree of two-mode squeezing (for details on squeezing see the recent review articles [5]).

Generally it is understood that by increasing the amount of energy transferred from the pump into the down-converted mode(s), the degree of squeezing can be enhanced. Nevertheless, in real processes the transfer of large amounts of energy from the pump to the down-converted field modes is associated with the depletion of the pump mode and the back action of the down-converted modes on the pump field. These two effects as well as the role of quantum fluctuations of the pump field are completely neglected in the parametric approximation, when the pump mode is assumed to be a classical field with a constant amplitude.

In a proper quantum-mechanical treatment one should take into account the quantum nature of the signal and the idler as well as the pump mode. In this case, instead of the Hamiltonians (1.1) and (1.2), we have to consider the following Hamiltonians [6–9]:

$$\hat{H}_D=\omega_a\hat{a}^\dagger\hat{a}+2\omega_a\hat{c}^\dagger\hat{c}+\lambda[(\hat{a}^\dagger)^2\hat{c}+\hat{a}^2\hat{c}^\dagger]\ , \tag{1.3}$$

$$\hat{H}_N=\omega_a\hat{a}^\dagger\hat{a}+\omega_b\hat{b}^\dagger\hat{b}+(\omega_a+\omega_b)\hat{c}^\dagger\hat{c}+\hat{H}_{\rm int}\ , \tag{1.4a}$$

where

$$\hat{H}_{\rm int}=\lambda(\hat{a}^\dagger\hat{b}^\dagger\hat{c}+\hat{a}\hat{b}\hat{c}^\dagger) , \quad (1.4b)$$

where we have assumed the pump mode c to be in the resonance with the signal and the idler modes (i.e., $\omega_c=2\omega_a$ or $\omega_a+\omega_b$).

We should note here that nonlinear dynamics described by the Hamiltonian (1.1) stimulated investigation in the direction of generalization to higher-order processes. It was expected that processes with Hamiltonians

$$\hat{H}_k=\omega\hat{a}^\dagger\hat{a}+\lambda_k\gamma[(\hat{a}^\dagger)^k\exp(-ki\omega t)+\hat{a}^k\exp(ki\omega t)] , \quad (1.5)$$

describing the production of a bunch of k photons from a classical current, can lead to the definition of new interesting states. Nevertheless, it was shown that the approach starting from such a generalization faces serious mathematical difficulties [9]. One possibility of how to overcome these mathematical problems was proposed by Hillery [9], who suggested taking into consideration explicitly the quantum nature of the pump mode. The inclusion of the quantum nature of the pump is also stimulated by the fact that the statistical properties of the pump can have serious effects on the nature of the generated down-converted modes. Such influence can be completely eliminated by an eventual parametric approximation beforehand (see, for instance, Ref. [10]). Moreover, the Hamiltonian (1.1) cannot be regarded as a parametric approximation of the Hamiltonian (1.3) except in the region of first instants $\sqrt{2}\gamma\lambda_D t<1$ (Ref. [10]).

The aim of the present paper is to study the entanglement between the modes in the process of the nondegenerate two-photon down-conversion with the quantized pump described by the Hamiltonian (1.4). We will show that due to the quantum nature of the dynamics, strong entanglement between the pump and the signal-idler subsystems can be observed. We will show that the higher the initial intensity of the pump mode, the stronger the entanglement between the pump and the signal-idler subsystem is established during the first instants of the time evolution. We will find that in the process under consideration the signal and the idler modes are strongly entangled (correlated), which results in a high degree of two-mode squeezing observable during the first instants of the time evolution. We will show that the back action of the signal-idler subsystem on the pump model leads to a strong single-mode squeezing of the pump mode. At the time interval during which squeezing of the pump mode can be observed this mode is far from being in the minimum uncertainty state. We will also analyze the long-time behavior of the quantum-optical system under consideration and we show that the interesting collapse-revival effect in the time evolution of a mean photon number and purity parameters of field modes can be observed. Finally, we will show that the degree of the entanglement between the modes in the nondegenerate quantum-optical down-conversion is highly sensitive with respect to the quantum-statistical properties of the initial state of the system under consideration.

II. QUANTUM DYNAMICS OF THE NONDEGENERATE TWO-PHOTON DOWN-CONVERSION

It has been shown recently that one cannot solve analytically in a closed form the von Neumann equation for the density $\hat{\rho}$ with the nonlinear Hamiltonians of the type (1.3) and (1.4) [11,12]. On the other hand, the numerical analysis based on a diagonalization of the Hamiltonians in the interaction picture turns out to be very efficient and allows one to study the time evolution of the quantum-mechanical systems described by trilinear Hamiltonians (see [13,14] and references quoted therein).

The purpose of this paper is to study quantum correlations between the pump, signal, and idler modes in the quantum nondegenerate two-photon down-conversion. As we will see later, the quantum dynamics described by the Hamiltonian (1.4) leads to a strong entanglement between the pump and the signal-idler subsystem and between the signal and the idler modes. The entanglement between the modes leads to the increase of marginal entropies of the pump, signal, and idler modes, respectively (even though the total entropy is constant and equal to zero, if we assume the system to be initially prepared in a pure state and neglect losses in the system). The quantum-mechanical entropy S_i (as defined by von Neumann) of the mode i ($i=a,b,c$) [15] (we assume the Boltzmann constant k_B to be equal to unity),

$$S_i=-{\rm Tr}_i(\hat{\rho}_i\ln\hat{\rho}_i) , \quad (2.1)$$

is defined through the reduced density operator $\hat{\rho}_i$ of the i mode. In particular, the reduced density operator of the pump mode $\hat{\rho}_c$ is given by the relation

$$\hat{\rho}_c={\rm Tr}_{ab}\hat{\rho} , \quad (2.2)$$

where ${\rm Tr}_{ab}$ denotes tracing over the signal and the idler variables.

In what follows we will utilize the Araki-Lieb theorem [16], which can be expressed in the form

$$|S_x-S_y|\le S\le S_x+S_y , \quad (2.3)$$

where S_x and S_y are the marginal entropies of two subsystems which compose the whole system. From (2.3) it follows that if the pump, signal, and idler modes are initially prepared in pure states, i.e.,

$$S_a|_{t=0}=S_b|_{t=0}=0=S_c|_{t=0} , \quad (2.4)$$

then the entropy S of the total system is initially equal to zero. Due to the fact that we do not take into account losses in our model, the total entropy is an integral of motion, i.e., $S=0$ for any $t>0$ and

$$S_{ab}=S_c \quad \text{for } t>0 , \quad (2.5)$$

where S_{ab} is the entropy of the signal-idler subsystem. Additionally, from the Araki-Lieb theorem it then follows that

$$|S_a-S_b|\le S_{ab}=S_c\le S_a+S_b . \quad (2.6)$$

One possibility of how to quantify the degree of the entanglement between two subsystems is to evaluate the index of correlation $I^{\text{corr}}_{x\text{-}y}$ defined as (see, for instance, [17])

$$I^{\text{corr}}_{x\text{-}y}=S_x+S_y-S_{xy}\ . \tag{2.7}$$

From the above it follows that in our case the index of correlation $I^{\text{corr}}_{ab\text{-}c}$ between the pump and the signal-idler system is equal to twice the marginal entropy of the pump mode:

$$I^{\text{corr}}_{\text{ab-}c}=2S_c\ . \tag{2.8}$$

The stronger the entanglement, the higher the value of the index of correlation and the larger the entropy of the pump mode. In this paper we will study the nondegenerate down-conversion process for such initial states for which the marginal entropies of the signal and idler modes are equal for any $t>0$. This additional assumption enables us to express the signal-idler index of correlation $I^{\text{corr}}_{a\text{-}b}$ in the form

$$I^{\text{corr}}_{a\text{-}b}=S_a+S_b-S_{ab}=2S_a-S_c\ , \tag{2.9}$$

from which it follows that for given values of the signal and idler entropies the index of correlation between the signal and idler decreases with the increase of the pump entropy.

The increase of the entropy of the pump mode in the two-photon down-conversion reflects the fact that the initial pure state of the pump mode is transformed into a statistical-mixture state. On the other hand, in the parametric approximation described by the Hamiltonian (1.2) no quantum entanglement between the pump and the signal-idler subsystem can appear; therefore, if the signal and idler modes are initially in pure states (for instance, in the vacuum state), then the entropy S_{ab} is identically equal to zero for any $t>0$ and both the signal and the idler are maximally correlated for any $t>0$ [17]. From here we conclude that the parametric approximation leads not only to neglecting of the important role of the pump fluctuations and the pump depletion but also to neglecting of the pump-signal entanglement.

In this paper, instead of evaluating entropies of the modes, we will evaluate the so-called purity parameters S_i^{pur} which are also functions of the reduced density matrix $\hat{\rho}_i$. The purity parameter S_i^{pur} is defined as

$$S_i^{\text{pur}}=1-\text{Tr}_i[(\hat{\rho}_i)^2]\ , \tag{2.10}$$

and it can be shown that the purity parameter represents a lower bound for the corresponding entropy S_i, i.e., $S_i^{\text{pur}}\le S_i$. For a pure state $S_i^{\text{pur}}=0$, otherwise $S_i^{\text{pur}}>0$. Moreover, the purity parameter (by the choice of our initial states) will satisfy relations valid for the entropy, such as, for instance, the Araki-Lieb theorem (2.6). The purity parameter of the chosen system is never greater than the purity parameter of its subsystem. For our particular case this implies

$$S^{\text{pur}}\equiv S^{\text{pur}}_{abc}\le S^{\text{pur}}_{ab}\le S^{\text{pur}}_{a}\ . \tag{2.11}$$

From a computational point of view, the parameter S_i^{pur} can be evaluated much more easily than S_i. Therefore we will analyze the entanglement between modes using the purity parameter and not the entropy S^i.

III. ENTANGLEMENT BETWEEN THE MODES: SHORT-TIME BEHAVIOR

In this section we will analyze the entanglement of the field modes in the case when the pump mode is initially prepared in a coherent state and the signal and the idler modes are in the vacuum state at $t=0$:

$$|\Psi(t=0)\rangle=|0\rangle_a|0\rangle_b|\gamma\rangle_c\ , \tag{3.1}$$

where $|\gamma\rangle_c$ describes the coherent state with a real amplitude γ:

$$|\gamma\rangle_c=\exp(-\gamma^2/2)\sum_{n=0}^{\infty}\frac{\gamma^n}{\sqrt{n!}}|n\rangle_c\ . \tag{3.2}$$

Using the numerical approach described in Refs. [13,14] we can study the time evolution of the initial-state vector (3.1) governed by the Hamiltonian (1.4). The purity parameter of the pump mode at the early stages of the evolution is shown in Fig. 1 for various initial intensities γ^2. From this picture we learn that the higher the intensity of the initial pump mode, the larger the purity parameter S_c^{pur} of the pump, i.e., the stronger the entanglement between the pump and the signal-idler system. Moreover, we see that after the first increase of the parameter S_c^{pur} a clear minimum is formed. These "oscillations" in S_c^{pur} are repeated few times before the purity parameter reaches an almost steady state with small oscillations (see the long-time behavior). From the time evolution of the purity parameter it is clearly seen that the pump mode remains in a pure state only at very early stages of the time evolution. In addition, the larger the intensity of the initial pump mode, the shorter the interval at which $S_c^{\text{pur}}\simeq 0$ [see Eq. (3.6)].

At the early stages of the time evolution the evolution operator

$$\hat{U}(t)=\exp(-i\hat{H}_N t) \tag{3.3}$$

can be well approximated by the first two terms of the

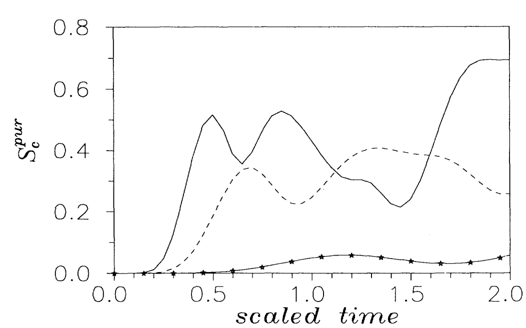

FIG. 1. The time evolution of the purity parameter S_c^{pur} for various intensities of the initial coherent state of the pump ($\gamma=1$, line with stars; $\gamma=3$, dashed line; $\gamma=5$, solid line).

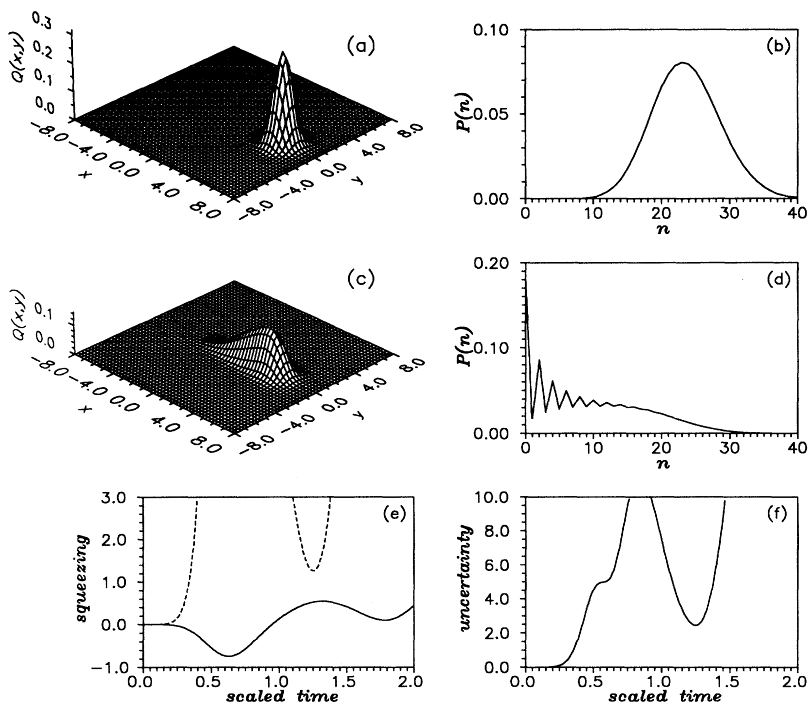

FIG. 2. Parameters of the pump mode ($\gamma=5$): (a) the Q_c function at $\lambda t=0.2$, (b) the photon-number distribution at $\lambda t=0.2$, (c) the Q_c function at $\lambda t=0.5$, (d) the photon-number distribution at $\lambda t=0.5$, (e) the time evolution of the single-mode squeezing, and (f) the time evolution of the MUS parameter.

formal Taylor expansion of the right-hand side of Eq. (3.3):

$$\hat{U}(t)\simeq 1-i\hat{H}_N t \ . \tag{3.4}$$

The time evolution of the state vector of the whole system is then given as (we drop the free evolution term)

$$\begin{aligned}|\psi(t)\rangle &= \hat{U}(t)|0\rangle_a|0\rangle_b|\gamma\rangle_c \\ &\approx (|0\rangle_a|0\rangle_b - it\lambda\gamma|1\rangle_a|1\rangle_b)|\gamma\rangle_c \\ &= |\xi\rangle_{ab}|\gamma\rangle_c \ .\end{aligned} \tag{3.5}$$

Such a factorization is possible only for times t for which [10]

$$t<t_c=\frac{1}{\gamma\lambda} \ . \tag{3.6}$$

From Eq. (3.5) it follows that for times $t<t_c$ the pump mode remains almost completely disentangled from the down-converted modes. In other words, during the time interval $t<t_c$ the statistical properties of the pump are not affected by the interaction with the signal-idler subsystem. To see this, we plot in Fig. 2 the Q function and the photon-number distribution of the pump mode. The Q_i function of the particular mode is defined as [18]

$$Q_i(\alpha)=\frac{1}{\pi}\langle\alpha|\hat{\rho}_i|\alpha\rangle \ , \tag{3.7}$$

and the photon-number distribution (PND) as

$$P_i(n)=\langle n|\hat{\rho}_i|n\rangle \ . \tag{3.8}$$

A. Statistical properties of the pump mode

In Fig. 2 we plot the Q function and the PND of the pump with $\gamma=5$ for two different values of time. In Figs. 2(a) and 2(b) we plot Q_c and $P_c(n)$ at time $\lambda t=0.2$. The shapes of these functions are almost identical to that of the initial coherent state, i.e., the Q function has a Gaussian profile and $P_c(n)$ is the Poissonian distribution. As an additional check of the behavior of the state of the pump mode during the early stages of the time evolution, we evaluate the deviation of the pump mode from the minimum uncertainty state (MUS). As a measure of the degree of deviation from a MUS state we can adopt the following parameter:

$$u_c=\langle(\Delta\hat{X}_c)^2\rangle\langle(\Delta\hat{Y}_c)^2\rangle-\tfrac{1}{16} \ , \tag{3.9}$$

where the quadrature operators $\hat{X}_c$ and $\hat{Y}_c$ are defined as

$$\begin{aligned}&\hat{X}_c=\frac{\hat{c}(t)+\hat{c}(t)^\dagger}{2}, \quad \hat{Y}_c=\frac{\hat{c}(t)-\hat{c}(t)^\dagger}{2i} \ , \\ &\hat{c}(t)=\hat{c}\exp(i\omega_c t) \ ,\end{aligned} \tag{3.10}$$

and

$$\langle(\Delta\hat{X}_c)^2\rangle=\langle\hat{X}_c^2\rangle-\langle\hat{X}_c\rangle^2, \quad \langle(\Delta\hat{Y}_c)^2\rangle=\langle\hat{Y}_c^2\rangle-\langle\hat{Y}_c\rangle^2 \ . \tag{3.11}$$

This function is zero for MUS states and is positive otherwise. The time evolution of the function u_c is shown in Fig. 2(f). We see that for short times the parameter u_c equals zero and then starts to grow rapidly, which means that under the influence of the down-converted modes the pump mode starts to deviate from the MUS. Comparing Figs. 1 and 2(f), we can conclude that the pump mode starts to deviate from the MUS at the same time that it starts to be strongly entangled with the signal-idler subsystem. We can conclude that the back action of the down-converted modes affects significantly the statistical properties of the pump mode. In Fig. 2(c) [2(d)] we plot the Q function (the PND) of the pump mode at time

$\lambda t=0.5$ (for $\gamma=5$). We see that the Q_c function is "squeezed" in one of the directions and stretched in the other. The photon-number distribution exhibits oscillations similar to those of the squeezed state [3]. Moreover, the PND becomes much broader when compared with the Poissonian distribution and it indicates a considerable increase in the probability of finding lower number states. From the above we can conclude that the back action of the down-converted modes leads to a squeezing of the pump mode. In Fig. 2(e) we plot the time evolution of squeezing parameters defined as [5]

$$q_1=4\langle(\Delta\hat{X}_c)^2\rangle-1,\quad q_2=4\langle(\Delta\hat{Y}_c)^2\rangle-1\ . \tag{3.12}$$

From this figure it follows that the back action of the down-converted modes leads to a considerable reduction of quadrature fluctuations of the pump mode. The maximum degree of squeezing is obtained at times around $\lambda t\approx 0.6$, after which quadrature fluctuations become rapidly superfluctuant and do not become squeezed again.

B. Statistical properties of down-converted modes

Now we turn our attention to the down-converted modes. The purity parameter S_{ab}^{pur} of the signal-idler subsystem is equal to that of the pump mode. This naturally means that at the very early stages of the time evolution the signal-idler subsystem is in a pure state. While the pump is still in a pure state, a two-mode squeezed vacuum state is generated in the signal-idler modes. The two-mode squeezing can be described by introducing two quadrature operators $\hat{Z}_i$:

$$\hat{Z}_1=\frac{\hat{d}(t)+\hat{d}^\dagger(t)}{\sqrt{2}},\quad \hat{Z}_2=\frac{\hat{d}(t)-\hat{d}^\dagger(t)}{i\sqrt{2}}\ , \tag{3.13}$$

where

$$\hat{d}(t)=\frac{\hat{a}\exp(i\omega_a t)+\hat{b}\exp(i\omega_b t)}{2}\ . \tag{3.14}$$

The degree of two-mode squeezing can be quantified using two parameters $z_i(t)$ ($i=1$ and 2):

$$z_i(t)=4\langle(\Delta\hat{Z}_i)^2\rangle-1\ , \tag{3.15}$$

where $\langle(\Delta\hat{Z}_i)^2\rangle=\langle\hat{Z}_i^2\rangle-\langle\hat{Z}_i\rangle^2$ and 100% squeezing is obtained for $z_i(t)=-1$. The time evolution of $z_i(t)$ is shown in Fig. 3(a). During the first instants of the time evolution a high degree of two mode squeezing is obtained. During the initial period the energy is transferred from the pump mode to the signal-idler modes. With the increase of the transferred energy the degree of squeezing becomes larger. This scenario is valid until the moment when the back action of the signal-idler modes on the pump mode becomes significant. Before that a pure two-mode squeezed state is generated in the signal-idler modes. This pure ($S_{ab}^{\mathrm{pur}}\simeq 0$) two-mode state is a minimum uncertainty state. One can check this by inspecting [see Fig. 3(b)] the time evolution of the MUS parameter $u_{ab}(t)$, which is defined as

$$u_{ab}=\langle(\Delta\hat{Z}_1)^2\rangle\langle(\Delta\hat{Z}_2)^2\rangle-\tfrac{1}{16}\ . \tag{3.16}$$

With the initial condition (3.1) the reduced density operators of the signal and idler modes $\hat{\rho}_a$ and $\hat{\rho}_b$ have the same form, from which it follows that the signal and idler modes have identical statistical properties, i.e.,

$$S_a^{\mathrm{pur}}=S_b^{\mathrm{pur}},\quad P_a(n)=P_b(n),\quad \text{and}\quad Q_a(\alpha)=Q_b(\alpha)\ . \tag{3.17}$$

The time evolution of the purity parameter S_a^{pur} of the

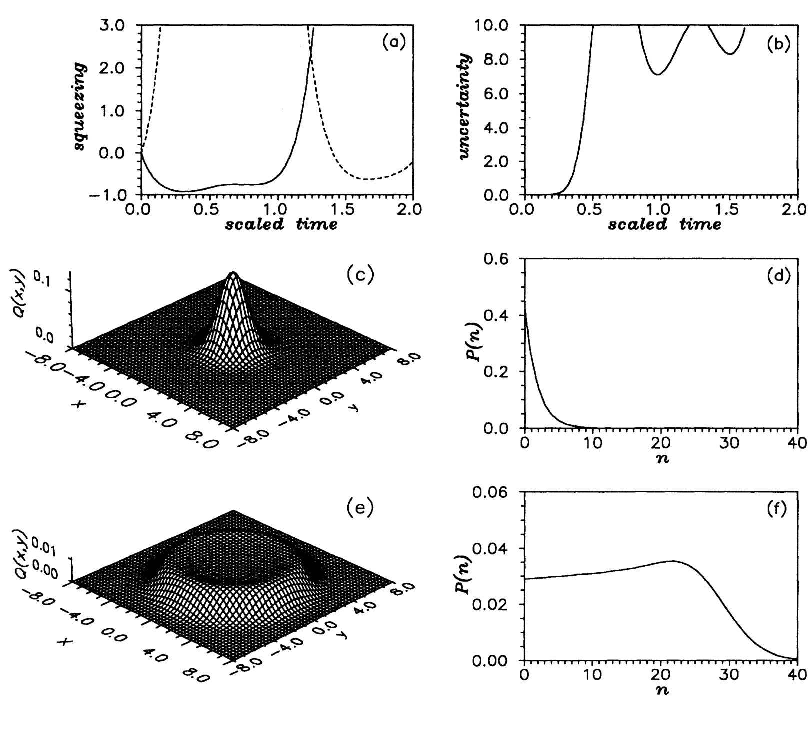

FIG. 3. Parameters of the down-converted modes ($\gamma=5$): (a) the time evolution of the two-mode squeezing, (b) the time evolution of the two-mode MUS parameter, (c) the Q_a function at $\lambda t=0.2$, (d) the photon-number distribution at $\lambda t=0.2$, (e) the Q_a function at $\lambda t=0.5$, (f) the photon-number distribution at $\lambda t=0.5$.

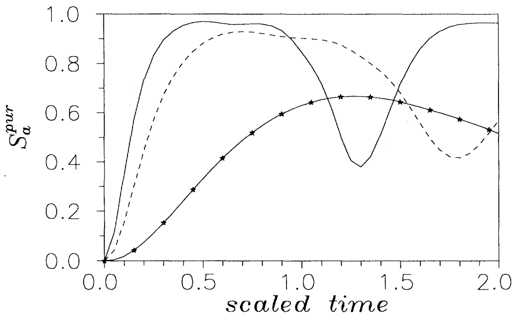

FIG. 4. The time evolution of the purity parameter $S_a^{\rm pur}$ for various intensities of the initial coherent state of the pump mode ($\gamma=1$, line with stars; $\gamma=3$, dashed line; $\gamma=5$, solid line).

signal mode is shown in Fig. 4. We see that for the given intensity of the initial pump mode the purity parameter $S_a^{\rm pur}$ increases during the initial period of the time evolution much more rapidly than the parameter $S_{ab}^{\rm pur}$ (compare Figs. 1 and 4). From here it follows that at initial stages of the time evolution the index of correlation $I_{a\text{-}b}^{\rm corr}$ takes its maximal values (i.e., $I_{a\text{-}b}^{\rm corr}\simeq 2S_a$), which is an additional proof that the two-mode squeezed vacuum is produced in the signal-idler modes [17]. As soon as the pump mode is affected by the action of the down-converted modes, the purity parameter $S_c^{\rm pur}$ becomes larger than zero, which results consequently in the deterioration of the degree of correlation between the signal and the idler.

In Fig. 3(c) [3(d)] we plot the Q function (the PND) of the signal mode at the moment $\lambda t=0.2$ (we assume $\gamma=5$). We clearly see the thermal-like character of the marginal photon-number distribution and the Q function, which is the characteristic feature of the two-mode squeezed vacuum state [5]. At later moments, when the pump and down-converted modes become entangled ($S_c>0$), the field statistics of the signal mode are significantly different from the thermal-like field. In particular, the photon-number distribution becomes very broad [compare Figs. 3(d) and 3(f)]. At times much longer than t_c [see Eq. (3.6)] the three modes under consideration become strongly entangled. They are not in pure states anymore, but the quantum nature of the dynamics leads to some new interesting features.

IV. ENTANGLEMENT BETWEEN THE MODES: LONG-TIME BEHAVIOR

If we compare the time evolution of the purity parameter of the pump mode $S_c^{\rm pur}$ [see Fig. 5(a)] and of the signal mode $S_a^{\rm pur}$ [Fig. 5(b)] we find that during the initial stages of the time evolution $S_a^{\rm pur}$ increases much more rapidly than $S_c^{\rm pur}$ [see the previous section and (2.11)]. Nevertheless, on the long-time scale the time evolution of both parameters is similar, i.e., both parameters exhibit small oscillation around some stationary value and a very significant decrease and significant oscillations at some particular moments. We can call this behavior the collapse and revival of the purity of the field mode. We should note here that the minima of the parameters $S_a^{\rm pur}$ and $S_c^{\rm pur}$ appear simultaneously and moments when they appear coincide with the "revivals" of the mean photon number of the signal (idler) mode (see Fig. 6) [14]. We see that due to the quantum dynamics we can observe a *partial* restoration of the initial purity of the modes under consideration.

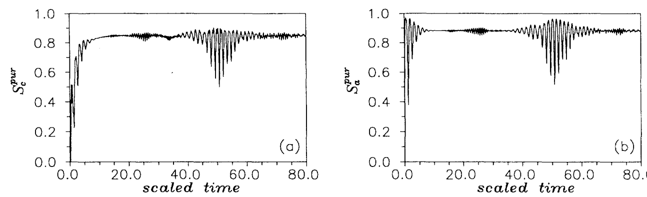

FIG. 5. The time evolution of the purity parameter $S_c^{\rm pur}$ of the pump mode (a) and $S_a^{\rm pur}$ of the down-converted mode (b) for $\gamma=5$ on the long-time scale.

Estimation of the "revival" time

The explanation of the collapse-revival behavior in our case can be done using the eigenvalues $E_i(n)$ and the eigenvectors $|E_i(n)\rangle$ of the Hamiltonian $\hat{H}_{\rm int}$ given by Eq. (1.4b):

$$\hat{H}_{\rm int}|E_i(n)\rangle=E_i(n)|E_i(n)\rangle\ , \tag{4.1}$$

where n is the eigenvalue of the number operator $\hat{N}$:

$$\hat{N}=\hat{b}^\dagger\hat{b}+\hat{c}^\dagger\hat{c}\ . \tag{4.2}$$

For a fixed value of n, the eigenvalues $E_i(n)$ and the eigenvectors $|E_i(n)\rangle$ are obtained via the diagonalization of the interaction Hamiltonian $\hat{H}_{\rm int}$ on the subspace $\mathcal{H}_n$ formed out of the set of vectors of the form $\{i=0,\dots,n;|i\rangle_a|i\rangle_b|n-i\rangle_c\}$. The dimension of the subspace $\mathcal{H}_n$ is $n+1$. The eigenvectors $|E_i(n)\rangle$ are given as superpositions of the basis vectors $|i\rangle_a|i\rangle_b|n-i\rangle_c$:

$$|E_j(n)\rangle=\sum_{i=0}^{n}a_{ji}(n)|i\rangle_a|i\rangle_b|n-i\rangle_c\ ,$$

where the actual values of $E_j(n)$ and $a_{ji}(n)$ can be found by numerical calculations.

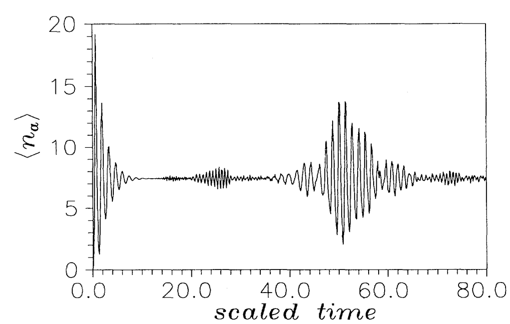

FIG. 6. The time evolution of the mean photon number $\langle n_a\rangle$ of the signal mode for $\gamma=5$.

We suppose the initial state of the whole system $|\psi(0)\rangle$ to be of the form

$$|\psi(0)\rangle=|0\rangle_a|0\rangle_b|\gamma\rangle_c$$
$$=\exp(-|\gamma|^2/2)\sum_{n=0}^{\infty}\frac{\gamma^n}{\sqrt{n!}}|0\rangle_a|0\rangle_b|n\rangle_c\ . \quad (4.3)$$

The state vector $|0\rangle_a|0\rangle_b|n\rangle_c$ can be decomposed in the basis of the eigenvectors $|E_i(n)\rangle$:

$$|0\rangle_a|0\rangle_b|n\rangle_c=\sum_{j=0}^{n}c_j(n)|E_j(n)\rangle\ . \quad (4.4)$$

In Fig. 7 we plot a distribution of probabilities $P(j)=|c_j(n)|^2$ [in fact, the coefficients $c_j(n)$ are real] which indicate the overlap between the state $|0\rangle_a|0\rangle_b|n\rangle_c$ and the eigenstate $|E_j(n)\rangle$.

From this figure it follows (for more details see Refs. [13,14]) that for $n=2l$ (even) the decomposition (4.4) can be well approximated as

$$|0\rangle_a|0\rangle_b|2l\rangle_c\approx c_0(2l)|E_0(2l)\rangle+c_1(2l)|E_1(2l)\rangle+c_2(2l)|E_2(2l)\rangle\ , \quad (4.5)$$

with $E_0(2l)=0$ and $E_2(2l)=-E_1(2l)$. In addition, from Fig. 7 follows that $c_1(n)=c_2(n)$. Consequently, the time evolution of the state (4.4) is given as (we drop the free evolution term proportional to $\exp[-i(\omega_a+\omega_b)2lt]$)

$$|\psi_{2l}(t)\rangle=\exp(-i\hat{H}_Nt)|0\rangle_a|0\rangle_b|2l\rangle_c$$
$$\approx c_0(2l)|E_0(2l)\rangle+c_1(2l)\exp[-iE_1(2l)t]|E_1(2l)\rangle+c_1(2l)\exp[iE_1(2l)t]|E_2(2l)\rangle\ . \quad (4.6)$$

For $n=2l+1$ (odd) we can express the expression (4.4) approximately as

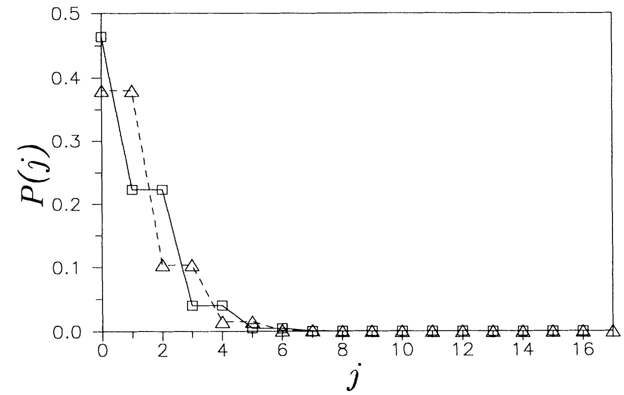

FIG. 7. Distributions of the probabilities $P(j)=|c_j(n)|^2$ indicating the overlap between the state $|0\rangle_a|0\rangle_b|n\rangle_c$ and the eigenstates $|E_j(n)\rangle$ for $n=16$ (line with squares) and $n=17$ (line with triangles). Increasing i indicates increasing absolute value of eigenvalues.

$$|0\rangle_a|0\rangle_b|2l+1\rangle_c\approx c_0(2l+1)|E_0(2l+1)\rangle+c_1(2l+1)|E_1(2l+1)\rangle\ , \quad (4.7)$$

where $E_0(2l+1)=-E_1(2l+1)$ and $c_0(2l+1)=c_1(2l+1)$ (see Fig. 7). The time evolution of this state is given as (we again drop the free evolution term $\exp[-i(\omega_a+\omega_b)(2l+1)t]$)

$$|\psi_{2l+1}(t)\rangle\approx\exp[-iE_0(2l+1)t]c_0(2l+1)|E_0(2l+1)\rangle+\exp[iE_0(2l+1)t]\times c_0(2l+1)|E_1(2l+1)\rangle\ . \quad (4.8)$$

Using the expressions (4.6) and (4.8), the time evolution of the whole system with the initial condition (4.3) can be written as

$$|\psi(t)\rangle\approx\sum_{l=0}^{\infty}Q(2l)\{c_0(2l)|E_0(2l)\rangle+c_1(2l)\exp[-iE_1(2l)t]|E_1(2l)\rangle+c_1(2l)\exp[iE_1(2l)t]|E_2(2l)\rangle\}$$
$$+\sum_{l=0}^{\infty}Q(2l+1)\{\exp[-iE_0(2l+1)t]c_0(2l+1)|E_0(2l+1)\rangle+\exp[iE_0(2l+1)t]c_0(2l+1)|E_1(2l+1)\rangle\}\ , \quad (4.9)$$

where $Q(n)=\exp(-\gamma^2/2)\gamma^n/\sqrt{n!}$. The time evolution of the mean photon number $\bar{n}_x$ of the particular mode x,

$$\bar{n}_x(t)=\langle\psi(t)|\hat{x}^\dagger\hat{x}|\psi(t)\rangle\ , \quad (4.10)$$

can be now well approximated as

$$\bar{n}_x(t) = \sum_{l=0}^{\infty} Q^2(2l) \Bigg[2c_0(2l)c_1(2l)\langle E_0(2l)|\hat{x}^\dagger\hat{x}|E_1(2l)\rangle \cos[E_1(2l)t] + 2c_0(2l)c_1(2l)\langle E_0(2l)|\hat{x}^\dagger\hat{x}|E_2(2l)\rangle \cos[E_1(2l)t] + 2c_1^2(2l)\langle E_1(2l)|\hat{x}^\dagger\hat{x}|E_2(2l)\rangle \cos[2E_1(2l)t] + \sum_{i=0}^{2} c_i^2(2l)\langle E_i(2l)|\hat{x}^\dagger\hat{x}|E_i(2l)\rangle \Bigg] + \sum_{l=0}^{\infty} Q^2(2l+1) \Bigg[2c_0^2(2l+1)\langle E_0(2l+1)|\hat{x}^\dagger\hat{x}|E_1(2l+1)\rangle \cos[2E_0(2l+1)t] + \sum_{i=0}^{1} c_i^2(2l+1)\langle E_i(2l+1)|\hat{x}^\dagger\hat{x}|E_i(2l+1)\rangle \Bigg] . \quad (4.11)$$

Terms in the expression (4.11) are proportional to cosines of the eigenvalues $E_1(2l)$, $2E_1(2l)$, and $2E_0(2l+1)$. Using arguments similar to those used for the derivation of the revival time for the coherent Jaynes-Cummings model [19], we can estimate the revival times for our case. For a sufficiently intense initial coherent state $|\gamma\rangle_c$ the relevant terms in the sums are those with $n \approx \bar{n} = \gamma^2$. The revival times t_{R_i} can be now estimated as times when two neighboring oscillations with $2\bar{l} \approx \gamma^2$ and $\gamma^2 + 2$ acquire a 2π phase difference [19]:

$$2[E_1(2\bar{l}+2) - E_1(2\bar{l})]t_{R_1} = 2\pi , \quad (4.12a)$$

$$[E_1(2\bar{l}+2) - E_1(2\bar{l})]t_{R_2} = 2\pi , \quad (4.12b)$$

$$2[E_0(2\bar{l}+1) - E_0(2\bar{l}-1)]t_{R_3} = 2\pi . \quad (4.12c)$$

A numerical analysis of the eigenvalues for n even and odd shows that $t_{R_2} \approx t_{R_3}$. The other two revivals are given by relations

$$t_{R_2} = \frac{2\pi}{E_1(2\bar{l}+2) - E_1(2\bar{l})} = 2t_{R_1} . \quad (4.13)$$

Using the actual values of the eigenvalues E_j (see Refs. [13,14]) given in Table I we find that $t_{R_1} \simeq 25$ and $t_{R_2} \simeq 50$, which is in agreement with numerical plots in Figs. 5 and 6 (we have set $\gamma = 5$). We see that during the first "revival" at $t \simeq t_{R_1}$ the envelope of the purity parameter indicates just a small reduction of S_c^{pur}. This reduction is much more pronounced at $t \simeq t_{R_2}$. Rapid oscillations of the purity parameter S_c^{pur} around t_{R_2} indicate a very rapid change of statistics of the pump (as well as the signal) mode. We illustrate this by plotting the $Q_c(\alpha)$ function and the corresponding photon-number distribution $P_c(n)$ of the pump mode at three subsequent moments in the vicinity of the revival time t_{R_2} (see Fig. 8). In Fig. 8(a) the Q_c function is plotted at the moment when the purity parameter reaches its local maximum.

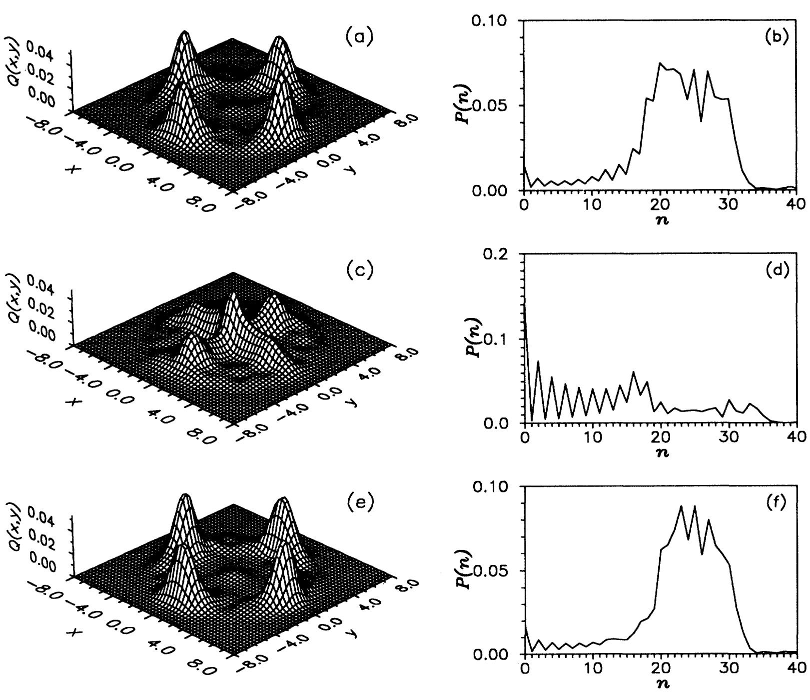

FIG. 8. The Q_c function [(a), (c), and (e)] and the photon-number distribution [(b), (d), and (f)] of the pump mode at times $\lambda t = 49.6$, 50.0, and 50.8, respectively.

TABLE I. The eigenvalues $E_j(n)$ of the interaction Hamiltonian (1.4b) for $n=23$–27. We adopt units $\hbar=1$ and $\lambda=1$.

n	E_0	E_1	E_2
23	2.168 76	−2.168 76	7.116 31
24	0.0	4.584 62	−4.584 62
25	2.231 32	−2.231 32	7.304 19
26	0.0	4.706 73	−4.706 73
27	2.291 14	−2.291 14	7.484 09

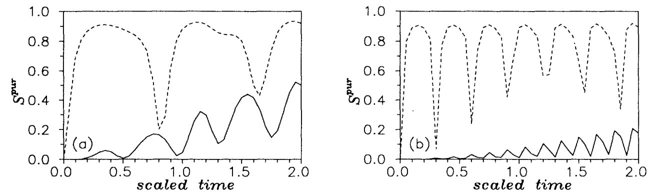

FIG. 9. (a) Time evolution of the purity parameter for $D=10$ (dashed line, a mode; solid line, c mode). In (b) the parameter $D=100$. In both cases $\gamma=3$.

We see that the Q_c function is formed by four identical component states which are mutually out of phase by 90°. The corresponding photon-number distribution has a Poissonian-like peak around the initial $\bar{n}$ [Fig. 8(b)]. These results indicate that at this particular moment the pump mode is in a statistical mixture state. On the contrary, at the moment when the purity parameter reaches its local minimum we can observe a significant peak of the Q_c function at the origin of the phase space [Fig. 8(c)]. Simultaneously the photon-number distribution exhibits significant oscillations [Fig. 8(d)], which can be explained as a consequence of quantum interference in the phase space (see [20] and references cited therein). At the subsequent moment when the purity parameter $S_c^{\rm pur}$ again reaches its maximum value the Q_c function is composed of four peaks [compare Figs. 8(e) and 8(a)] and the photon-number distribution has a peak around $\bar{n}$ [compare Figs. 8(f) and 8(b)].

V. ROLE OF INITIAL STATISTICS OF DOWN-CONVERTED MODES

We can expect that the entanglement between the modes will depend not only on the initial statistics of the pump mode but also on the initial statistics of the down-converted modes [10]. In this section we illustrate this dependence on a simple example when the idler mode is initially prepared in a number state $|D\rangle_b$, i.e., the total state vector at $t=0$ has the form

$$|\psi(0)\rangle=|0\rangle_a|D\rangle_b|\gamma\rangle_c \ . \tag{5.1}$$

In this section we will assume the number of photons in mode b to be comparable or larger than in mode c. Therefore, in what follows we will not denote mode c as the pump mode.

It can be shown easily that for the case $D\neq 0$ the following equations hold for the parameters of the a and b modes:

$$S_a^{\rm pur}=S_b^{\rm pur} \ , \tag{5.2}$$

$$P_a(n)=P_b(n+D) \ . \tag{5.3}$$

The entanglement parameters of the c and a modes for various values of D are shown in Fig. 9. We see that the time evolution of $S_a^{\rm pur}$ depends on the value of D. Namely, the larger the value of D, the faster the purity parameter $S_a^{\rm pur}$ increases. The other interesting feature we can see in Fig. 9 is the quasiperiodic time evolution of the purity parameters $S_a^{\rm pur}$ and $S_c^{\rm pur}$ for various values of D.

At the early stages of the time evolution the state vector of the whole system can be naively (see below) approximated as (we drop the free evolution term)

$$|\psi(t)\rangle\approx(|0\rangle_a|D\rangle_b-it\sqrt{D+1}\lambda\gamma|1\rangle_a|D+1\rangle_b)|\gamma\rangle_c$$
$$=|\xi\rangle_{ab}|\gamma\rangle_c \ . \tag{5.4}$$

Such a factorization can be done only for times

$$0\le t<\frac{1}{\lambda\sqrt{D+1}\gamma}=t_c \ , \tag{5.5}$$

at which the purity parameter $S_c^{\rm pur}$ is approximately

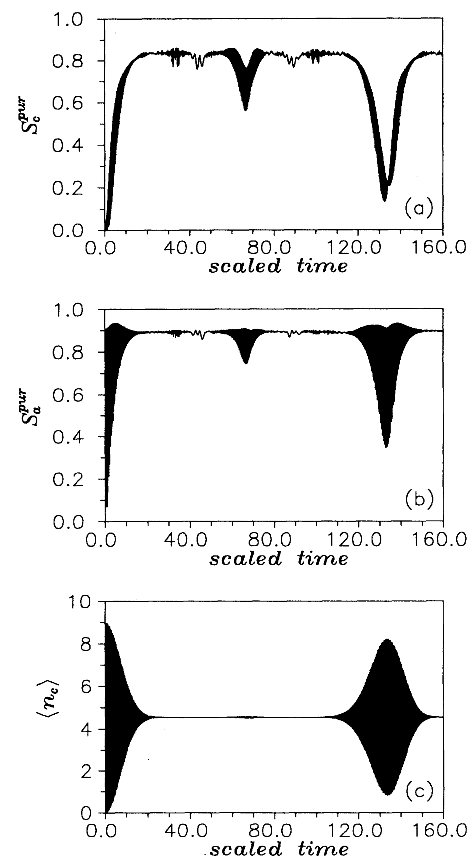

FIG. 10. The time evolution of the purity parameter of the pump mode (a) and the signal mode (b) on the long-time scale for $D=100$ and $\gamma=3$. In (c) we see the time evolution of the mean photon number.

equal to zero. On the other hand, at times longer than time t_c the c mode becomes entangled with modes a and b and the application of the parametric approximation is questionable [10]. From Eqs. (5.4) and (5.5) it follows that the initial time interval during which the modes are still not entangled should become shorter with increasing D. Nevertheless, the exact calculation shows that in the case when $D \gg \gamma^2$ the entanglement between the c mode to the a and b modes (i.e., the parameter S_c^{pur}) is much weaker compared with the case when $D=0$ [see Fig. 9(b)], i.e., the modes become entangled much later than is indicated by t_c. Let us note here that while in the case $D=0$ at the early stages of the time evolution the two-mode squeezed vacuum is produced in the $(a+b)$ subsystem, in the case when $D \neq 0$ a displaced two-mode squeezed vacuum state [21] is produced in these modes.

In the previous section we have shown that in the long-time scale our system exhibits a purely quantum effect of collapses and revivals of the purity parameters. In other words, the system under consideration exhibits "spontaneous" disentanglement between the pump mode and the down-converted modes. A similar behavior can be observed when the idler mode is initially prepared in the Fock state $|D\rangle_b$. Moreover, with the increase of the initial number of idler photons the "collapse-revival" effect becomes even more pronounced (see Figs. 9 and 10). In the limit $D \gg \gamma^2$ there are moments during which the purity parameter of the c mode approaches zero, which means that two "down-converted" modes become disentangled from the c mode. Simultaneously, the purity parameter S_a^{pur} becomes reduced, but the index of correlation $I_{a\text{-}b}^{\text{corr}}$ is still larger than zero, i.e., the "down-converted" modes are still correlated.

In the case with $D>0$, the estimation of the "revival" time is much more difficult compared with the case $D=0$. This is mainly because of the fact that the decomposition of states $|0\rangle_a|D\rangle_b|n\rangle_c$ into the eigenvectors $|E_i(n)\rangle$ is not as simple as in the case where $D=0$. Nevertheless, for the limit case $D \gg \gamma^2$ an estimation for the revival times of the mean photon number can be given [22]. These revival times t_{R_k} are

$$t_{R_k} \simeq 4\pi k\sqrt{D}\,, \quad k=1,2,3,\ldots, \tag{5.6}$$

which is in good agreement with Fig. 10(c). There are some additional "small revivals" of the purity parameters at times $t \simeq t_{R_1}/2$ [see Figs. 10(a) and 10(b)]. It turns out that it is very difficult to estimate these additional revival times.

VI. CONCLUDING REMARKS

In this paper we have studied the entanglement between the field modes in the process of nondegenerate two-photon down-conversion with a quantized pump. We have shown that due to the quantum nature of the dynamics the strong entanglement between the pump and the signal-idler subsystems can be observed. We have found that the higher the initial intensity of the pump mode, the stronger the entanglement between the pump and the signal-idler subsystem is established during the first instants of the time evolution. We have also shown that the signal and idler modes are strongly entangled (correlated). This entanglement is much stronger than the entanglement between the pump and the signal-idler subsystem. Correlations between the signal and the idler modes lead to a high degree of two-mode squeezing which can be observed during the first instants of the time evolution when the pump mode is still approximately in a pure state. On the other hand, the back action of the signal-idler subsystem on the pump mode leads to a strong single-mode squeezing of the pump mode. At the time interval during which squeezing of the pump mode can be observed, the pump mode is far from being in the minimum uncertainty state. We have also analyzed the long-time behavior of the quantum-optical system under consideration and have shown that the interesting collapse-revival effect in the time evolution of the mean photon number and of the purity parameters of the field modes can be observed. We have also shown that the degree of entanglement between modes in the nondegenerate quantum-optical down-conversion strongly depends on the initial state of the system. In particular, we have studied the situation when the idler mode is initially prepared in the Fock state $|D\rangle_b$. We have found that with the increase of D ($D \gg \gamma^2$) the entanglement between the signal and the pump modes becomes weaker. This strong dependence on the initial conditions requires a more detailed analysis and is studied in Ref. [22].

On the other hand, in the long-time scale very significant disentanglement between the c ("pump") mode and the down-converted modes can be observed. This "spontaneous" disentanglement has quantum-mechanical origin, and therefore it is difficult to say whether it can be observed in real systems in which dissipative processes in the long-time scale play a "destructive" role, i.e., they rapidly destroy quantum coherences.

ACKNOWLEDGMENTS

This work was supported by the Grant Agency of the Slovak Academy of Sciences Grant No. GA SAV 2/999143/92. One of us (V.B.) would like to acknowledge financial support from the Ilkovič Foundation, and to thank Professor Gershon Kurizki for the hospitality at the Weizmann Institute, where part of this work was done.

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