PHYSICAL REVIEW A VOLUME 48, NUMBER 6 DECEMBER 1993

Effect of atomic coherence on the second- and higher-order squeezing in a two-photon three-level cascade atomic system

Nadeem A. Ansari
Departimento di Scienze Fisiche, Università di Napoli "Federico II," Mostra d'Oltremare, Pad. 20, 80125 Napoli, Italy
(Received 26 May 1993)

A three-level cascade atomic system is considered where atomic coherence can be achieved either by applying an intense pump field or initially preparing the atoms in a coherent superposition of the states. It is predicted that the system, under certain conditions, exhibits almost perfect squeezing outside the cavity. By using the steady-state Q solution it is also shown that certain higher-order squeezing can also be achieved inside the cavity, under a suitable choice of different parameters.

PACS number(s): 42.50.Dv

I. INTRODUCTION

A three-level atomic system in a cascade configuration shows some important features where the crucial role is played by the atomic coherence. As an interesting example, Scully and Zubairy [1] considered a two-photon noise-free amplification where atomic coherence is introduced when atoms are initially prepared in the coherent superposition of the top and bottom level. They predicted that under certain conditions the added noise in one of the quadrature of the field vanishes and in this manner it is possible to amplify a super-Poisson signal into a sub-Poisson output signal with some gain. In coherent superposition of the states the two-photon three-level cascade atomic system also exhibits the property of correlated emission laser (CEL) [2]. In a two-photon CEL the spontaneously emitted photons are highly correlated and the photon diffusion coefficient vanishes under different suitable conditions. Scully *et al.* [2] have recently shown that such a system can produce not only light which is free from the spontaneous emission fluctuations but also phase squeezing can be obtained. Another possible way to introduce atomic coherence in a two-photon three-level cascade atomic system is to couple the top and the bottom level by applying an intense field [3]. Ansari, Gea-Banacloche, and Zubairy [4] considered this situation and they predicted that by applying an intense field, the system shows phase-sensitive amplification and the added noise in both quadratures of the field modes vanishes and the system behaves as a degenerate parametric amplifier, under certain conditions. They also showed different possible situations for a two-photon CEL. An and Sargent [5] have developed a quantum theory of multiwave mixing in which they considered such a three-level atomic system in the presence of two field modes and the top to bottom level transitions are made possible with the help of two photons of an externally applied intense field. They predicted that such a system can give perfect second-order squeezing outside the cavity. It is also shown that this system gives phase-sensitive amplification and when it is considered inside a cavity it shows certain nonclassical effects [6,7].

The present aim of this paper is to analyze the effect of atomic coherence when the top and bottom levels of a three-level cascade atomic system are coupled by the intense field and when they are initially prepared in the coherent superposition of these levels, on the second- and higher-order squeezing. Squeezed states of the electromagnetic fields, because of the reduced quantum fluctuations below the standard quantum limit, in one quadrature, at the expense of increased fluctuations in the other quadrature, have an important application in communication, weak-signal detection, optical amplification, etc. [8]. It is also possible to obtain certain higher-order squeezing and such a type of squeezing has been proposed and predicted in many systems by Hong and Mandel [9,10]. Another type of higher-order squeezing, namely, amplitude-square squeezing, has been introduced by Hillery [11] and has been predicted in second-harmonic generation [11]. Such a type of squeezing has also been predicted in a harmonic oscillator [12] and the multi photon Jaynes-Cumming model [13,14].

In this paper we consider a three-level cascade atomic system with both the above-mentioned conditions for the atomic coherence. The transitions from the top to the bottom level via the intermediate level comes out in the form of two photons of equal frequency. We derive the equations of motion of the expectation values of different second-order moments, from the density-matrix equation of motion for the field and evaluate their steady-state expectation values. With the help of these expectation values we predict that such a system in both cases of atomic coherence can give almost perfect squeezing outside the cavity under suitable conditions of detuning, intense pump phase choice, and initial coherence phase choice when they are prepared in coherent superposition of the states and the linear gain coefficient. We also evaluate an exact steady-state solution of the Fokker-Planck equation of motion in the Q representation and by using that solution we predict that higher-order squeezing and amplitude-square squeezing along with the second-order squeezing can also be obtained. In addition, we evaluate the probability distribution function for finding n photons in the field in such a system and show the behavior of this function when the system behaves classically and when it gives squeezing.

1050-2947/93/48(6)/4686(11)/$06.00

Our paper is arranged in five sections. In Sec. II, we will briefly discuss the system under consideration and find out the steady-state solution of the Fokker-Planck equation in Q representation. Section III is divided into three subsections which describe the second- and higher-order squeezing and the amplitude-square squeezing. In Sec. IV we discuss the photon statistics of the present model and show its behavior. Finally in Sec. V, we will conclude our discussion.

II. FOKKER-PLANCK EQUATION OF MOTION FOR THE FIELD

We consider a three-level atomic system in cascade configuration as shown in Fig. 1. The transitions from level $|a\rangle$ to $|b\rangle$ and $|b\rangle$ to $|c\rangle$ are dipole-allowed transitions and they result in the production of two photons of frequency ν. The transitions from level $|a\rangle$ to $|c\rangle$ are dipole-forbidden transitions and there are two ways to introduce atomic coherence: (a) by applying an external classical driving field [4] and (b) by injecting the atoms into the cavity in the coherent superposition of levels $|a\rangle$ and $|c\rangle$ [2]. In the remaining section we will discuss briefly these two conditions and derive an equivalent Fokker-Planck equation of motion in Q representation and calculate its steady-state solution.

A. Classical pump field

The transitions from levels $|a\rangle$ to $|c\rangle$ can be made possible by applying a sufficiently strong pump field which couples these levels. Then the transitions from $|a\rangle$ to $|b\rangle$ and $|b\rangle$ to $|c\rangle$ are treated quantum mechanically up to second order in the coupling constant and $|a\rangle$ to $|c\rangle$ classically up to all orders. The Hamiltonian of the system can be written in terms of the unperturbed and the perturbed parts as

$$H=H_0+V\ , \tag{1}$$

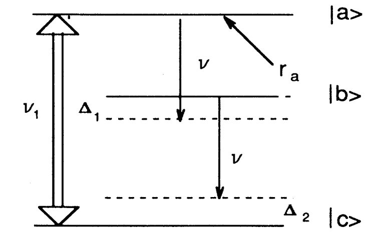

FIG. 1. Systematic diagram for a three-level atomic system in cascade configuration in the presence of the classical field.

where the interaction Hamiltonian is

$$V=\hbar g[(|a\rangle\langle b|+|b\rangle\langle c|)a+a^{\dagger}(|b\rangle\langle a|+|c\rangle\langle b|) -\frac{\hbar\Omega}{2}(e^{-i\phi-i\nu_1 t}|a\rangle\langle c|+e^{i\phi+i\nu_1 t}|c\rangle\langle a|)]\ . \tag{2}$$

Also a ($a^{\dagger}$) are the annihilation (creation) operator for the field of frequency ν, Ω is the Rabi frequency of the driving classical field, ϕ and ν_1 are its phase and the frequency, respectively, and g is the atom-field coupling constant which is taken to be equal for both the transitions. By defining the atom-field states

$$\begin{aligned}|1\rangle&=|a,n-2\rangle\ ,\\ |2\rangle&=|b,n-1\rangle\ ,\\ |3\rangle&=|c,n\rangle\ ,\end{aligned} \tag{3}$$

here, for example, $|3\rangle=|c,n\rangle$ implies that there are n photons in the field of frequency ν and the atom is in the ground state, and $|2\rangle=|b,n-1\rangle$ describes that the atom absorbs a photon and is excited to the intermediate state. Then by taking the trace over the atomic state, the reduced density matrix equation of motion for the field is of the form [4]

$$\begin{aligned}\dot{\rho}_F=&-[\beta_{11}^{*}aa^{\dagger}\rho_F+\beta_{11}\rho_F aa^{\dagger}-(\beta_{11}+\beta_{11}^{*})a^{\dagger}\rho_F a]-[(\beta_{22}^{*}+\Gamma_1)a^{\dagger}a\rho_F+(\beta_{22}+\Gamma_1)\rho_F a^{\dagger}a-(\beta_{22}+\beta_{22}^{*}+2\Gamma_1)a\rho_F a^{\dagger}]\\ &-[\beta_{12}^{*}aa\rho_F+\beta_{21}\rho_F aa-(\beta_{21}+\beta_{12}^{*})a\rho_F a]e^{-i\Phi}-[\beta_{21}^{*}a^{\dagger}a^{\dagger}\rho_F+\beta_{12}\rho_F a^{\dagger}a^{\dagger}-(\beta_{12}+\beta_{21}^{*})a^{\dagger}\rho_F a^{\dagger}]e^{i\Phi}\ ,\end{aligned} \tag{4}$$

where Γ_1 is the cavity-loss term and different coefficients which appeared in Eq. (4) are

$$\begin{aligned}\beta_{11}&=\frac{g^2r_a}{4}\left[\left[\frac{1}{\gamma}+\frac{1}{\gamma-i\Omega}\right]\frac{1}{\gamma+i(\Delta_1-\Omega/2)}+\left[\frac{1}{\gamma}+\frac{1}{\gamma+i\Omega}\right]\frac{1}{\gamma+i(\Delta_1+\Omega/2)}\right]\ ,\\ \beta_{12}&=\frac{g^2r_a}{4}\left[\left[\frac{1}{\gamma}-\frac{1}{\gamma-i\Omega}\right]\frac{1}{\gamma-i(\Delta_2+\Omega/2)}-\left[\frac{1}{\gamma}-\frac{1}{\gamma+i\Omega}\right]\frac{1}{\gamma-i(\Delta_2-\Omega/2)}\right]\ ,\\ \beta_{21}&=\frac{g^2r_a}{4}\left[\left[\frac{1}{\gamma}+\frac{1}{\gamma-i\Omega}\right]\frac{1}{\gamma+i(\Delta_1-\Omega/2)}-\left[\frac{1}{\gamma}+\frac{1}{\gamma+i\Omega}\right]\frac{1}{\gamma+i(\Delta_1+\Omega/2)}\right]\ ,\\ \beta_{22}&=\frac{g^2r_a}{4}\left[\left[\frac{1}{\gamma}-\frac{1}{\gamma-i\Omega}\right]\frac{1}{\gamma-i(\Delta_2+\Omega/2)}+\left[\frac{1}{\gamma}-\frac{1}{\gamma+i\Omega}\right]\frac{1}{\gamma-i(\Delta_2-\Omega/2)}\right]\ ,\end{aligned} \tag{5}$$

where it is considered that atoms are injected to level $|a\rangle$ with a rate r_a and γ is the equal decay rate for all the three levels. Also $\Delta_1=\omega_a-\omega_b-\nu$, $\Delta_2=\omega_b-\omega_c-\nu$. The physical interpretation of these terms are as follows. The first and the last terms represent the gain and losses in the system. The second and third terms are phase-sensitive terms because of the coupling of levels $|a\rangle$ and $|c\rangle$ by the external classical field. Also $\Phi=\phi+(\nu_1-2\nu)t$. In case of exact two-photon resonance, i.e., $\nu_1=2\nu$ we get $\Phi=\phi$ which is the reference phase of the classical field.

We define the Q representation for the field as the expectation value of the density matrix in coherent state

$$Q(\alpha,\alpha^*)=\frac{1}{\pi}\langle\alpha|\rho_F|\alpha\rangle \ , \tag{6}$$

where $|\alpha\rangle$ is the eigenstate of the annihilation operator a for the field with eigenvalue α, i.e.,

$$\begin{aligned} a|\alpha\rangle&=\alpha|\alpha\rangle \ , \\ a^\dagger|\alpha\rangle&=\left[\frac{\partial}{\partial\alpha}+\frac{\alpha^*}{2}\right]|\alpha\rangle \ . \end{aligned} \tag{7}$$

Using Eqs. (4), (6), and (7), the Fokker-Planck equation of motion in the Q representation for the field becomes

$$\dot{Q}=\left[-\beta_{11}\frac{\partial}{\partial\alpha}\alpha+\beta_{22}''^*\left[\frac{\partial}{\partial\alpha}\alpha+\frac{\partial^2}{\partial\alpha\,\partial\alpha^*}\right]+\beta_{21}^*e^{i\phi}\frac{\partial}{\partial\alpha}\alpha^*-\beta_{12}e^{i\phi}\left[\frac{\partial}{\partial\alpha}\alpha^*+\frac{\partial^2}{\partial\alpha^2}\right]+\text{c.c.}\right]Q \ , \tag{8}$$

where $\beta_{22}''=\beta_{22}+\Gamma_1$. In order to get a steady-state solution of the above equation in Q, we define the complex quantities in terms of real and imaginary parts

$$\begin{aligned} \alpha&=x_1+ix_2 \ , \\ \frac{\partial}{\partial\alpha}&=\frac{1}{2}\left[\frac{\partial}{\partial x_1}-i\frac{\partial}{\partial x_2}\right] \ , \end{aligned} \tag{9}$$

and

$$\begin{aligned} \beta_{11}&=y_1+iy_2 \ , \\ \beta_{22}''&=y_3+iy_4 \ , \\ \beta_{21}e^{-i\phi}&=y_5+iy_6 \ , \\ \beta_{12}e^{i\phi}&=y_7+iy_8 \ . \end{aligned} \tag{10}$$

Then replacing $Q(\alpha,\alpha^*)$ by $M(x_1,x_2)$ the resultant Fokker-Planck equation of motion reads as

$$\frac{\partial M(x_1,x_2)}{\partial t}=\left\{\frac{\partial}{\partial x_1}\left[A_1x_1+A_2x_2+A_7\frac{\partial}{\partial x_1}+A_6\frac{\partial}{\partial x_2}\right]+\frac{\partial}{\partial x_2}\left[A_3x_1+A_4x_2+A_5\frac{\partial}{\partial x_2}+A_6\frac{\partial}{\partial x_1}\right]\right\}M(x_1,x_2) \ , \tag{11}$$

where

$$\begin{aligned} A_1&=-y_1+y_3+y_5-y_7 \ , \\ A_2&=y_2+y_4-y_6-y_8 \ , \\ A_3&=-(y_2+y_4+y_6+y_8) \ , \\ A_4&=-y_1+y_3-y_5+y_7 \ , \\ A_5&=\frac{y_3+y_7}{2} \ , \\ A_6&=\frac{y_8}{2} \ , \\ A_7&=\frac{y_3-y_7}{2} \ . \end{aligned} \tag{12}$$

In steady state $\partial M(x_1,x_2)/\partial t=0$ Eq. (11) reduces to equations

$$\begin{aligned} \frac{\partial M}{\partial x_1}&=-(B_1x_1+B_2x_2)M \ , \\ \frac{\partial M}{\partial x_2}&=-(B_3x_1+B_4x_2)M \ , \end{aligned} \tag{13}$$

where

$$\begin{aligned} B_1&=\frac{A_6A_3-A_5A_1}{A_6^2-A_5A_7} \ , \\ B_2&=\frac{A_6A_4-A_5A_2}{A_6^2-A_5A_7} \ , \\ B_3&=\frac{A_6A_1-A_7A_3}{A_6^2-A_5A_7} \ , \\ B_4&=\frac{A_6A_2-A_4A_7}{A_6^2-A_5A_7} \ . \end{aligned} \tag{14}$$

The exact solution of Eq. (13) is of the form

$$M(x_1,x_2)=\frac{1}{N}\exp\left[-\left[\frac{B_1^2}{2}x_1^2+\frac{B_4^2}{2}x_2^2+\frac{B_2+B_3}{2}x_1x_2\right]\right] \ , \tag{15}$$

under the condition $B_2=B_3$. To obtain this condition we proceed as follows.

The two-photon resonance condition implies that $\Delta_1+\Delta_2=0$, and if we consider zero detuning, i.e., $\Delta_1=\Delta_2=0$, which will be true if the intermediate level $|b\rangle$ is exactly at the middle, and $\omega_a-\omega_b=\omega_b-\omega_c=\nu$ is satisfied. Also if we choose the phase $\phi=\pi/2$ of the pump field, then $B_2=B_3=0$ and the steady-state solution for the Q representation takes the form

$$M(x_1,x_2)=\frac{1}{N}\exp\left[-\frac{1}{2}(B_1x_1^2+B_4x_2^2)\right] . \qquad (16)$$

For zero detuning and $\phi=\pi/2$, B_1 and B_4 become

$$B_1=\frac{4(4-2\kappa'+\kappa'\Omega'+5\Omega'^2+\kappa'\Omega'^2+\kappa'\Omega'^3+\Omega'^4)}{8-2\kappa'\Omega'+10\Omega'^2+3\kappa'\Omega'^2+\kappa'\Omega'^3+2\Omega'^4} ,$$
$$B_4=\frac{-4(4-2\kappa'-\kappa'\Omega'+5\Omega'^2+\kappa'\Omega'^2-\kappa'\Omega'^3+\Omega'^4)}{-8-2\kappa'\Omega'-10\Omega'^2-3\kappa'\Omega'^2+\kappa'\Omega'^3-2\Omega'^4} . \qquad (17)$$

The normalization constant N is

$$N=\frac{2\pi}{\sqrt{B_1B_4}} , \qquad (18)$$

where $\kappa'=\kappa/\Gamma_1$ and $\kappa=2r_ag^2/\gamma^2$ is the linear gain coefficient, Ω' is defined as the Rabi frequency divided by the laval decay constant γ. In Q representation,

$$Q(\alpha,\alpha^*)=\frac{\sqrt{B_1B_4}}{2\pi}\exp\left[-\frac{B_1}{8}(\alpha+\alpha^*)^2+\frac{B_4}{8}(\alpha-\alpha^*)^2\right] . \qquad (19)$$

The expectation value of any high-order antinormal moment can be calculated with the help of this equation as

$$\langle F(a,a^\dagger)\rangle=\int F(\alpha,\alpha^*)Q(\alpha,\alpha^*)d^2\alpha . \qquad (20)$$

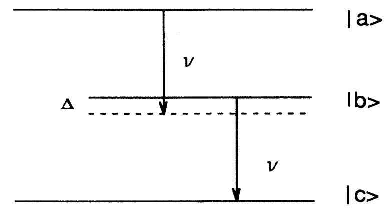

FIG. 2. Diagram for the system when the atoms are prepared in the coherent superposition of the states.

B. Coherent superposition of the states

Consider the same three-level atomic system in cascade configuration but the intermediate level $|b\rangle$ is now detuned with respect to the one photon of frequency ν by an amount Δ, i.e., $\Delta=\omega_{bc}-\nu$ and $\omega_{ac}=2\nu$. The exact two-photon resonant condition is satisfied as shown in Fig. 2. Again, $|a\rangle\to|b\rangle$ and $|b\rangle\to|c\rangle$ transitions are dipole allowed and come out in two photons of frequency ν and $|a\rangle\to|c\rangle$ is dipole forbidden. The atoms are injected into the cavity in a coherent superposition of the states $|a\rangle$ and $|c\rangle$ with a rate r. Then we can define the wave function as [15]

$$|\psi(t)\rangle=c_a|a\rangle+c_ce^{i\mu}|c\rangle , \qquad (21)$$

where c_a and c_c are the probability amplitudes of the atoms in levels, $|a\rangle$ and $|c\rangle$, respectively, and μ is the relative phase. The interaction Hamiltonian in the rotating-wave approximation at exact resonance is

$$V=\hbar g[a^\dagger(|b\rangle\langle a|+|c\rangle\langle b|)+(|a\rangle\langle b|+|b\rangle\langle c|)a] . \qquad (22)$$

As before, a ($a^\dagger$) are the annihilation (creation) operators for the field of frequency ν and g is the atom-field coupling constant (equal for the three levels). Then, the equation of motion for the reduced density matrix of the field with frequency ν is

$$\dot\rho_F=-[\beta_{11}'^*aa^\dagger\rho_F+\beta_{11}'\rho_Faa^\dagger-(\beta_{11}'+\beta_{11}'^*)a^\dagger\rho_Fa]-[(\beta_{22}'^*+\Gamma_1)a^\dagger a\rho_F+(\beta_{22}'+\Gamma_1)\rho_Fa^\dagger a-(\beta_{22}'+\beta_{22}'^*+2\Gamma_1)a\rho_Fa^\dagger]$$
$$-[\beta_{12}'^*aa\rho_F+\beta_{21}'\rho_Faa-(\beta_{21}'+\beta_{12}'^*)a\rho_Fa]-[\beta_{21}'^*a^\dagger a^\dagger\rho_F+\beta_{12}'\rho_Fa^\dagger a^\dagger-(\beta_{12}'+\beta_{21}'^*)a^\dagger\rho_Fa^\dagger] . \qquad (23)$$

The new coefficients of Eq. (23) are

$$\beta_{11}'=\frac{\kappa}{2}\xi\rho_{aa} ,\quad \beta_{22}'=\frac{\kappa}{2}\xi\rho_{cc} ,\quad \beta_{12}'=\frac{\kappa}{2}\xi\rho_{ca} ,\quad \beta_{21}'=\frac{\kappa}{2}\xi\rho_{ac} . \qquad (24)$$

The probabilities ρ_{aa} and ρ_{cc} are the initial populations of levels $|a\rangle$ and $|c\rangle$, respectively, and the condition for initial coherence is $\rho_{ac}=\rho_{ca}^*=|\rho_{ac}|e^{i\mu}$. Also $\xi=1/(1+i\Delta/\gamma)$ and $\kappa=2rg^2/\gamma^2$ is the linear gain coefficient, γ being the equal level decay constant for the three levels.

The steady-state solution of the Fokker-Planck equation of motion can be obtained by following the same steps as in Sec. II A and it has the same form as in Eq. (19) but with the condition of zero detuning, which corresponds to level $|b\rangle$ being exactly at the middle, and $\mu=0$. The new coefficients are now of the form

$$B_1=\frac{2\left[\frac{\kappa'}{2}(\rho_{cc}-\rho_{aa})+1\right]}{\frac{\kappa'}{2}(\rho_{cc}-|\rho_{ac}|)+1} ,\quad B_4=\frac{2\left[\frac{\kappa'}{2}(\rho_{cc}-\rho_{aa})+1\right]}{\frac{\kappa'}{2}(\rho_{cc}+|\rho_{ac}|)+1} . \qquad (25)$$

In the following sections we will use this steady-state solution to evaluate certain high-order squeezing and the photon distribution function.

III. SECOND- AND HIGHER-ORDER SQUEEZING

In this section we will evaluate the expressions for the second- and higher-order squeezing for the two cases, (i) in the presence of externally applied intense field and (ii) in the coherent superposition of the top and bottom levels of the atoms. We will also predict the regions where we can get strong second- and higher-order squeezing.

A. Second-order squeezing

In order to calculate the minimum variance we define the quadratures for the field of frequency ν as

$$d_1=\frac{1}{2}(a+a^\dagger)\ ,\qquad d_2=\frac{1}{2i}(a-a^\dagger)\ . \tag{26}$$

The condition for second-order squeezing is

$$(\Delta d_i)^2=\langle d_i^2\rangle-\langle d_i\rangle^2<\tfrac{1}{4}\ . \tag{27}$$

By using Eq. (26), (27) becomes

$$s_1\equiv(\Delta d_1)^2=\tfrac{1}{4}[2\langle a^\dagger a\rangle+\langle a^2\rangle+\langle a^{\dagger 2}\rangle]+\tfrac{1}{4}\ ,\qquad s_2\equiv(\Delta d_2)^2=\tfrac{1}{4}[2\langle a^\dagger a\rangle-\langle a^2\rangle-\langle a^{\dagger 2}\rangle]+\tfrac{1}{4}\ . \tag{28}$$

Different second-order moments in Eq. (28) can be calculated from the density matrix equation of motion [Eqs. (4) and (23)],

$$\frac{d}{dt}\langle a^\dagger a\rangle=\mathrm{Tr}(\rho a^\dagger a)=(\rho_1+\rho_1^*)\langle a^\dagger a\rangle+\rho_2\langle a^{\dagger 2}\rangle+\rho_2^*\langle a^2\rangle+\rho_{11}\ , \tag{29}$$

similarly

$$\frac{d}{dt}\langle a^2\rangle=2\rho_1\langle a^2\rangle+2\rho_2\langle a^\dagger a\rangle-2\beta_{21}^*e^{i\phi}\ , \tag{30}$$

where

$$\rho_1=\beta_{11}-\beta_{22}^*-\Gamma_1\ ,\qquad \rho_2=(\beta_{12}-\beta_{21}^*)e^{i\phi}\ ,\qquad \rho_{11}=\beta_{11}+\beta_{11}^*\ . \tag{31}$$

The equation of motion for $\langle a^{\dagger 2}\rangle$ can be obtained by taking the complex conjugate of the Eq. (30). In steady state the time dependence of Eqs. (29) and (30) vanishes and we are left with three equations. After solving them simultaneously we get the steady-state expressions of different second-order moments as

$$\langle a^\dagger a\rangle=\frac{(\rho_1\rho_2\beta_{21}e^{-i\phi}+\text{c.c.})+\rho_{11}|\rho_1|^2}{(\rho_1+\rho_1^*)(|\rho_2|^2-|\rho_1|^2)}\ ,\qquad \langle a^2\rangle=\frac{|\rho_2|^2\beta_{21}^*e^{i\phi}-\rho_1^*(\rho_1+\rho_1^*)\beta_{21}^*e^{i\phi}-\rho_2^2\beta_{21}e^{-i\phi}-\rho_{11}\rho_1^*\rho_2}{(\rho_1+\rho_1^*)(|\rho_2|^2-|\rho_1|^2)}\ . \tag{32}$$

The steady-state expression for $\langle a^{\dagger 2}\rangle$ can be obtained by taking the complex conjugate expression of $\langle a^2\rangle$.

1. *Outside the cavity*

The steady-state expression for the spectral density of the second-order moments outside the cavity can be calculated along the same lines as discussed by Holm and Sargent [16].

The equation of motion for the expectation value of the annihilation operator for the field mode is

$$\frac{d}{dt}\langle a\rangle=\rho_1\langle a\rangle+\rho_2\langle a^\dagger\rangle\ . \tag{33}$$

Equation (33) and its complex conjugate can be written in matrix form as

$$\frac{d}{dt}\langle J(t)\rangle=A\langle J(t)\rangle\ , \tag{34}$$

where

$$J(t)=\begin{bmatrix} a\\ a^\dagger\end{bmatrix}\ ,\qquad A=\begin{bmatrix}\rho_1 & \rho_2\\ \rho_2^* & \rho_1^*\end{bmatrix}\ . \tag{35}$$

Then, the direct product of J and its transpose form a 2×2 matrix

$$J\times J^T\equiv JJ\ , \tag{36}$$

where $\times$ corresponds to the direct product. The equation of motion for $\langle JJ\rangle$ can be obtained by expressing Eqs. (29) and (30) into the matrix form

$$\frac{d}{dt}\langle JJ\rangle=-A\langle JJ\rangle-\langle JJ\rangle A^T+D\ , \tag{37}$$

where A is given by Eq. (35) and

$$D=\begin{bmatrix} -2\beta_{21}^{*}e^{i\phi} & \rho_{11} \\ \rho_{11} & -2\beta_{21}e^{-i\phi} \end{bmatrix} . \tag{38}$$

The spectral density for the steady-state expectation values of the second-order moments for the field is calculated from the relation [16]

$$L_{\text{in}}=(A+i\omega)^{-1}D(A^{T}-i\omega)^{-1} . \tag{39}$$

The different elements of the matrix L_{in}, corresponding to the spectral density of $\langle a^{\dagger}a\rangle$, $\langle a^{2}\rangle$, etc., are $(\omega=0)$

$$\begin{aligned} L_{11}&=\frac{-2(\rho_1^{*2}\beta_{21}^{*}e^{i\phi}+\rho_1^{*}\rho_2\rho_{11}+\rho_2^{2}\beta_{21}e^{-i\phi})}{(|\rho_1|^2-|\rho_2|^2)^2} , \\ L_{12}&=\frac{2(\rho_1\rho_2\beta_{21}e^{-i\phi}+\text{c.c.})+(|\rho_1|^2+|\rho_2|^2)\rho_{11}}{(|\rho_1|^2-|\rho_2|^2)^2} , \\ L_{22}&=L_{11}^{*} . \end{aligned} \tag{40}$$

Outside the cavity the spectral density can be obtained by multiplying L_{in} by the damping constant which is the cavity line width $2\Gamma_1$, i.e.,

$$L_{\text{out}}=2\Gamma_1 L_{\text{in}} . \tag{41}$$

The expression for the second-order squeezing outside the cavity is

$$(S_1)_{\text{out}}=\frac{\Gamma_1}{2}(2L_{12}+L_{11}+L_{22})+\frac{1}{4} . \tag{42}$$

Different matrix elements are given in Eq. (40).

2. *Classical pump field*

In case of classical pump field we notice that the exact resonance condition implies that $\Delta_1+\Delta_2=0$ or $\Delta_1=-\Delta_2=\Delta$, i.e., the middle level is not exactly one-photon resonant but the two-photon resonance condition is still valid. In Fig. 3 we have plotted s_1 from Eq. (28) by using Eqs. (5) and (32) against dimensionless detuning Δ/γ for the phase choice of the pump field $\phi=\pi/2$ and 0, respectively, $\Omega'=5$ and $\kappa'=2$. It is evident from the figure that the strong squeezing is obtained around a small region at $\Delta/\gamma=0$ and $\phi=\pi/2$, while for $\phi=0$ very small squeezing is obtained for the positive values of the detuning. In order to see the dependence of the strong squeezing on the dimensionless Rabi frequency Ω', at zero detuning and for $\phi=\pi/2$, in Fig. 4 we have plotted s_1 vs Ω', for $\kappa'=2$. This figure shows that first amount of squeezing increases with Ω' and has a maximum value at $\Omega'=2.5$ and then it starts decreasing for further increasing values of Ω'.

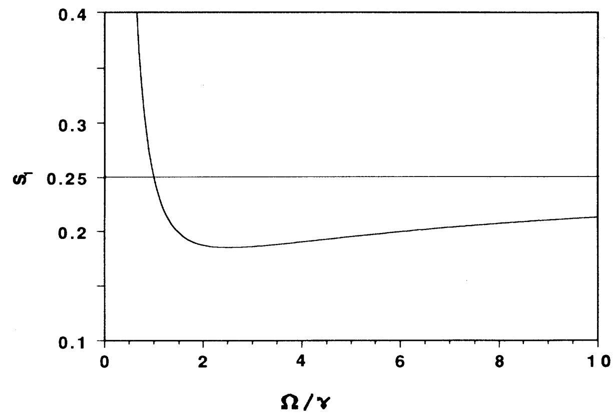

FIG. 4. Second-order squeezing s_1 vs Rabi frequency of the classical field Ω', for $\kappa'=2$, $\Delta/\gamma=0$, and $\phi=\pi/2$.

The steady-state expression for second-order squeezing outside the cavity may be obtained by using Eqs. (5), (40), and (42). In Fig. 5, we have plotted the steady-state squeezing outside the cavity against Ω' for zero detuning, $\kappa'=50,75$ and $\phi=\pi/2$. The plot illustrates the region where we can get almost perfect squeezing outside the cavity. Thus in this manner we can predict that the system exhibits almost perfect squeezing outside the cavity for the particular choice of the Rabi frequency and the reference phase of the intense driving field, when the intermediate level of the atomic system is exactly one-photon resonant.

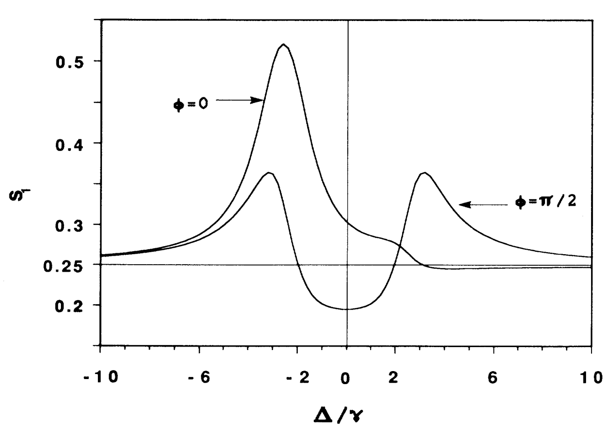

FIG. 3. Second-order squeezing s_1 vs dimensionless detuning Δ/γ, for $\kappa'=2$, $\Omega'=5$, and $\phi=0$ and $\pi/2$.

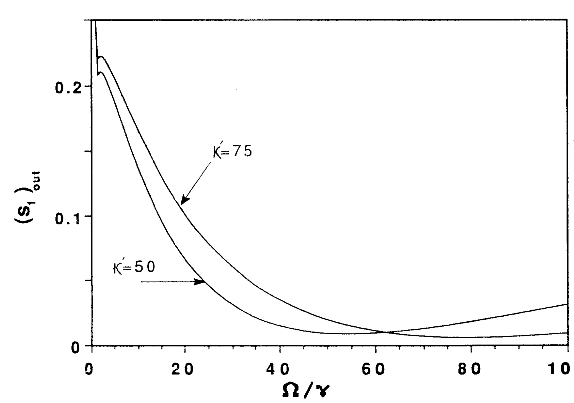

FIG. 5. Second-order squeezing $(s_1)_{\text{out}}$ outside the cavity vs Ω', for $\kappa'=50$ and 75 and for the same parameters as used in Fig. 4.

3. *Coherent superposition of the states*

For the second case when atoms are injected in a coherent superposition of the top and bottom levels into the cavity, we get the same equation as Eq. (32) for the steady-state expectation values of different second-order moments except that now different $\beta'_{i,j}$'s are given in Eq. (24). Also we define probabilities of having the atom in different levels in terms of a parameter ϵ, such that

$$\rho_{aa}=\frac{1+\epsilon}{2}\ ,\qquad \rho_{cc}=\frac{1-\epsilon}{2}\ ,\qquad |\rho_{ac}|=\frac{\sqrt{1-\epsilon^2}}{2}\ , \tag{43}$$

where $-1\le\epsilon\le1$. For the lower bound limit when $\epsilon=-1$, $\rho_{aa}=|\rho_{ac}|=0$ and $\rho_{cc}=1$, i.e., the atoms are in the bottom level, for the upper bound limit the atoms are in the top level $|a\rangle$, and in between values of ϵ correspond to superposition of the top and bottom levels and give the phase sensitivity in the system.

Using Eqs. (28), (32), and (24), the expressions of steady-state squeezing inside the cavity are obtained. In Fig. 6 we have plotted such a second-order squeezing against Δ/γ for $\kappa'=2$, $\epsilon=-0.6$ and for $\mu=0$ and $\pi/2$. From the graph we can easily verify that strong squeezing can be obtained for the phase choice of initial coherence $\mu=0$, around a small region at zero detuning. Also for $\mu=\pi/2$ some amount of squeezing can be obtained for negative values of the detuning and for its positive values the quadrature exhibits no squeezing. In Fig. 7 we have plotted such a second-order squeezing against ϵ, for $\mu=0$ and zero detuning. The maximum squeezing in this case is achieved for $\epsilon=-0.5$ and there is no squeezing for $\epsilon>0$. This situation can be visualized by the fact that for the positive values of ϵ, the probability of the atoms in the top level increases and this destroys the steady-state squeezing inside the cavity.

The steady-state squeezing outside the cavity is obtained by using Eqs. (24), (40), and (42). In Fig. 8 we have plotted such a steady-state squeezing outside the cavity versus ϵ for zero detuning, $\kappa'=50$ and 75. Again we can predict the regions of the perfect squeezing outside the cavity.

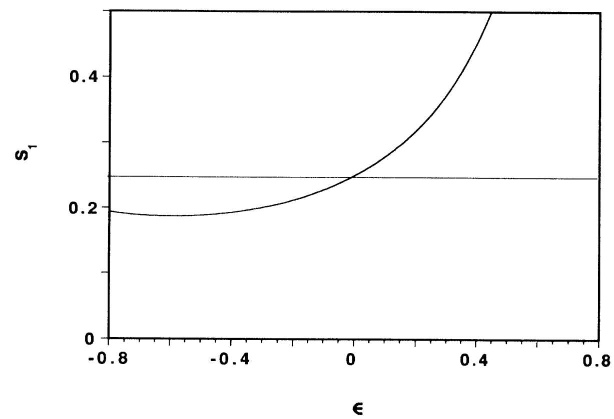

FIG. 7. Second-order squeezing s_1 vs ϵ, for $\kappa'=2$, $\Delta'=0$, and $\mu=0$.

Figures 3 and 6 also illustrate that the three-level atomic system in cascade configuration with two photons of equal frequency ν in the presence of an intense driving field and in the coherent superposition of the states shows identical behavior at zero detuning and for the proper choice of the different parameters in order to give the maximum squeezing inside and outside the cavity.

We have obtained the steady-state Q solution inside the cavity which will be used to evaluate higher-order squeezing and photon statistics. Thus, in the remaining part of this paper we will restrict ourselves to discuss the behavior of the system inside the cavity only.

B. Fourth- and sixth-order squeezing

Hong and Mandel [9] have obtained the relation for the nth-order variance by using the Campbell-Baker-Hausdorff identity,

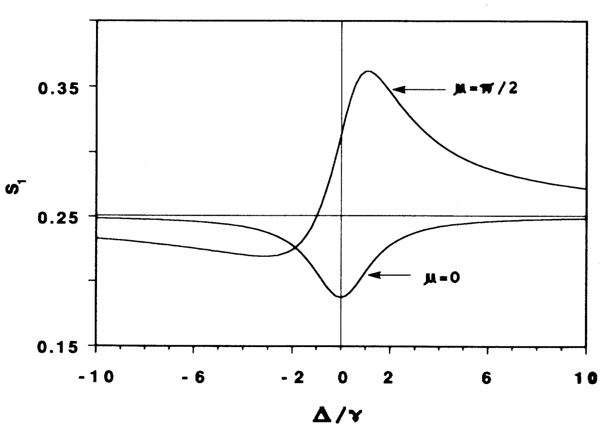

FIG. 6. Second-order squeezing s_1 vs detuning Δ/γ, for $\kappa'=2$, $\epsilon=-0.6$, and $\mu=0$ and $\pi/2$.

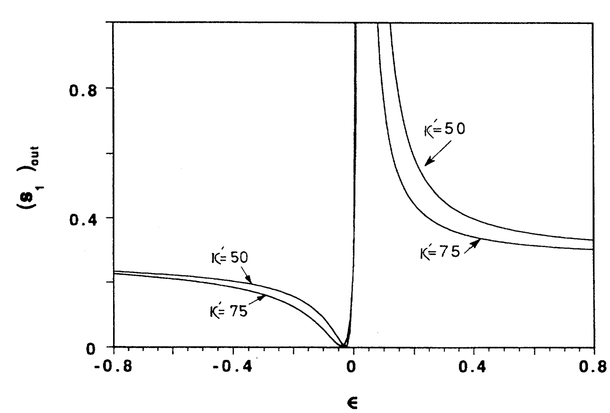

FIG. 8. Second-order squeezing $(s_1)_{\text{out}}$ outside the cavity vs ϵ for $\kappa'=50$ and 75 and for the same parameters as used in Fig. 7.

$$\langle \exp(xd_i)\rangle = \langle :\exp(xd_i):\rangle \exp(x^2 s/2) , \qquad (44)$$

where $::$ denotes the normal ordering and $s\equiv(1/2i)[d_1,d_2]=\frac{1}{4}$. Expanding both sides of the above equation as a power series of x and comparing the coefficient of $x^n/n!$ gives the nth-order variance. Hence the state of the field is squeezed to the fourth order if it satisfies

$$a_4 \equiv [\langle :(\Delta d_i)^4:\rangle + \tfrac{6}{4}\langle :(\Delta d_i)^2:\rangle] < 0 \quad (i=1,2) , \qquad (45)$$

where

$$\langle :(\Delta d_1)^4:\rangle = \tfrac{1}{16}[\langle d_1^4\rangle + \langle d_1^{\dagger 4}\rangle + 6\langle d_1^{\dagger 2} d_1^2\rangle + 4(\langle d_1^\dagger d_1^3\rangle + \langle d_1^{\dagger 3} d_1\rangle)] , \qquad (46)$$

and $\langle :(\Delta d_1)^2:\rangle$ can be obtained from Eq. (28).

By using the steady-state solution for the Q representation we can calculate the fourth-order moments which are essential for the fourth-order squeezing. At this stage it is important to mention that the exact steady-state solution of the Fokker-Planck equation of motion is obtained at zero detuning and for the classical pump phase choice $\phi=\pi/2$ and the initial coherence phase choice $\mu=0$, in case of superposition of the states. For such choices we can also obtain the strong second-order squeezing (which is clear from Figs. 3 and 6). This implies that by using the Q solution we have the higher-order squeezing in the region of strong second-order squeezing. By using the Q solution from Eq. (19) to evaluate the higher-order moments as in Eq. (20), we get

$$a_2 \equiv \langle :\Delta d_1^2:\rangle = \frac{1}{B_1} - \frac{1}{2} ,$$
$$\langle :\Delta d_1^4:\rangle = 3\left[\frac{1}{B_1^2} - \frac{1}{B_1} + \frac{1}{4}\right] . \qquad (47)$$

Also for the sixth-order squeezing we define the relation

$$a_6 \equiv |\langle :\Delta d_1^6:\rangle + \frac{15}{4}\langle :\Delta d_i^4:\rangle + \frac{45}{16}\langle :\Delta d_i^2:\rangle < 0 , \qquad (48)$$

where

$$\langle :\Delta d_i^6:\rangle = \frac{15}{B_1^3} - \frac{45}{2B_1^2} + \frac{45}{4B_1} - \frac{15}{8} . \qquad (49)$$

In Fig. 9(a) we have plotted such higher-order squeezing along with the second-order squeezing, by using Eqs. (17), (47), and (49), against Ω' for $\kappa'=2$. This figure predicts that such higher-order squeezing are possible when we drive the three-level atomic system by some externally applied intense field and the regions for the fourth- and sixth-order squeezing are the same as of the strong second-order squeezing. Figure 9(b) represents such higher-order squeezing versus ϵ, for $\kappa'=2$, when atoms are prepared in the coherent superposition of the states and by using Eqs. (25), (47), and (49). Again the regions for the higher-order squeezing are the same as for the strong second-order squeezing and follow the same conditions as discussed for the second-order squeezing.

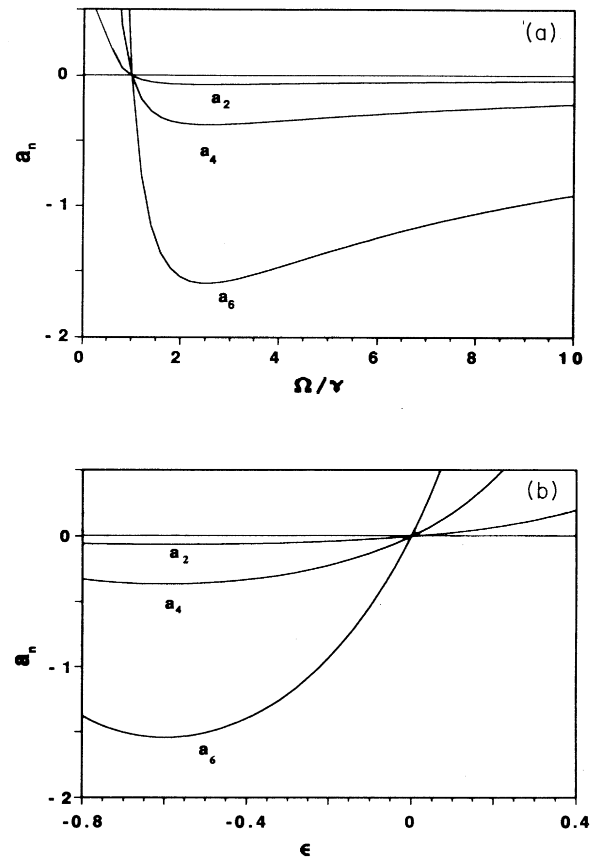

FIG. 9. (a) Higher-order squeezing a_n vs Ω', for the same parameters as used in Fig. 4. (b) Higher-order squeezing a_n vs ϵ, for the same parameters as used in Fig. 7.

C. Amplitude-square squeezing

Another type of higher-order squeezing, amplitude-square squeezing, can be obtained by defining the quadratures in terms of the real and imaginary parts of the square of the field, i.e.,

$$X_1 = \frac{a^2 + a^{\dagger 2}}{2} ,$$
$$X_2 = \frac{a^2 - a^{\dagger 2}}{2i} . \qquad (50)$$

The commutation relation for these operators reads as

$$[X_1, X_2] = i(2\langle a^\dagger a\rangle + 1) , \qquad (51)$$

and the uncertainty relation for the quadrature for the square of the field's amplitude is

$$\Delta X_1 \Delta X_2 \geq \langle a^\dagger a\rangle + \tfrac{1}{2} . \qquad (52)$$

Then the condition for the amplitude-square squeezing follows

$$\Delta X_i^2 < \langle a^\dagger a\rangle + \tfrac{1}{2} \quad (i=1,2) , \qquad (53)$$

the variances into the two quadratures are

$$\Delta X_1^2 = \tfrac{1}{4}[\langle (a^2+a^{\dagger 2})^2\rangle - \langle (a^2+a^{\dagger 2})\rangle^2] ,$$
$$= \tfrac{1}{4}[\langle a^{\dagger 2}a^2\rangle + \langle a^2 a^{\dagger 2}\rangle + \langle a^4\rangle + \langle a^{\dagger 4}\rangle - (\langle a^2\rangle^2 + \langle a^{\dagger 2}\rangle^2 + 2\langle a^2\rangle\langle a^{\dagger 2}\rangle)] , \quad (54)$$

and

$$\Delta X_2^2 = \tfrac{1}{4}[\langle a^{\dagger 2}a^2\rangle + \langle a^2 a^{\dagger 2}\rangle - \langle a^4\rangle - \langle a^{\dagger 4}\rangle + \langle a^2\rangle^2 + \langle a^{\dagger 2}\rangle^2 - 2\langle a^2\rangle\langle a^{\dagger 2}\rangle] . \quad (55)$$

After evaluating different higher-order moments with the help of $Q(\alpha,\alpha^*)$, the exact steady-state expressions for the variances are

$$\Delta X_1^2 = \frac{2}{B_1^2} + \frac{2}{B_4^2} - \frac{2}{B_1} - \frac{2}{B_4} + 1 \quad (56)$$

and

$$\Delta X_2^2 = \frac{4}{B_1 B_4} - \frac{2}{B_1} - \frac{2}{B_4} + 1 . \quad (57)$$

When the atomic system is driven by the classical field then different coefficients of Eqs. (56) and (57) are given by Eq. (17), and second quadrature satisfies the condition for the amplitude-square squeezing. In Fig. 10(a), we have plotted ΔX_2^2 against Ω' for $\kappa' = 1,2$. From the graph we predict that when such a system is driven by the classical field, it exhibits amplitude-square squeezing, along with the second- and higher-order squeezing. As discussed before, the region for the amplitude-square squeezing is the same as for the strong second-order squeezing inside the cavity.

In Fig. 10(b), we have plotted ΔX_2^2 against ϵ, for $\kappa' = 1,2$, by using Eqs. (57) and (25), which corresponds to the situation in which atoms are prepared in a coherent superposition of the top and bottom levels. It is clear from the graph that we can also obtain the amplitude-square squeezing under the same condition for ϵ as for the second- and higher-order squeezing.

IV. PROBABILITY DISTRIBUTION FUNCTION

With the help of the steady-state solution of the Fokker-Planck equation, we can also work out another interesting property of the present system under consideration, which is the probability of finding n photons in the field. As has already been discussed, the solution $Q(\alpha,\alpha^*)$ is obtained under the same conditions of the detuning and the phase of the classical field and phase of the initial coherence, when the atoms are considered in superposition of the states, as in the case of strong second-order squeezing. So we can analyze the photon statistics in the region of strong second-order squeezing.

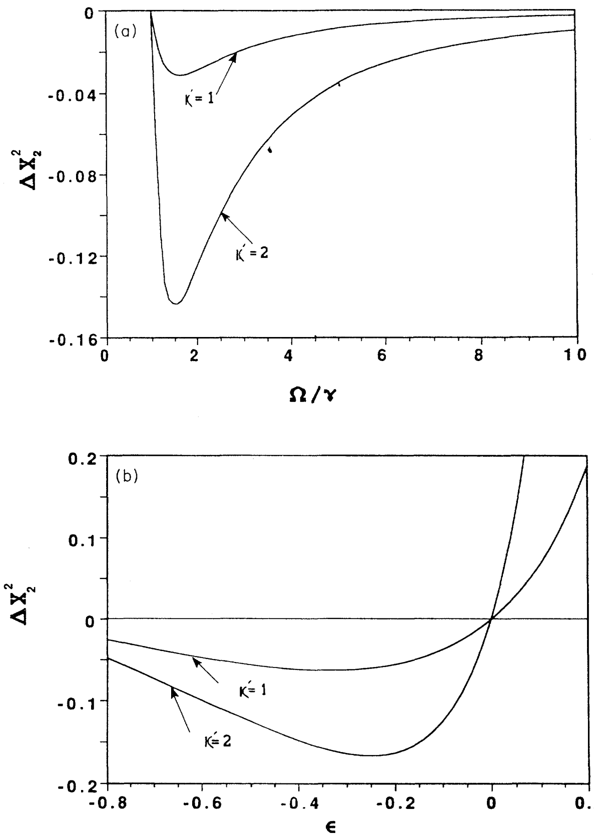

FIG. 10. (a) Amplitude-square squeezing ΔX_2^2 vs Ω', for $\kappa' = 1,2$, and for the same parameters as used in Fig. 4. (b) Amplitude-square squeezing ΔX_2^2 vs ϵ, for $\kappa' = 1,2$, and for the same parameters as used in Fig. 7.

The probability $p(n)$ of finding n photons in the field can be determined by using the relation

$$p(n) = \langle n|\rho_F|n\rangle . \quad (58)$$

In terms of the Q representation Eq. (58) becomes

$$p(n) = \frac{\pi}{n!}\left[\frac{\partial^{2n}}{\partial\alpha^n \partial\alpha^{*n}}[Q(\alpha,\alpha^*)e^{|\alpha|^2}]\right]_{\alpha=\alpha^*=0} . \quad (59)$$

On substituting the values of $Q(\alpha,\alpha^*)$ from Eq. (19) into Eq. (59), it follows

$$p(n) = \frac{\pi}{n!}\left\{\frac{\partial^{2n}}{\partial\alpha^n\partial\alpha^{*n}}\left[\exp\left[-\frac{B_1}{8}(\alpha+\alpha^*)^2 + \frac{B_4}{8}(\alpha-\alpha^*)^2 + |\alpha|^2\right]\right]\right\}_{\alpha=\alpha^*=0} . \quad (60)$$

When we define $Q(\alpha,\alpha^*)$ in terms of $M(x_1,x_2)$, and using Eqs. (9) and (16), we can write

$$p(n) = \frac{\sqrt{B_1 B_4}}{2^{(2n+1)}n!}[(D_1^2 + D_2^2)^n \exp(a_1 x_1^2 + a_2 x_2^2)]_{x_1=x_2=0} , \quad (61)$$

where

$$D_i = \frac{\partial}{\partial x_i} \quad (i=1,2) \ ,$$
$$a_1 = 1 - B_1/2, \quad a_2 = 1 - B_4/2 \ . \tag{62}$$

After performing the nth-order differentiation of Eq. (61) and applying the condition of $x_1 = x_2 = 0$, we finally get

$$p(n) = \frac{\sqrt{B_1 B_4}(-1)^n}{2^{(2n+1)}} \sum_{r=0}^{n} \frac{1}{r!(n-r)!} H_{2(n-r)}(0) H_{2r}(0)(a_2)^r (a_1)^{n-r} \ , \tag{63}$$

H being the Hermite polynomial.

In case of intense field, B_1 and B_4 are given by Eq. (17) and in case of coherent superposition of the states, by Eq. (25). In Fig. 11(a), we have plotted $p(n)$ versus n, by using Eqs. (63) and (17), for $\Omega' = 0.5$, 2.5, *and* 5 and $\kappa' = 2$. The plot shows that the probability of finding n photons decreases smoothly with the increasing n for $\Omega' = 0.5$. At this value of Ω', the system shows no squeezing of any kind and behaves classically. But for $\Omega' = 2.5, 5$ it shows the behavior of the photon statistics when there is some amount of squeezing present.

In Fig. 11(b), we have plotted $p(n)$ vs n by using Eqs. (63) and (25), for $\epsilon = -0.5$, -0.2, *and* 0.4. Again we have the same dependence of $p(n)$ on n when the system behaves classically and when it shows some amount of squeezing.

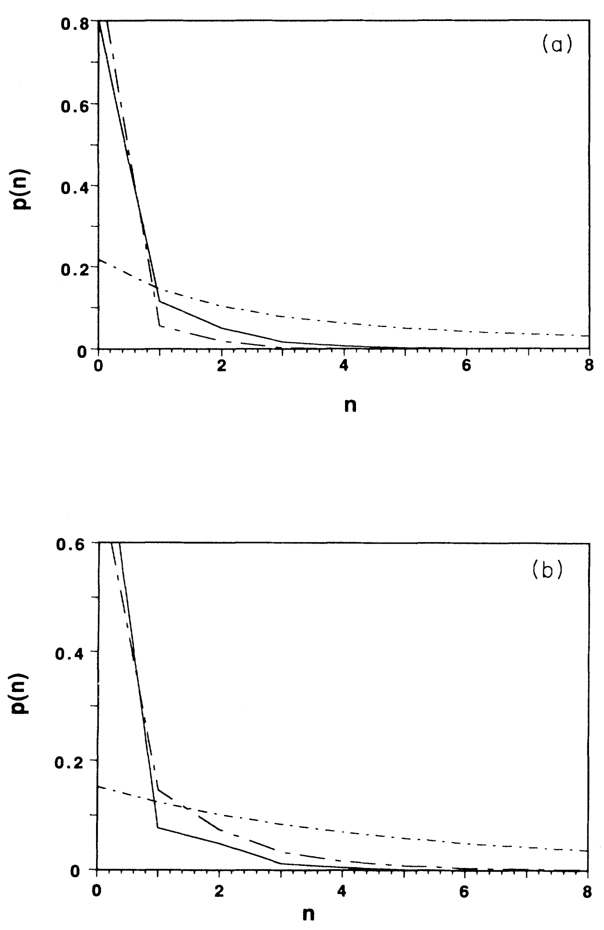

FIG. 11. (a) $p(n)$ vs n for $\Omega' = 0.5$ (— · — · —·), 2.5 (— — —), and 5 (——) and for the same values of parameters as used in Fig. 4. (b) $p(n)$ vs n for $\epsilon = -0.5$ (— · — · — ·), -0.2 (— — —), and 0.4 (——) and for the same values of parameters as used in Fig. 7.

V. DISCUSSIONS

In this paper we have considered a three-level atomic system in a cascade configuration where top to bottom level transitions are dipole forbidden. There are two ways to introduce atomic coherence, either by coupling these levels by externally applied intense field or by preparing the atomic system in the initial coherent superposition of the top and bottom levels. The transitions from the top level through the intermediate level result in the production of two photons of equal frequency. By using the density matrix equation of motion, which is obtained under both the conditions of atomic coherence, we derived an equivalent Fokker-Planck equation of motion. The exact steady-state solution in the Q representation is obtained under zero detuning and the proper choice of the intense pump field's phase and phase of the initial coherence. We have predicted that such system can give almost perfect squeezing outside the cavity for some particular values of Rabi frequency of the pump field, the parameter ϵ which gives the probability for the atoms being in different levels, and the linear gain coefficient. We have also shown the region for strong second-order squeezing inside the cavity. In addition, we have also predicted that such a system under both conditions of atomic coherence exhibits certain higher-order and amplitude-square squeezing. With the help of the Q solution we have also determined the probability distribution function for finding n photons and predicted its behavior when the system behaves classically and when it shows a certain amount of squeezing inside the cavity.

ACKNOWLEDGMENTS

The author wishes to thank Professor S. Solimeno and his group for some useful discussions and cooperation during the preparation of this work. This research was supported by the International Centre for Theoretical Physics programme for research and training in Italian laboratories, Trieste, Italy.

[1] M. O. Scully and M. S. Zubairy, Opt. Commun. **66**, 303 (1988).

[2] M. O. Scully, K. Wodkiewicz, M. S. Zubairy, J. Bergou, N. Lu, and J. Meyer ter Veh, Phys. Rev. Lett. **60**, 1832 (1988).

[3] M. O. Scully, Phys. Rev. Lett. **55**, 2802 (1985).

[4] N. A. Ansari, J. Gea-Banacloche, and M. S. Zubairy, Phys. Rev. A **41**, 5179 (1990).

[5] Sunghyuck An and M. Sargent III, Phys. Rev. A **39**, 1841 (1989).

[6] N. A. Ansari, Phys. Rev. A **46**, 1560 (1992).

[7] N. A. Ansari, J. Mod. Opt. **40**, 413 (1993).

[8] See, for example, K. Zaheer and M. S. Zubairy, in *Advances in Atomic and Molecular and Optical Physics,* edited by D. R. Bates and B. Bederson (Academic, New York, 1990), Vol. 28, p. 143.

[9] C. K. Hong and L. Mandel, Phys. Rev. Lett. **54**, 323 (1985).

[10] C. K. Hong and L. Mandel, Phys. Rev. A **32**, 974 (1985).

[11] M. Hillery, Opt. Commun. **62**, 135 (1987); Phys. Rev. A **36**, 3796 (1987).

[12] C. C. Gerry and E. R. Vrscay, Phys. Rev. A **37**, 1779 (1988).

[13] X. Yang and X. Zhang, Opt. Commun. **138**, 409 (1989).

[14] M. H. Mahran and A.-S. F. Obada, Phys. Rev. A **42**, 1718 (1990).

[15] See, for example, M. Sargent III, M. O. Scully, and W. E. Lamb, Jr., *Laser Physics* (Addison-Wesley, Reading, MA, 1974).

[16] D. A. Holm and M. Sargent III, Phys. Rev. A **35**, 2150 (1987).