Bremsstrahlung in laser-assisted scattering

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(Received 16 November 1992)

The process of spontaneous radiation accompanying the scattering of a charged particle by a center of force is studied under conditions in which the scattering takes place in an intense, low-frequency laser field. Approximations are developed which provide expressions for both the amplitude and the total rate for spontaneous bremsstrahlung in the presence of the external field, given in terms of the physical (on-shell) amplitude for bremsstrahlung in the absence of the field. Of particular interest is the effect of the external field on the radiation probability when there is a narrow resonance in the field-free scattering. An explicit analytic approximation is obtained for the spectrum of spontaneous radiation in the resonant case; a series of peaks is found, with spacing between adjacent peaks equal to the frequency of the external field. The effect may be thought of as a continuum analog of the process of intense-field harmonic generation. A simple model, constructed with the aid of a low-frequency approximation for the amplitude for bremsstrahlung in the absence of the external field, is evaluated numerically in order to illustrate this resonance-replication effect.

PACS number(s): 34.80.Qb, 03.65.Nk

I. INTRODUCTION

The development of computational techniques appropriate for the treatment of scattering that takes place in a radiation field, either external or spontaneously produced, has long been a subject of interest. The analysis of spontaneous bremsstrahlung radiation provides a useful tool in probing the dynamics of the scattering process, while the nonlinear multiphoton effects encountered in the study of scattering in the presence of an intense external field, generated by the high-powered lasers now available, present both opportunities and challenges to the theorist. Here we shall be concerned with a scattering event accompanied by both spontaneous and stimulated bremsstrahlung. A scattered electron emits photons spontaneously in the absence of the laser field. With a laser field present, the radiation energy is redistributed through *stimulated* photon exchanges with the field. This in turn alters the *spontaneous* radiation spectrum dramatically. Of particular interest to us will be the situation where the scattering supports a relatively long-lived resonant state. In this case a series of peaks is expected in the spectrum of the spontaneous radiation under the combined influence of both the resonance and the laser field. We shall develop approximations appropriate to single-photon spontaneous radiation by a charged particle undergoing resonant scattering in the presence of a low-frequency laser field.

There is a physical similarity between the process under consideration and that of bound-state harmonic generation, which has been studied fairly extensively in recent years [1]. A resonant state with a lifetime much greater than the period of the laser field may be considered as a quasi-bound state. Thus the discrete, replicated, nature of the bremsstrahlung spectrum for laser-assisted resonant scattering may reasonably be thought of as "quasiharmonic" radiation. In spite of the analogy between these two processes significant differences in their spectra are expected, and this will be discussed below.

Our objective is to develop an approximation for the radiation spectrum that can be expressed in a fairly simple analytic form. We therefore confine our attention to external fields of low frequency since accurate trial functions, providing a nonperturbative representation of the laser-assisted scattering, are then available [2]. It is a remarkable property of these approximate wave functions that they lead to expressions for the radiation cross section that require for their evaluation only a knowledge of the spontaneous radiation rate in the *absence* of the low-frequency laser field. This will be recognized as a characteristic feature of a wide class of low-frequency approximations, one of the earliest being Low's result [3] for spontaneous bremsstrahlung. More recent examples are contained in the sum rules derived for the total cross sections for laser-assisted scattering [4] and "two-color" ionization [5].

The low-frequency approximation is based on the assumption that the external field is slowly varying—for the case of a monochromatic field considered here the frequency is required to be small compared with any of the other characteristic energies in the problem. In the absence of scattering resonances the wave function introduced by Kroll and Watson [2] provides an adequate approximation for large enough scattering energies (the relative error being of first order in laser frequency, as discussed further below). In the resonant case, the Kroll-Watson (KW) function is not suitable and a modification, of the form introduced earlier [5] in connection with a different problem, is adopted here.

In Sec. II the approximate wave functions to be used in the calculation are introduced and a measure of their accuracy is provided. The use of these wave functions to derive a low-frequency approximation for the spontaneous bremsstrahlung amplitude in laser-assisted scattering

1050-2947/93/48(1)/505(11)/$06.00

is described in Sec. III, with separate treatments given to the nonresonant and resonant cases. A simple model is constructed and evaluated numerically to illustrate the main features of the spectrum in the resonant case. The model makes use of a low-frequency approximation for single-photon bremsstrahlung in the absence of the field, of the form derived some time ago by Feshbach and Yennie [6]. An alterative derivation of the Feshbach-Yennie approximation is given in the Appendix. We conclude in Sec. IV with a discussion and summary of results.

II. APPROXIMATE WAVE FUNCTIONS

As preparation for the derivation given below we shall introduce approximate wave functions describing a process in which an electron of mass μ scatters from a center of force, represented by a short-range potential $V(\mathbf{r})$. The scattering takes place in the presence of an intense laser field with vector potential, in the dipole approximation, given by $\mathbf{A}(t)=\mathbf{a}\cos\omega t$. (Effects of target structure can be accounted for within the low-frequency approximation, and a more general representation of the field can be adopted [4]; for simplicity these possible generalizations are not considered here.) The Schrödinger equation for the system (in units with $\hbar=1$) is

$$\left[\frac{1}{2\mu}\left[-i\nabla-\frac{e}{c}\mathbf{A}(t)\right]^2+V(\mathbf{r})-i\frac{\partial}{\partial t}\right]\Psi_{\mathbf{p}}^{(\pm)}(\mathbf{r},t)=0\ , \tag{2.1}$$

where $\Psi_{\mathbf{p}}^{(+)}$ is the solution that develops from a plane wave of momentum $\mathbf{p}$ in the remote past and $\Psi_{\mathbf{p}}^{(-)}$ is the solution corresponding a plane wave in the distant future. The wave equation cannot in general be solved analytically. However, if the frequency of the field is sufficiently low, the wave functions can be approximated fairly accurately. The low-frequency condition requires that the energy of the laser photon be much lower than any characteristic atomic energy—in the present problem it is the scattering energy. (Here we assume, temporarily, that there are no scattering resonances.) Some time ago Kroll and Watson [2] introduced an approximate solution of Eq. (2.1) of the form

$$\tilde{\Psi}_{\mathbf{p}}^{(\pm)}(\mathbf{r},t)=\exp\left[-i\int^t dt'\frac{p(\omega t')^2}{2\mu}+i\frac{e}{c}\mathbf{A}(t)\cdot\mathbf{r}\right]u_{\mathbf{p}(\omega t)}^{(\pm)}(\mathbf{r})\ , \tag{2.2}$$

where the field-modified momentum is

$$\mathbf{p}(\omega t)=\mathbf{p}-\frac{e}{c}\mathbf{a}\cos\omega t\ , \tag{2.3}$$

and the field-free wave functions are defined as

$$|u_{\mathbf{p}}^{(\pm)}\rangle=|\mathbf{p}\rangle+G_0^{(\pm)}(E_{\mathbf{p}})V|\mathbf{p}\rangle\ , \tag{2.4}$$

with $E_{\mathbf{p}}=p^2/2\mu$. The time-independent Green's function is represented formally as $G_0^{(\pm)}(E)=(E\pm i\eta+\nabla^2/2\mu-V)^{-1}$, with η a positive infinitesimal.

The accuracy of the KW function just defined may be tested by inserting it into Eq. (2.1); one finds that

$$\left[H-i\frac{\partial}{\partial t}\right]\tilde{\Psi}_{\mathbf{p}}^{(\pm)}(\mathbf{r},t)=\exp\left[-i\int^t dt'\frac{p(\omega t')^2}{2\mu}+i\frac{e}{c}\mathbf{A}(t)\cdot\mathbf{r}\right]\times e\mathbf{E}(t)\cdot(-i\nabla_{\mathbf{p}(\omega t)}-\mathbf{r})u_{\mathbf{p}(\omega t);\mathrm{sc}}^{(\pm)}(\mathbf{r})\ , \tag{2.5}$$

where $u_{\mathbf{p};\mathrm{sc}}^{(\pm)}$ is the scattered part of the full wave [the second term on the right-hand side in Eq. (2.4)] and the electric field is given by

$$\mathbf{E}(t)=\frac{\omega\mathbf{a}}{c}\sin\omega t\ . \tag{2.6}$$

The significance of the phase term $(e/c)\mathbf{A}\cdot\mathbf{r}$ in the approximation (2.2) is clear; it serves as a gauge-transformation factor, changing the laser-atom interaction from the $\mathbf{p}\cdot\mathbf{A}$ form (velocity gauge) to the $\mathbf{E}\cdot\mathbf{r}$ form (length gauge). As a result the error in the wave function, as measured by the remainder term on the right-hand side in Eq. (2.5), is proportional to the electric-field strength, and hence the frequency. This is the basis for the expectation that the KW approximation is accurate at low frequencies. (A limitation on the strength of the field is implicit here. The analysis of Ref. [4] leads to the requirement that the parameter ea/pc be of order unity or smaller.) The argument for the validity of the KW approximation breaks down in the case of resonant scattering since the wave function will be rapidly varying and the momentum derivative can make the magnitude of the remainder exceptionally large. A generalization of the KW function, one that retains its validity even for wave functions that vary rapidly in the neighborhood of a resonance, was introduced in a recent treatment of multiphoton ionization [5]. This function, when expanded in a Fourier series, takes on the form

$$\tilde{\Psi}_{\mathbf{p}}^{(\pm)}(\mathbf{r},t)=\sum_n\exp[-iE_{\mathbf{p}n}t+i\frac{e}{c}\mathbf{A}(t)\cdot\mathbf{r}]f_n^{(\pm)}(\mathbf{r})\ . \tag{2.7}$$

The energy $E_{\mathbf{p}n}$ is defined as

$$E_{\mathbf{p}n}=E_{\mathbf{p}}+\Delta+n\omega\ , \tag{2.8}$$

with $\Delta=e^2a^2/4\mu c^2$ recognized as the pondermotive energy of the field. The Fourier coefficients in Eq. (2.7) are the functions

$$f_n^{(\pm)}(\mathbf{r})=\int_0^{2\pi}\frac{d\theta}{2\pi}\exp[i\Theta_{\mathbf{p}n}(\theta)]u_{\mathbf{p}(\theta)}^{(\pm)}(E_{\mathbf{p}n};\mathbf{r})\ , \tag{2.9}$$

where the phase is defined by

$$\Theta_{\mathbf{p}n}(\theta)=n\theta+\rho_{\mathbf{p}}\sin\theta-\gamma\sin2\theta\ , \tag{2.10}$$

with

$$\rho_{\mathbf{p}}=\frac{e\mathbf{p}\cdot\mathbf{a}}{\mu c\omega},\quad \gamma=\frac{e^2a^2}{8\mu c\omega}\ . \tag{2.11}$$

The off-shell field-free wave functions in Eq. (2.9) are represented formally as

$$|u_{\mathbf{p}}^{(\pm)}(E)\rangle=|\mathbf{p}\rangle+G_0^{(\pm)}(E)V|\mathbf{p}\rangle \ . \tag{2.12}$$

This reduces to the on-shell version, Eq. (2.4), for $E=p^2/2m$. A measure of the error in the approximation (2.7) is provided by the relation

$$\left[H-i\frac{\partial}{\partial t}\right]\tilde{\Psi}_{\mathbf{p}}^{(\pm)}(\mathbf{r},t)=-e\mathbf{E}(t)\cdot\mathbf{r}\tilde{\Psi}_{\mathbf{p};\mathrm{sc}}^{(\pm)}(\mathbf{r},t) \ , \tag{2.13}$$

where $\tilde{\Psi}_{\mathbf{p};\mathrm{sc}}^{(\pm)}$ is the scattered-wave component of the full wave function. It is defined as in Eq. (2.7), but with the replacement of the field-free solution $u_{\mathbf{p}(\theta)}^{(\pm)}(E_{\mathbf{p}n})$ in Eq. (2.9) by the corresponding scattered wave $u_{\mathbf{p}(\theta);\mathrm{sc}}^{(\pm)}(E_{\mathbf{p}n})$ given by the second term in Eq. (2.12).

The accuracy of the function (2.7) is indicated by the magnitude of the right-hand side of Eq. (2.13). This term is proportional to the electric-field strength [owing to the retention of the gauge-transforming phase factor in (2.7)] and hence to the frequency ω. The important feature of Eq. (2.13) is that the remainder does not contain the momentum derivative present in the corresponding equation (2.5) for the KW function. Thus a rapid variation of the wave function in the domain of a resonance will not introduce an unusually large error. This improvement is achieved, evidently, through the introduction of the Fourier expansion leading to a sampling of the wave function over a range of energies, as indicated in Eqs. (2.8) and (2.9).

In the following section these wave functions will be used as approximations to the exact initial- and final-state distorted waves, describing the combined interaction of the electron with the target and the laser field, in the matrix element for single-photon bremsstrahlung.

III. BREMSSTRAHLUNG RADIATION

A. Formulation

We consider the process in which an electron with initial momentum $\mathbf{p}$ undergoes laser-assisted scattering into a final state with momentum $\mathbf{p}'$, accompanied by the spontaneous emission of a photon of momentum $\mathbf{\Omega}$ and polarization $\boldsymbol{\epsilon}$, with $\mathbf{\Omega}\cdot\boldsymbol{\epsilon}=0$. The S matrix element is $S=-iM$, where, in the dipole approximation,

$$M(\mathbf{p}',\mathbf{p};\mathbf{\Omega},\boldsymbol{\lambda})=\int_{-\infty}^{\infty}dt\int d\mathbf{r}[\Psi_{\mathbf{p}'}^{(-)}(\mathbf{r},t)]^* \times(\boldsymbol{\lambda}\cdot\mathbf{r}e^{i\Omega t})\Psi_{\mathbf{p}}^{(+)}(\mathbf{r},t) \ , \tag{3.1}$$

and $\boldsymbol{\lambda}=i(e/c)\Omega\,\mathbf{A}_{\mathrm{sp}}$. We have introduced the amplitude $\mathbf{A}_{\mathrm{sp}}=(2\pi c^2/\Omega L^3)^{1/2}\boldsymbol{\epsilon}$ of the component of the vector potential of the spontaneous radiation field corresponding to photon emission, with L^3 representing the quantization volume. It will be convenient to begin by examining the amplitude for bremsstrahlung in the absence of the external field. In that case we may write

$$\Psi_{\mathbf{p}}^{(\pm)}(\mathbf{r},t)=e^{-iE_{\mathbf{p}}t}u_{\mathbf{p}}^{(\pm)}(\mathbf{r}) \ , \tag{3.2}$$

where $E_{\mathbf{p}}=p^2/2\mu$ is the scattering energy and the $u_{\mathbf{p}}^{(\pm)}(\mathbf{r})$ are the spatial wave functions defined earlier in Eq. (2.4). The transition matrix then takes the form

$$M(\mathbf{p}',\mathbf{p};\mathbf{\Omega},\boldsymbol{\lambda})=2\pi\delta(E_{\mathbf{p}'}-E_{\mathbf{p}}+\Omega)\boldsymbol{\lambda}\cdot\mathbf{m}(\mathbf{p}',\mathbf{p}) \ , \tag{3.3}$$

where

$$\mathbf{m}(\mathbf{p}',\mathbf{p})=\langle u_{\mathbf{p}'}^{(-)}|\mathbf{r}|u_{\mathbf{p}}^{(+)}\rangle \tag{3.4}$$

is the matrix element related to the single-photon bremsstrahlung. We observe that if there is a narrow scattering resonance at energy E_r, then the magnitude of the final-state wave function in Eq. (3.4) will rise rapidly as $E_{\mathbf{p}'}$ passes through E_r and the amplitude $\mathbf{m}(\mathbf{p}',\mathbf{p})$ will show a peak at this energy. That is, for a fixed value of $E_{\mathbf{p}}$ lying above E_r, the spontaneous radiation probability is enhanced for Ω close to the value $E_{\mathbf{p}}-E_r$, owing to the existence of the scattering resonance.

The peaking effect may be seen more explicitly by examining the form of the low-frequency approximation for spontaneous bremsstrahlung obtained by Feshbach and Yennie [6], valid for resonant scattering. An alternative (and simplified) derivation of the Feshbach-Yennie formula is presented in the Appendix. The expression contained in Eqs. (A6) and (A11) is convenient for our present purpose; it reads

$$\begin{aligned}\mathbf{m}(\mathbf{p}',\mathbf{p})\cong{}&\frac{i}{\Omega}\left[\frac{\mathbf{p}'}{\mu\Omega}+2(\mathbf{p}'-\mathbf{p})\frac{\partial}{\partial\tau}\right]t(E_{\mathbf{p}},\tau)|_{\tau=(\mathbf{p}'-\mathbf{p})^2}\\&-\frac{i}{\Omega}\left[\frac{\mathbf{p}}{\mu\Omega}+2(\mathbf{p}'-\mathbf{p})\frac{\partial}{\partial\tau}\right]\\&\times t(E_{\mathbf{p}}-\Omega,\tau)|_{\tau=(\mathbf{p}'-\mathbf{p})^2} \ ,\end{aligned} \tag{3.5}$$

where the t's are on-shell scattering matrices as functions of scalar variables. For resonant scattering the second term in Eq. (3.5) peaks at the radiation frequency $\Omega_{\mathrm{peak}}=E_{\mathbf{p}}-E_r$. The representation (3.5) will be adapted later in Sec. III D to illustrate the calculational procedure to be used when low-frequency approximations are appropriate for the description of both stimulated *and* spontaneous bremsstrahlung.

We now return to the general case of arbitrary Ω. According to Eq. (3.3) and the relation between the vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\epsilon}$, the cross section for laser-assisted scattering accompanied by the spontaneous emission of a photon into any one of the discrete states in a range $\Delta\mathbf{\Omega}$ may be written as

$$\Delta\sigma(\mathbf{\Omega})=\frac{(2\pi)^4\mu}{p}\sum_{\Delta\Omega}\int d\mathbf{p}'\delta(E_{\mathbf{p}'}-E_{\mathbf{p}}+\Omega) \times\sum_{i=1}^{2}|\boldsymbol{\epsilon}_i\cdot\mathbf{m}(\mathbf{p}',\mathbf{p})|^2\frac{2\pi e^2\Omega}{L^3} \ . \tag{3.6}$$

The sum over polarization states for a given propagation direction $\hat{\mathbf{\Omega}}$ can be expressed as

$$\sum_{i=1}^{2}|\boldsymbol{\epsilon}_i\cdot\mathbf{m}|^2=|\mathbf{m}|^2-|\hat{\mathbf{\Omega}}\cdot\mathbf{m}|^2 \ . \tag{3.7}$$

Following standard procedures we may now pass to the limit of infinite quantization volume, sum over polarization states, and integrate over all the directions of the propagation vector $\mathbf{\Omega}$. In this way we arrive at an expression for the cross section, differential in the photon frequency Ω, of the form

$$\frac{d\sigma(\Omega)}{d\Omega}=(2\pi)^3\frac{4\mu e^2\Omega^3}{3pc^3}\int d\mathbf{p}'\delta(E_{\mathbf{p}'}-E_{\mathbf{p}}+\Omega)|\mathbf{m}(\mathbf{p}',\mathbf{p})|^2 . \tag{3.8}$$

We are now in position to examine the spectrum of single-photon spontaneous radiation when the system considered above is exposed to an intense low-frequency laser field. For this purpose we adopt the generalized approximate wave functions introduced in Eq. (2.7). With these functions replacing the exact solutions in the transition matrix (3.1) we obtain the approximation

$$M(\mathbf{p}',\mathbf{p};\Omega,\boldsymbol{\lambda})\cong\int_{-\infty}^{\infty}dt\int d\mathbf{r}[\tilde{\Psi}_{\mathbf{p}'}^{(-)}(\mathbf{r},t)]^* \times(\boldsymbol{\lambda}\cdot\mathbf{r}e^{i\Omega t})\tilde{\Psi}_{\mathbf{p}}^{(+)}(\mathbf{r},t) . \tag{3.9}$$

The error in this approximation is of order (ωa), that is, of first order in the frequency ω, and in the amplitude a of the vector potential. This property is indicated by Eq. (2.13) (and can be put on a firmer basis by examining an explicit formal expression for the error in the transition-matrix element). After integrating over the time variable in Eq. (3.9) we obtain

$$M(\mathbf{p}',\mathbf{p};\Omega,\boldsymbol{\lambda})\cong 2\pi\sum_l \delta(E_{\mathbf{p}'}-E_{\mathbf{p}}+\Omega+l\omega)\boldsymbol{\lambda}\cdot\mathbf{M}_l(\mathbf{p}',\mathbf{p}) . \tag{3.10}$$

$\boldsymbol{\lambda}\cdot\mathbf{M}_l$ represents the amplitude for spontaneous emission of a photon $(\boldsymbol{\Omega},\boldsymbol{\epsilon})$ from the scattering channel in which l low-frequency photons are exchanged with the external field. (The frequency Ω is not assumed to be small at this stage.) We have

$$\mathbf{M}_l(\mathbf{p}',\mathbf{p}) =\sum_n\int_0^{2\pi}\frac{d\theta'}{2\pi}\int_0^{2\pi}\frac{d\theta}{2\pi}\exp[-i\Theta_{\mathbf{p}'l+n}(\theta')+i\Theta_{\mathbf{p}n}(\theta)] \times\bar{\mathbf{m}}(\mathbf{p}'(\theta'),E_{\mathbf{p}'l+n};\mathbf{p}(\theta),E_{\mathbf{p}n}) , \tag{3.11}$$

where $\bar{\mathbf{m}}$ is defined as

$$\bar{\mathbf{m}}(\mathbf{p}',E';\mathbf{p},E)=\langle u_{\mathbf{p}'}^{(-)}(E')|\mathbf{r}|u_{\mathbf{p}}^{(+)}(E)\rangle . \tag{3.12}$$

The bar over $\mathbf{m}$ is inserted to distinguish it from the on-shell amplitude $\mathbf{m}$ defined earlier in Eq. (3.4). For $E'=p'^2/2\mu$ and $E=p^2/2\mu$, $\bar{\mathbf{m}}$ reduces to $\mathbf{m}$. The probability of spontaneous radiation can be derived from the expression in Eq. (3.10). In the following we first examine the nonresonant case and then generalize the discussion to allow for resonances.

B. Nonresonant scattering

The expression for the transition amplitude $\mathbf{M}_l$ can be simplified considerably when the scattering is nonresonant. In that case the amplitude $\bar{\mathbf{m}}$ in Eq. (3.11) may be assumed to be a smooth function of its energy arguments. We begin by replacing $\bar{\mathbf{m}}$ in Eq. (3.11) by its Taylor series expansion

$$\bar{\mathbf{m}}(\mathbf{p}(\theta),E_{\mathbf{p}n})=\sum_k\frac{1}{k!}\frac{\partial^k\bar{\mathbf{m}}}{\partial E_{\mathbf{p}n}^k}\bigg|_{E_{\mathbf{p}n}=E_{\mathbf{p}0}}(n\omega)^k , \tag{3.13}$$

with respect to the energy $E_{\mathbf{p}n}$ about the value $E_{\mathbf{p}0}$. (Final-state and photon variables are held fixed and the dependence of $\bar{\mathbf{m}}$ on these variables is temporarily suppressed.) An n-fold integration-by-parts procedure—developed and described in Ref. [5]—establishes the relation

$$\int_0^{2\pi}\frac{d\theta}{2\pi}\exp[i\Theta_{\mathbf{p}n}(\theta)]F(\mathbf{p}(\theta))(n\omega)^k =\int_0^{2\pi}\frac{d\theta}{2\pi}\exp[i\Theta_{\mathbf{p}n}(\theta)]F(\mathbf{p}(\theta)) \times(-\omega\rho_{\mathbf{p}}\cos\theta+2\omega\gamma\cos 2\theta)^k+O(\omega a) , \tag{3.14}$$

where $F(\mathbf{p})$ is an arbitrary smooth function of momentum $\mathbf{p}$. The error incurred in this procedure is of the same order (ωa) as that introduced by the underlying approximation for the wave functions. The above relation is now used to replace $(n\omega)^k$ by $(-\omega\rho_{\mathbf{p}}\cos\theta+2\omega\gamma\cos 2\theta)^k$ in Eq. (3.13). After resumming the expansion and recognizing that

$$E_{\mathbf{p}(\theta)}=E_{\mathbf{p}0}-\omega\rho_{\mathbf{p}}\cos\theta+2\omega\gamma\cos 2\theta , \tag{3.15}$$

we see that the initial-state momentum has been placed on the energy shell. A similar procedure applied to the arguments $\mathbf{p}'(\theta')$ and $E_{\mathbf{p}'l+n}$ of $\bar{\mathbf{m}}$ puts the final-state momentum on shell as well. The result is summarized by the rule

$$\bar{\mathbf{m}}(\mathbf{p}'(\theta'),E_{\mathbf{p}'l+n};\mathbf{p}(\theta),E_{\mathbf{p}n})\to\mathbf{m}(\mathbf{p}'(\theta'),\mathbf{p}(\theta)) . \tag{3.16}$$

$\mathbf{m}$ is the on-shell amplitude defined earlier in Eq. (3.4) and is independent of the index n. The sum over n in Eq. (3.11) may then be performed with the aid of the identity

$$\sum_{n=-\infty}^{\infty}e^{in(\theta'-\theta)}=2\pi\delta(\theta'-\theta) . \tag{3.17}$$

The integration over the phase variable θ' is then trivial and we find that the approximation (3.11) has been reduced to

$$\mathbf{M}_l(\mathbf{p}',\mathbf{p})=\int_0^{2\pi}\frac{d\theta}{2\pi}\exp[-il\theta-i\rho\sin\theta]\mathbf{m}(\mathbf{p}'(\theta),\mathbf{p}(\theta)) , \tag{3.18}$$

with

$$\rho=\frac{e(\mathbf{p}'-\mathbf{p})\cdot\mathbf{a}}{\mu\omega c} . \tag{3.19}$$

We have just shown that the low-frequency approximation for the bremsstrahlung amplitude in the presence of the laser field takes on a very simple form in the absence of a scattering resonance. The result, in the form (3.18), could have been derived directly using the KW approximate wave function (2.2) valid for nonresonant scattering. The derivation described here was based on the generalized wave function (2.7), allowing for resonances, in order to prepare the way for the discussion in Sec. III C and to have available a uniform procedure that is more generally applicable.

An expression for the differential cross section for the spontaneous radiation of a photon can be obtained from the representation Eq. (3.10) for the transition amplitude. We find that

$$\frac{d\sigma(\Omega)}{d\Omega}=(2\pi)^3\frac{4\mu e^2\Omega^3}{3pc^3}\sum_l\int d\mathbf{p}'\delta(E_{\mathbf{p}'}-E_{\mathbf{p}}+\Omega+l\omega)\times|\mathbf{M}_l(\mathbf{p}',\mathbf{p})|^2\ . \tag{3.20}$$

A sum over polarization states and an integration over propagation directions of the spontaneously radiated photon has been performed following the procedure leading to Eq. (3.8). To perform the sum over the index l (in the nonresonant case) we first expand the δ function in its Taylor series

$$\delta(E_{\mathbf{p}'}-E_{\mathbf{p}}+\Omega+l\omega)=\sum_k\frac{1}{k!}\delta^{(k)}(E_{\mathbf{p}'}-E_{\mathbf{p}}+\Omega)(l\omega)^k\ . \tag{3.21}$$

We now employ a procedure of repeated partial integration, of the type referred to earlier. Thus, with $\mathbf{M}_l$ given by Eq. (3.18), we may represent $|\mathbf{M}_l|^2$ as a double integral over phase angles θ and θ'. In the integrand of one of them, say, the one with θ dependence, we may make the replacement $(l\omega)^k\rightarrow(-\omega\rho\cos)^k$ for each term in the sum generated by the expansion (3.21). The sum over l may now be carried out using the identity (3.17). The cross section, differential with respect to the photon frequency, is then obtained in the form

$$\frac{d\sigma(\Omega)}{d\Omega}=(2\pi)^3\frac{4\mu e^2\Omega^3}{3pc^3}\times\int_0^{2\pi}\frac{d\theta}{2\pi}\int d\mathbf{p}'\delta(E_{\mathbf{p}'(\theta)}-E_{\mathbf{p}(\theta)}+\Omega)\times|\mathbf{m}(\mathbf{p}'(\theta),\mathbf{p}(\theta))|^2\ . \tag{3.22}$$

The essential feature of this result is that the cross section for spontaneous emission in the presence of the laser field is represented in terms of the corresponding cross section in the absence of the field. The effect of the low-frequency field is accounted for approximately through an averaging of the initial and final momenta over the variable θ representing the phase of the field. This is a general feature of a class of sum rules associated with the low-frequency approximation [4,5].

The cross section (3.22) can be rearranged in a different form which may be convenient for applications in some cases. Using a relation valid for an arbitrary function $g(\theta)$,

$$\sum_n\int_0^{2\pi}\frac{d\theta'}{2\pi}\int_0^{2\pi}\frac{d\theta}{2\pi}\exp[-i\Theta_{\mathbf{p}n}(\theta')+i\Theta_{\mathbf{p}n}(\theta)]g(\theta)=\int_0^{2\pi}\frac{d\theta}{2\pi}g(\theta)\ , \tag{3.23}$$

we rewrite Eq. (3.22) as

$$\frac{d\sigma(\Omega)}{d\Omega}=(2\pi)^3\frac{4\mu e^2\Omega^3}{3pc^3}\times\sum_n\int_0^{2\pi}\frac{d\theta'}{2\pi}\int_0^{2\pi}\frac{d\theta}{2\pi}\exp[-i\Theta_{\mathbf{p}n}(\theta')+i\Theta_{\mathbf{p}n}(\theta)]\times\int d\mathbf{p}'\delta(E_{\mathbf{p}'}-E_{\mathbf{p}(\theta)}+\Omega)|\mathbf{m}(\mathbf{p}',\mathbf{p}(\theta))|^2\ , \tag{3.24}$$

where we have made the variable transformation $\mathbf{p}'(\theta)\rightarrow\mathbf{p}'$ in the integral over momentum. Adopting the partial integration procedure for the variable θ that was used in the derivation of (3.14), we may replace $\mathbf{p}(\theta)$ in Eq. (3.24) by $\mathbf{p}(\theta_n)$, with the fixed phase determined by the condition

$$\left.\frac{d[\Theta_{\mathbf{p}n}(\theta)]}{d\theta}\right|_{\theta=\theta_n}=0\ . \tag{3.25}$$

With the generalized Bessel function defined as

$$J_{-n}(\rho,\gamma)=\int_0^{2\pi}\frac{d\theta}{2\pi}\exp[i(n\theta+\rho\sin\theta-\gamma\sin2\theta)]\ , \tag{3.26}$$

we express the differential cross section in the form

$$\frac{d\sigma(\Omega)}{d\Omega}=\sum_n J_{-n}^2(\rho_{\mathbf{p}},\gamma)\left[(2\pi)^3\frac{4\mu e^2\Omega^3}{3pc^3}\times\int d\mathbf{p}'\delta(E_{\mathbf{p}'}-E_{\mathbf{p}(\theta_n)}+\Omega)\times|\mathbf{m}(\mathbf{p}',\mathbf{p}(\theta_n))|^2\right]\ . \tag{3.27}$$

This expression will be shown in Sec. III C to have wider applicability; it is valid for resonant scattering as well.

C. Resonant scattering

If there is a resonance in the field-free scattering, whose width is comparable to or smaller than the laser frequency, the introduction of an expansion of the type shown in Eq. (3.13) will no longer be useful owing to the rapid variation of the radiation amplitude $\bar{\mathfrak{m}}(\mathbf{p}',E';\mathbf{p},E)$ when either of the energy variables is close to the resonance energy E_r. Since we are interested in the effect of the resonance on the bremsstrahlung radiation, we shall assume that none of the energies $E_{\mathbf{p}n}$ lies inside the resonance region; this allows us to avoid consideration of resonance effects in the absence of bremsstrahlung. Laser-induced resonances occur for $E_{\mathbf{p}'l+n}$ near E_r. We wish to simplify the expression (3.11) while properly accounting for the resonance. Toward this end we decompose the amplitude $\bar{\mathfrak{m}}$ into two parts,

$$\bar{\mathfrak{m}}=\bar{\mathfrak{m}}^b+\bar{\mathfrak{m}}^r\ , \tag{3.28}$$

allowing us to separate the resonant contribution $\bar{\mathfrak{m}}^r$

from the slowly varying background $\overline{\mathfrak{m}}^{b}$. We take $\overline{\mathfrak{m}}^{r}$ to be negligible outside the resonance region. Turning to the resonant contribution first, we define an amplitude $\mathbf{M}_l^r$ as in Eq. (3.11), but with $\overline{\mathfrak{m}}$ replaced by its resonant component $\overline{\mathfrak{m}}^{r}$. This expression may be simplified, without loss of accuracy, through a multiple integration-by-parts procedure with respect to the variable θ (of the type employed earlier). This leads to the form

$$\mathbf{M}_l^r(\mathbf{p}',\mathbf{p})=\sum_n \int_0^{2\pi}\frac{d\theta'}{2\pi}\exp[-i\Theta_{\mathbf{p}'l+n}(\theta')] \times J_{-n}(\rho_{\mathbf{p}},\gamma) \times \overline{\mathfrak{m}}^{r}(\mathbf{p}'(\theta'),E_{\mathbf{p}'l+n};\mathbf{p}(\theta_n),E_{\mathbf{p}n}) , \tag{3.29}$$

where the generalized Bessel function J_{-n} and fixed phase θ_n are defined in Eqs. (3.26) and (3.25), respectively. This result may be used to evaluate the resonant contribution, $d\sigma^r(\Omega)/d\Omega$, to the cross section for emission of a photon into the frequency range $(\Omega,\Omega+d\Omega)$. It takes the form shown in Eq. (3.20), with the amplitude $\mathbf{M}_l$ replaced by its resonant part $\mathbf{M}_l^r$. The energy conservation condition $E_{\mathbf{p}'}=E_{\mathbf{p}}-\Omega-l\omega$ fixes the energy variable $E_{\mathbf{p}'l+n}$ appearing in the function $\overline{\mathfrak{m}}^{r}$ as

$$E_{\mathbf{p}'l+n}=E_{\mathbf{p}}-\Omega+n\omega+\Delta=E_{\mathbf{p}n}-\Omega . \tag{3.30}$$

The spectrum of spontaneous radiation will then have peaks near those photon frequencies Ω determined by the condition $E_r=E_{\mathbf{p}n}-\Omega$ since only here does the amplitude $\overline{\mathfrak{m}}^{r}$ contribute appreciably to $d\sigma^r/d\Omega$. Under the assumption that the width of the resonance is small compared to the laser photon energy, contributions to the cross section corresponding to different values of the index n do not overlap. Thus, with terms in Eq. (3.29) associated with different n values treated separately (and with the polarization sum and angular integration carried out), we rewrite the expression for the differential cross section $d\sigma^r(\Omega)/d\Omega$ as the incoherent sum

$$\frac{d\sigma^r(\Omega)}{d\Omega}=\sum_n J_{-n}^2(\rho_{\mathbf{p}},\gamma)\frac{d\sigma_n^r(\Omega)}{d\Omega} , \tag{3.31}$$

where, with the aid of Eq. (3.30), we have

$$\frac{d\sigma_n^r(\Omega)}{d\Omega}=(2\pi)^3\frac{4\mu e^2\Omega^3}{3pc^3}\int d\mathbf{p}'\sum_l \delta(E_{\mathbf{p}'}-E_{\mathbf{p}}+\Omega+l\omega) \times\left|\int_0^{2\pi}\frac{d\theta'}{2\pi}\exp[-i\Theta_{\mathbf{p}'l+n}(\theta')]\overline{\mathfrak{m}}^{r}(\mathbf{p}'(\theta'),E_{\mathbf{p}n}-\Omega;\mathbf{p}(\theta_n),E_{\mathbf{p}n})\right|^2 . \tag{3.32}$$

We now expand the energy-conservation δ function in powers of $(l\omega)$. As in the procedure leading to Eq. (3.22), we are then able to perform the sum over photon-number index l and, with the aid of relations (2.8) and (3.15), to express the cross section as

$$\frac{d\sigma_n^r(\Omega)}{d\Omega}=(2\pi)^3\frac{4\mu e^2\Omega^3}{3pc^3}\int_0^{2\pi}\frac{d\theta'}{2\pi}\int d\mathbf{p}'\delta(E_{\mathbf{p}'(\theta')}-E_{\mathbf{p}n}+\Omega)|\overline{\mathfrak{m}}^{r}(\mathbf{p}'(\theta'),E_{\mathbf{p}n}-\Omega;\mathbf{p}(\theta_n),E_{\mathbf{p}n})|^2 . \tag{3.33}$$

From energy conservation, implied by the δ function appearing in the above expression, we have $E_{\mathbf{p}'(\theta')}=E_{\mathbf{p}n}-\Omega$. Furthermore, the condition given in Eq. (3.25) is readily shown to lead to the relation $E_{\mathbf{p}(\theta_n)}=E_{\mathbf{p}n}$. Therefore $\overline{\mathfrak{m}}^{r}$ is reduced to its on-shell version $\mathbf{m}^r$. As a result, we have

$$\frac{d\sigma_n^r(\Omega)}{d\Omega}=(2\pi)^3\frac{4\mu e^2\Omega^3}{3pc^3}\int d\mathbf{p}'\delta(E_{\mathbf{p}'}-E_{\mathbf{p}(\theta_n)}+\Omega) \times|\mathbf{m}^r(\mathbf{p}',\mathbf{p}(\theta_n))|^2 . \tag{3.34}$$

Introduction of the variable transformation $\mathbf{p}'(\theta')\to\mathbf{p}'$ has enabled us to remove the average over the phase θ'.

We now outline the generalization of the above procedure that takes into account the interference between the smooth background and the resonance. Written in a form similar to the expression (3.20), this contribution to the cross section is

$$\frac{d\sigma^i(\Omega)}{d\Omega}=(2\pi)^3\frac{4\mu e^2\Omega^3}{3pc^3}\int d\mathbf{p}'\sum_l \delta(E_{\mathbf{p}'}-E_{\mathbf{p}}+\Omega+l\omega) \times 2\,\mathrm{Re}[(\mathbf{M}_l^b)^*\cdot\mathbf{M}_l^r] \tag{3.35}$$

where, by referring to Eq. (3.18) valid for nonresonant scattering, we see that

$$\mathbf{M}_l^b(\mathbf{p}',\mathbf{p})=\int_0^{2\pi}\frac{d\theta}{2\pi}\exp[-il\theta-i\rho\sin\theta]\mathbf{m}^b(\mathbf{p}'(\theta),\mathbf{p}(\theta)) , \tag{3.36}$$

and $\mathbf{M}_l^r$ is obtained from Eq. (3.29). In an analysis similar to that which led to Eq. (3.34) we find (with details of the derivation omitted) that

$$\frac{d\sigma^i(\Omega)}{d\Omega}=\sum_n J_{-n}^2(\rho_{\mathbf{p}},\gamma)\frac{d\sigma_n^i(\Omega)}{d\Omega} , \tag{3.37}$$

where

$$\frac{d\sigma_n^i(\Omega)}{d\Omega}=(2\pi)^3\frac{4\mu e^2\Omega^3}{3pc^3} \times\int d\mathbf{p}'\delta(E_{\mathbf{p}'}-E_{\mathbf{p}(\theta_n)}+\Omega) \times 2\,\mathrm{Re}\{[\mathbf{m}^b(\mathbf{p}',\mathbf{p}(\theta_n))]^*\cdot\mathbf{m}^r(\mathbf{p}',\mathbf{p}(\theta_n))\} . \tag{3.38}$$

For the contribution to the cross section from the background amplitude $\mathbf{M}^b$, we may use the nonresonant result (3.27) derived in Sec. III B. Explicitly, we write it as

$$\frac{d\sigma^b(\Omega)}{d\Omega}=\sum_n J^2_{-n}(\rho_{\mathbf{p}},\gamma)\frac{d\sigma^b_n(\Omega)}{d\Omega}\ , \tag{3.39}$$

where

$$\frac{d\sigma^b_n(\Omega)}{d\Omega}=(2\pi)^3\frac{4\mu e^2\Omega^3}{3pc^3}\int d\mathbf{p}'\delta(E_{\mathbf{p}'}-E_{\mathbf{p}(\theta_n)}+\Omega)\times|\mathbf{m}^b(\mathbf{p}',\mathbf{p}(\theta_n))|^2\ . \tag{3.40}$$

Combining the contributions (3.34), (3.38), and (3.40), and recognizing that

$$|\mathbf{m}|^2=|\mathbf{m}^r|^2+2\,\mathrm{Re}[(\mathbf{m}^b)^*\cdot\mathbf{m}^r]+|\mathbf{m}^b|^2\ ,$$

we are able to express the total cross section in the form

$$\frac{d\sigma(\Omega)}{d\Omega}=\sum_n J^2_{-n}(\rho_{\mathbf{p}},\gamma)\frac{d\sigma_n(\Omega)}{d\Omega}\ , \tag{3.41}$$

where

$$\frac{d\sigma_n(\Omega)}{d\Omega}=(2\pi)^3\frac{4\mu e^2\Omega^3}{3pc^3}\int d\mathbf{p}'\delta(E_{\mathbf{p}'}-E_{\mathbf{p}(\theta_n)}+\Omega)\times|\mathbf{m}(\mathbf{p}',\mathbf{p}(\theta_n))|^2\ . \tag{3.42}$$

This last expression, according to Eq. (3.8), is just the singly differential cross section for *spontaneous* emission of a photon of frequency Ω accompanying field-free scattering with initial scattering energy $E_{\mathbf{p}(\theta_n)}$. [The formal correspondence between this result and that derived previously in Eq. (3.27) should be noted.] The cross section for scattering in the presence of the laser field, given by Eq. (3.41), takes the form of a sum of terms each weighted by the square of a generalized Bessel function. For a given term corresponding to the index n, the frequency for which there is a peak in the spontaneous radiation spectrum is given by

$$\Omega_{\mathrm{peak}}=E_{\mathbf{p}n}-E_r=E_{\mathbf{p}}+\Delta+n\omega-E_r\ ; \tag{3.43}$$

(we have made use of the relation $E_{\mathbf{p}(\theta_n)}=E_{\mathbf{p}n}$ established earlier). We see that these peak positions are not simply multiples of the laser frequency but rather are displaced by a variable amount that depends on the incident energy and (by virtue of the appearance of the ponderomotive shift) the laser intensity. Note that the resonance leads to an enhancement of the radiation probability even when the incident energy $E_{\mathbf{p}}$ lies below the resonance energy since the electron can absorb enough energy from the field to bring it into resonance. This is in distinction to the situation encountered earlier in the study of nonresonant scattering where the field plays a rather minor role, as seen from the sum rule (3.22).

D. An application

As an aid in visualizing the effect of a resonance on laser-assisted bremsstrahlung we shall adopt a very simple model of the scattering system and evaluate the cross section (3.41). The frequency Ω of the spontaneous radiation will be restricted to a range for which the low-frequency approximation for the amplitude $\mathbf{m}$ in Eq. (3.42) is valid. Then $\mathbf{m}$ may be represented in terms of the physical (on-shell) scattering amplitude, as shown in Eq. (3.5), but with $\mathbf{p}$ replaced by $\mathbf{p}(\theta_n)$ and $E_{\mathbf{p}}$ by $E_{\mathbf{p}n}=E_{\mathbf{p}}+\Delta+n\omega$.

We assume that the scattering is dominated by a resonance, of width Γ at energy E_r, in the lth partial wave. The partial-wave scattering amplitude is then represented as the sum of a smooth background, taken simply to be a constant C^l, and a resonant term of the Breit-Wigner form; we have

$$t^l(E_{\mathbf{p}},\tau)=-(2\pi)^2\frac{\mu}{p}\left[C^l+(2l+1)\frac{\Gamma/2}{E_r-E_{\mathbf{p}}-i\Gamma/2}\right]\times P_l(x)\ , \tag{3.44}$$

where P_l is the lth order Legendre polynomial and $x=1-\tau/4\mu E_{\mathbf{p}}$. To focus on the essential features of this example we assume that the resonance is in the s wave and we ignore contributions from other partial waves. (These restrictions can be lifted.) We then find that

$$\begin{aligned}|\mathbf{m}(\mathbf{p}',\mathbf{p}(\theta_n))|^2&=\frac{1}{\mu^2\Omega^4}\{|p't^0(E_{\mathbf{p}n})|^2+|p(\theta_n)t^0(E_{\mathbf{p}n}-\Omega)|^2\\&\quad-2\cos\xi\,\mathrm{Re}[p'p(\theta_n)t^0(E_{\mathbf{p}n})^*\times t^0(E_{\mathbf{p}n}-\Omega)]\}\ ,\end{aligned} \tag{3.45}$$

where the angle-independent s-wave amplitude t^0 is now expressed as a function of a single energy variable and where ξ is the angle formed between vectors $\mathbf{p}'$ and $\mathbf{p}(\theta_n)$. After carrying our the integral over final-state momenta in Eq. (3.42) we obtain the result

$$\begin{aligned}\frac{d\sigma_n(\Omega)}{d\Omega}&=(2\pi)^4\frac{16\mu e^2}{3c^3\Omega}\left[\frac{E_{\mathbf{p}n}-\Omega}{E_{\mathbf{p}}}\right]^{1/2}\times[(E_{\mathbf{p}n}-\Omega)|t^0(E_{\mathbf{p}n})|^2\\&\quad+E_{\mathbf{p}n}|t^0(E_{\mathbf{p}n}-\Omega)|^2]\ .\end{aligned} \tag{3.46}$$

This expression is now substituted into Eq. (3.41), yielding an explicit formula, for this model problem, for the laser-assisted bremsstrahlung cross section as a function of radiation frequency Ω.

In our numerical evaluation of the expression for $d\sigma/d\Omega$ just obtained we choose the resonance energy E_r and with Γ to be 10 and 0.01 eV, respectively. The background amplitude C^0 in Eq. (3.44) is taken to be 0.2, a value that leads to a relatively sharp resonance. The system is exposed to a laser field, specified by the vector potential $\mathbf{A}=\mathbf{a}\cos\omega t$, with $\omega=0.1$ eV and $\mathbf{a}$ parallel to the incident momentum $\mathbf{p}$. We present numerical results for two representative situations, in which the incident energy is either above or below the resonant energy. In choosing the intensity of the field, $I=9.0\times10^7$ W/cm^2, we take care to ensure that the range of frequency Ω for

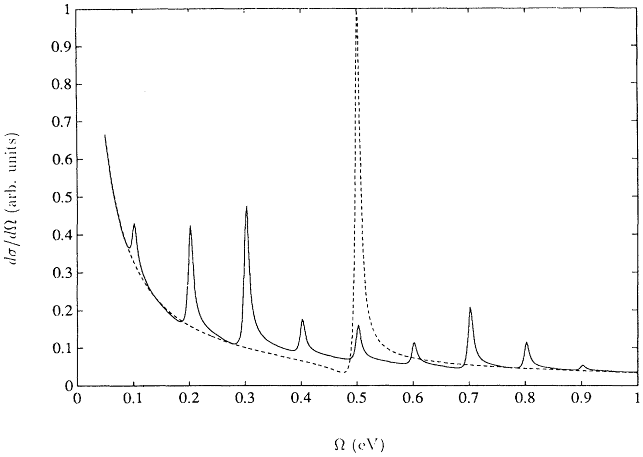

FIG. 1. Plots of the spontaneous bremsstrahlung spectrum for field-free resonant scattering (dashed curve) and for laser-assisted scattering (solid curve). The scattering energy is 10.5 eV, slightly above the resonance energy of 10 eV. The remaining parameters (resonance width, laser frequency, and laser intensity) are given in the text.

which the bremsstrahlung cross section is appreciable lies well below the scattering energy, as required for the validity of the low-frequency approximation.

Figure 1 shows two plots of $d\sigma/d\Omega$, corresponding to scattering with and without the laser field present. In the absence of the field the spontaneous radiation rate peaks for frequencies near the energy 0.5 eV, reflecting the existence of the scattering resonant state. Additional peaks appear when the field is turned on. In this case the scattered electron either gains or loses energy in quanta that are multiples of laser frequency ω; this is revealed by the positions of the peaks in the spectrum of the bremsstrahlung radiation. The field-free bremsstrahlung cross section shows no resonance peaking if the scattering energy is below the resonant energy. (The rise at very low frequencies signals the familiar infrared divergence effect, a general feature not directly relevant to this discussion.) However, one expects that with the field turned on the electron may absorb enough energy from the field to bring it into the resonance region, thereby inducing a peaking structure in the bremsstrahlung spectrum. This is indeed confirmed by the calculations, as shown in Fig. 2.

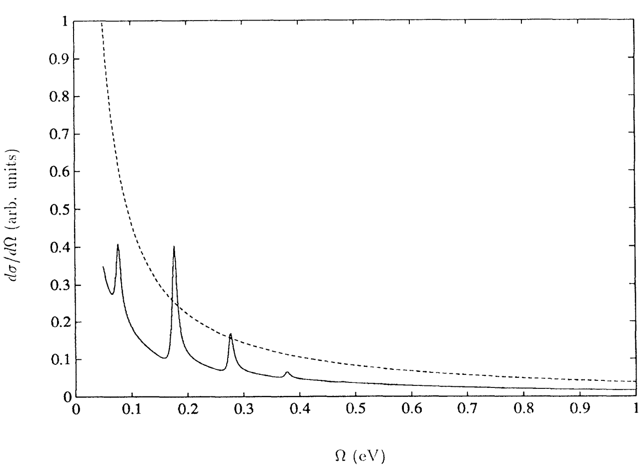

FIG. 2. Plots of the spontaneous bremsstrahlung spectrum for field-free resonant scattering (dashed curve) and for laser-assisted scattering (solid curve). Peaks are seen even though the scattering energy 9.975 eV is below the resonance energy of 10 eV.

As remarked above, the choice of field intensity in this model problem is limited by the requirement that the significant portion of the spontaneous radiation spectrum be confined to the low-frequency domain. This restriction can be relaxed in those cases for which the bremsstrahlung amplitude $\mathbf{m}$ can be evaluated without the aid of the low-frequency approximation. Then, in the presence of fields of higher intensity, higher-order harmonics of the laser frequency will appear in the spectrum.

IV. SUMMARY

The study of spontaneous bremsstrahlung accompanying the resonant scattering of a charged particle has been a subject of theoretical and experimental interest since the time it was recognized [6] that such studies could lead to effective procedures for determining resonance parameters. The nature of the radiation spectrum is radically altered when the scattering takes place in the presence of an external field. Specifically, the resonance-peak structure will in that case show a dependence on field intensity; moreover, there will be a replication of peaks (harmonic generation in the continuum), and resonance peaks will be induced by the field. In order to be able to examine these interesting effects using relatively simple analytic approximations we have limited ourselves here to a consideration of laser fields of low frequency and have adopted a variational description of the process. We take advantage of the availability of a trial function that accurately represents laser-assisted *resonant* scattering in the low-frequency regime. This trial function [5] reduces to the well-known Kroll-Watson wave function [2] when the scattering is nonresonant (and to the Volkov solution when there is no scattering potential at all). Trial functions of this type account nonperturbatively for the virtual absorption and emission of photons in the initial or final state of scattering events, or in the final state of ionization processes, and have been applied frequently to multiphoton physics. We emphasize that, while the amplitude for spontaneous radiation is evaluated in first order, the interaction with the laser field is treated nonperturbatively throughout the paper. In our numerical results, this is evidenced by the fact that the peak intensities for different orders are comparable, rather than rapidly decreasing in higher orders as would be the case in the perturbative regime. The ponderomotive energy was included formally in Eq. (2.8) to ensure the accuracy of the trial function in the domain of intense fields. While that energy shift may be negligible in some cases (as it was in the model studied numerically in Sec. III D) its inclusion in the trial function is justified since the alterative treatment based on perturbation theory is invalidated by the appearance of divergences.

It is a characteristic feature of low-frequency approximations that the cross sections of interest can be represented in terms of the measurable cross section for a process, of a simpler nature, taking place in the absence of the low-frequency field. This feature is apparent in the result obtained here. The bremsstrahlung spectrum in low-frequency approximation, shown in Eqs. (3.41) and (3.42), takes the form of a sum of terms, each containing as a factor the cross section for spontaneous bremsstrahlung accompanying field-*free* scattering. Two separate derivation of this result were given, one suitable for nonresonant scattering and the second adapted specifically (through an appropriate choice of trial function) to account for the effect of a resonance. Peaks in the radiation spectrum are predicted to lie at frequency values given in Eq. (3.43). These values are not simply multiples of the laser frequency, as they are in the case of harmonic generation from bound systems [1], but are shifted by an amount that depends on the separation between the incident energy and the energy of the resonance; there is also a dependence on laser intensity through the appearance of the ponderomotive energy shift. The resonant energy of the (field-free) scattering process can be indirectly deduced from an observation of the peak positions in the bremsstrahlung signals since Eq. (3.43) provides a relation between peak frequencies and resonant energy. The magnitude of the cross section at each peak value is controlled by a weighting factor that takes the form of the square of a generalized Bessel function; in particular, it is this (intensity-dependent) factor that introduces a cutoff in the spectrum for high-enough values of the order-index n appearing in Eq. (3.41).

The numerical example presented in Sec. III D serves the purpose of exhibiting some of the general features discussed above, such as the generation of resonance peaks under circumstances in which none would appear in the absence of the field, as shown in Fig. 2. The calculation was simplified considerably by the use of the Feshbach-Yennie low-frequency approximation for the field-free bremsstrahlung amplitude and the adoption of a simple (Breit-Wigner) representation of the field-free scattering amplitude. When calculations are performed for fields that are of greater intensity than assumed in this example, the spectrum will extend beyond the low-frequency domain and this will require a direct evaluation of the bremsstrahlung matrix element. Well-established numerical procedures are available for such an evaluation [7]. Alternatively, if measured spontaneous bremsstrahlung cross sections are available they may be used as input to Eq. (3.41), thereby providing estimates of the modification of the radiation cross section arising from the presence of a low-frequency external field.

We remarked earlier on the connection between laser-assisted bremsstrahlung radiation in resonant scattering and high-order harmonic radiation from bound states. We expect that the rate of bremsstrahlung emission in electron scattering from a beam of atoms will be much lower than that of harmonic generation by atoms in a gas. Nevertheless, we anticipate that the effect in some situations is still within the reach of observation. Bremsstrahlung in field-free scattering at electron energies above 100 eV has been well studied [8], and the application of an intense laser field does not substantially change the emission rate. This is indicated by the result shown in Eq. (3.41), which expresses the laser-assisted bremsstrahlung rate in terms of the corresponding field-free rate. The weighting factors (Bessel functions) in the sum are of order unity for parameters corresponding to experimentally available laser sources. (This is the case, for example, for the parameters chosen in our numerical calculation.) In addition, the existence of resonance peaks rising above the background is an aid in experimental detection. The numerical example we used serves to demonstrate calculational procedures. The scale of parameters adopted (e.g., 10 eV for the electron energy) is not restrictive. In order to achieve rates that are sufficiently high it may be appropriate to study scattering from heavy atoms or highly stripped ions, with resonant energies above 100 eV. Another potential application of the theory may lie in the study of plasma heating by intense laser radiation [9]. It is conceivable that the energy loss from bremsstrahlung in resonant electron scattering from ions in the plasma must be accounted for in order to achieve a full understanding of the efficiency of the laser-heating process, and the availability of theoretical estimates of energy-loss rates, such as the one developed here, may be useful in this regard.

ACKNOWLEDGMENT

This work was supported in part by the National Science Foundation under Grant No. PHY-9019745.

APPENDIX

Here we derive a low-frequency approximation for spontaneous radiation applicable to resonant scattering. The result is equivalent to that given originally by Feshbach and Yennie [6], but is obtained using a different approach. We present the alternative derivation for completeness and for the consistency of our formulation. At the same time we are able to provide another illustration, in addition to those given in the text, of the interesting relationship that exists between stimulated and spontaneous radiation processes in the low-frequency regime. We begin by considering the scattering of an electron in an external field of arbitrary polarization. The vector potential is taken to be $\mathbf{A}(t)=\mathbf{a}_1\cos\Omega t+\mathbf{a}_2\cos(\Omega t+\alpha)$, with α being the phase difference. We then calculate, in low-frequency approximation and in the weak-field limit, the single-photon stimulated emission amplitude whose analytic form is identical to the spontaneous amplitude of interest here. This procedure for generating approximations for spontaneous-emission amplitudes from an analysis of scattering in an external field was introduced by Brown and Goble [10]; additional applications were found subsequently [11].

The amplitude for laser-assisted scattering may be represented as [12]

$$T(\mathbf{p}',\mathbf{p};\mathbf{A}) = \int_{-\infty}^{\infty} dt \int d\mathbf{r} \left\{ \Phi^*_{\mathbf{p}'} V \tilde{\Psi}^{(+)}_{\mathbf{p}} + [\Psi^{(-)}_{\mathbf{p}';\mathrm{sc}}]^* \left[H - i\frac{\partial}{\partial t} \right] \tilde{\Psi}^{(+)}_{\mathbf{p}} \right\}, \quad (\mathrm{A1})$$

where $\Phi_{\mathbf{p}'}$ is the Volkov asymptotic solution

$$\Phi_{\mathbf{p}'}(\mathbf{r},t) = (2\pi)^{-3/2} \exp\left[i\mathbf{p}'\cdot\mathbf{r} - i\int^t \frac{[p'(\Omega t')]^2}{2\mu} dt' \right], \quad (\mathrm{A2})$$

and $\tilde{\Psi}^{(+)}_{\mathbf{p}}$ is an approximation to the exact solution. We shall adopt the trial function introduced earlier in Eq. (2.7). Then, as seen from Eq. (2.13), the second term in Eq. (A1) is proportional to the electric-field amplitude. We may therefore replace the exact scattered wave $\Psi^{(-)}_{\mathbf{p}';\mathrm{sc}}$ by the corresponding component of the trial function (2.7). This leaves an error of second order in the field amplitude, which may be ignored in the determination of the single-photon emission amplitude. We now carry out these substitutions, gather terms that are of first order in a_1 or a_2, and select those corresponding to the emission process. With this latter amplitude denoted (in the notation of Ref. [4]) as T_{-1}, we find that

$$T_{-1}(\mathbf{p}',\mathbf{p};\mathbf{A}) = 2\pi\delta(E_{\mathbf{p}'} - E_{\mathbf{p}} + \Omega)\frac{e}{2c}\mathbf{a}\cdot\hat{\mathbf{m}}(\mathbf{p}',\mathbf{p}), \quad (\mathrm{A3})$$

where $\mathbf{a} = \mathbf{a}_1 + \mathbf{a}_2 e^{i\alpha}$ and

$$\hat{\mathbf{m}}(\mathbf{p}',\mathbf{p}) = \left[-\frac{\mathbf{p}'}{\mu\Omega} - \nabla_{\mathbf{p}'} \right] t(\mathbf{p}',\mathbf{p};E_p) + \left[\frac{\mathbf{p}}{\mu\Omega} - \nabla_{\mathbf{p}} \right] t(\mathbf{p}',\mathbf{p};E_{\mathbf{p}'}) - i\Omega\langle u^{(-)}_{\mathbf{p}';\mathrm{sc}} | \mathbf{r} | u^{(+)}_{\mathbf{p},\mathrm{sc}} \rangle . \quad (\mathrm{A4})$$

The t matrix is formally defined as

$$t(\mathbf{p}',\mathbf{p};E) = \langle \mathbf{p}' | V | u^{(+)}_{\mathbf{p}}(E) \rangle , \quad (\mathrm{A5})$$

where the function $u^{(+)}_{\mathbf{p}}(E)$ has the representation given in Eq. (2.12).

The result shown in Eq. (A3) may be applied to the case of spontaneous emission by replacing the external-field amplitude $\mathbf{a}/2$ by the appropriate amplitude $\mathbf{A}_{\mathrm{sp}} = (2\pi c^2/\Omega L^3)^{1/2}\boldsymbol{\epsilon}$. [The factor 2 accounts for the choice of the cosine forms used for the vector potential of the external field $\mathbf{A}(t)$ while the spontaneous field has the time dependence $\mathbf{A}_{\mathrm{sp}} e^{i\Omega t}$ for photon emission.] Comparing Eq. (A3) with the corresponding spontaneous bremsstrahlung amplitude (3.3), we may make the identification

$$\mathbf{m}(\mathbf{p}',\mathbf{p}) = -\frac{i}{\Omega}\hat{\mathbf{m}}(\mathbf{p}',\mathbf{p}) . \quad (\mathrm{A6})$$

We emphasize that the resultant expression (A6) provides an exact representation of the matrix element for spontaneous emission, valid for arbitrary photon energy Ω. The expression may be simplified for $\Omega \ll E_{\mathbf{p}}$ since, if one ignores terms of first order in Ω, the last term in Eq. (A4) may be dropped. Expanding the off-shell t matrices about the corresponding on-shell values, we get

$$t(\mathbf{p}',\mathbf{p};E_{\mathbf{p}}) \cong t(p\hat{\mathbf{p}}',\mathbf{p}) - \frac{\mu\Omega}{p'}\hat{\mathbf{p}}'\cdot\nabla_{\mathbf{q}'} t(\mathbf{q}',\mathbf{p})|_{\mathbf{q}'=p\hat{\mathbf{p}}'} , \quad (\mathrm{A7})$$

$$t(\mathbf{p}',\mathbf{p};E_{\mathbf{p}'}) \cong t(\mathbf{p}',p'\hat{\mathbf{p}}) + \frac{\mu\Omega}{p}\hat{\mathbf{p}}\cdot\nabla_{\mathbf{q}} t(\mathbf{p}',\mathbf{q})|_{\mathbf{q}=p'\hat{\mathbf{p}}} , \quad (\mathrm{A8})$$

where we have used the approximation $p'-p \cong -\mu\Omega/p' \cong -\mu\Omega/p$ to keep only the first two terms in the expansions. The low-frequency approximation to Eq. (A4) can then be written as

$$\hat{\mathbf{m}}(\mathbf{p}',\mathbf{p}) \cong -\left[\frac{\mathbf{p}'}{\mu\Omega} - \hat{\mathbf{p}}'(\hat{\mathbf{p}}'\cdot\nabla_{\mathbf{q}'}) + \nabla_{\mathbf{q}'} \right] t(\mathbf{q}',\mathbf{p})|_{\mathbf{q}'=p\hat{\mathbf{p}}'} + \left[\frac{\mathbf{p}}{\mu\Omega} + \hat{\mathbf{p}}(\hat{\mathbf{p}}\cdot\nabla_{\mathbf{q}}) - \nabla_{\mathbf{q}} \right] t(\mathbf{p}',\mathbf{q})|_{\mathbf{q}=p'\hat{\mathbf{p}}} . \quad (\mathrm{A9})$$

The single-photon bremsstrahlung amplitude $\boldsymbol{\epsilon}\cdot\hat{\mathbf{m}}$, with $\boldsymbol{\epsilon}$ an arbitrary polarization vector, may now be expressed as

$$\boldsymbol{\epsilon}\cdot\hat{\mathbf{m}}(\mathbf{p}',\mathbf{p}) \cong -\frac{\boldsymbol{\epsilon}\cdot\mathbf{p}'}{\mu\Omega} t(p\hat{\mathbf{p}}',\mathbf{p}) + \hat{\mathbf{p}}'\times(\boldsymbol{\epsilon}\times\hat{\mathbf{p}}')\cdot\nabla_{\mathbf{q}'} t(\mathbf{q}',\mathbf{p})|_{\mathbf{q}'=p\hat{\mathbf{p}}'} + \frac{\boldsymbol{\epsilon}\cdot\mathbf{p}}{\mu\Omega} t(\mathbf{p}',p'\hat{\mathbf{p}}) + \hat{\mathbf{p}}\times(\boldsymbol{\epsilon}\times\hat{\mathbf{p}})\cdot\nabla_{\mathbf{q}} t(\mathbf{p}',\mathbf{q})_{\mathbf{q}=p'\hat{\mathbf{p}}} , \quad (\mathrm{A10})$$

which is just the Feshbach-Yennie result [6] evaluated in the dipole approximation. Here we have simplified the derivation of this result by employing a method consistent with that used in the text.

The above Feshbach-Yennie result can be put in an alternative form which may be convenient for certain applications. We obtain this form by expressing the momentum gradient in terms of derivatives with respect to the scalar variables $E_{\mathbf{p}}$ and $\tau = (\mathbf{p}'-\mathbf{p})^2$. The energy derivative makes no contribution (by virtue of the orientation of the triple-vector products) and we find, after some rearrangement and with the vector $\boldsymbol{\epsilon}$ factored out, that Eq. (A10) is equivalent to

$$\hat{\mathbf{m}}(\mathbf{p}',\mathbf{p}) \cong -\left[\frac{\mathbf{p}'}{\mu\Omega} + 2(\mathbf{p}'-\mathbf{p})\frac{\partial}{\partial\tau} \right] t(E_{\mathbf{p}},\tau)|_{\tau=(\mathbf{p}'-\mathbf{p})^2} + \left[\frac{\mathbf{p}}{\mu\Omega} + 2(\mathbf{p}'-\mathbf{p})\frac{\partial}{\partial\tau} \right] t(E_{\mathbf{p}}-\Omega,\tau)|_{\tau=(\mathbf{p}'-p)^2} . \quad (\mathrm{A11})$$

The error is of first order in Ω.

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