Realistic optical homodyne measurements and quasiprobability distributions

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The deteriorating effect of low-efficiency detectors in different schemes suitable for a direct measurement of the Q function and in optical homodyne tomography is studied in some detail. It turns out that this effect amounts to smoothing the respective quasiprobability distribution that would be measured with unit-efficiency detectors. Our main result is that those smoothed distributions can be identified with certain s-parametrized quasiprobability distributions. Thus the latter gain direct experimental significance as distributions measurable under realistic experimental conditions.

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I. INTRODUCTION

Quasiprobability distributions such as Glauber's P function, the Wigner function, and the Q function originally introduced into quantum optics as valuable mathematical tools [1], have now actually become accessible to measurements. First of all, this applies to the Q function which can be measured directly in various ways [2–5]. A tomographic scheme for an indirect determination of quasiprobability functions has been suggested by Vogel and Risken [6], and along these lines a pioneering experimental work was just recently performed by Smithey *et al.* [7].

It is well known that low-efficiency detectors deteriorate the results of such measurements. In the present paper, the role of the detector efficiency will be studied in some detail. Specifically, it will be shown that the distributions actually measured or tomographically reconstructed are smoothed quasiprobability distributions that can be identified with certain s-parametrized quasiprobability distributions [1].

The paper is organized as follows. In Sec. II a convenient equivalent model for the description of balanced homodyne detection using nonideal detectors is established. This is used in Sec. III to evaluate the measured distributions in case of beam splitting and amplification followed by, or combined with, beam splitting, and in Sec. IV to specify the quasiprobability distribution that is reconstructed, in realistic experiments, with the technique of optical homodyne tomography. Finally, the results are discussed in Sec. V.

II. BALANCED HOMODYNE DETECTION

Since the measurements we are dealing with are usually based on balanced homodyne detection, we will first describe this technique, thereby allowing for nonunit detection efficiencies η [8]. Balanced homodyne detection using a strong coherent field as a local oscillator was shown to provide a practical means of directly measuring the quadrature component of the field with respect to the local oscillator, provided unit-efficiency detectors are employed. Choosing the phase of the local oscillator as 0 or $\pi/2$, one thus can measure, in particular, the variables x and p (analogous to position and momentum), respectively, that appear as arguments in the quasiprobability distributions. (Strictly speaking, x and p are the quadrature components of the electric field strength, defined as

$$\hat{x}=2^{-1/2}(\hat{a}+\hat{a}^\dagger)\ ,\quad \hat{p}=-i2^{-1/2}(\hat{a}-\hat{a}^\dagger)\ , \tag{1}$$

where $\hat{a}^\dagger$ and $\hat{a}$ are the familiar creation and annihilation operators.) As is well known [9], a realistic detector can be modeled by an ideal one ($\eta=1$) with a partly transmitting mirror (a special form of an attenuator) in front of it. Hence, in the homodyne detection scheme such a fictitious beam splitter has to be placed before each of the two detectors [see Fig. 1(a)]. We will first show that this setup can actually be simplified: One beam splitter attenuating the signal before it is mixed with the local oscillator has the same effect [see Fig. 1(b)].

Let us start from the model with two independent attenuators. We denote the photon annihilation operators for the incident signal, the local oscillator, and the two mixed fields leaving the 50:50 beam splitter by $\hat{a}$, $\hat{a}_{\rm LO}$, $\hat{b}_1$, and $\hat{b}_2$, respectively. Then the action of the mixer can be described as follows:

$$\begin{aligned}\hat{b}_1&=2^{-1/2}(\hat{a}+\hat{a}_{\rm LO})\ ,\\ \hat{b}_2&=2^{-1/2}(-\hat{a}+\hat{a}_{\rm LO})\ .\end{aligned} \tag{2}$$

The subsequent damping processes give rise to the transformations [9]

$$\begin{aligned}\hat{b}_1'&=\cos\Theta\,\hat{b}_1+\sin\Theta\,\hat{c}_1\ ,\\ \hat{b}_2'&=\cos\Theta\,\hat{b}_2+\sin\Theta\,\hat{c}_2\ .\end{aligned} \tag{3}$$

Here $\hat{c}_1$ and $\hat{c}_2$ are the photon annihilation operators for the vacuum fields entering the respective beam splitter via the "unused" port, and the angle Θ is related to the detector efficiency η (assumed equal for both detectors) by

$$\cos^2\Theta=\eta\ . \tag{4}$$

Formally, it is very convenient to introduce operators representing the photocurrents produced by unit-efficiency broad-band detectors [10]

$$\hat{I}_1=\hat{b}_1'^{\dagger}\hat{b}_1' \ , \quad \hat{I}_2=\hat{b}_2'^{\dagger}\hat{b}_2' \ . \tag{5}$$

What is measured in balanced homodyne detection is the difference of the two photocurrents:

$$\Delta\hat{I}=\hat{b}_1'^{\dagger}\hat{b}_1'-\hat{b}_2'^{\dagger}\hat{b}_2' \ . \tag{6}$$

Making the usual assumption that the local oscillator is a very intense coherent field being in a Glauber state $|\alpha_{\rm LO}\rangle$, we may replace, to a good approximation, the annihilation operator $\hat{a}_{\rm LO}$ by the complex amplitude $\alpha_{\rm LO}$. With the help of Eqs. (2) and (3) we thus obtain from Eq. (6), retaining only the relevant terms which are linear in $\alpha_{\rm LO}$ and $\alpha^*_{\rm LO}$, the relation

$$\Delta\hat{I}=\alpha_{\rm LO}\cos\Theta[\cos\Theta\,\hat{a}+2^{-1/2}\sin\Theta(\hat{c}_1-\hat{c}_2)]+\text{H.c.}, \tag{7}$$

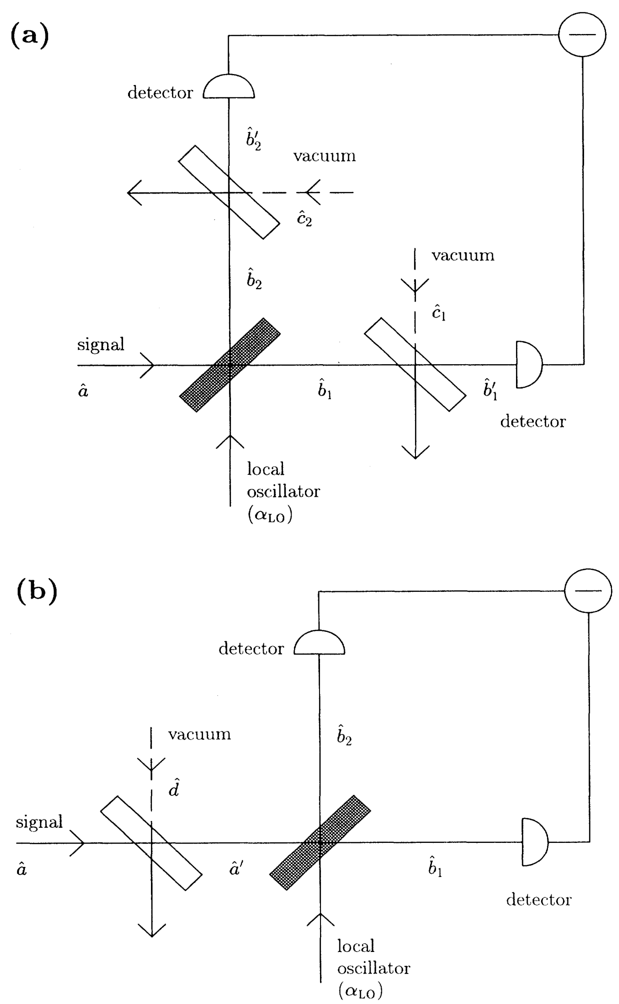

FIG. 1. (a) Balanced homodyne detection scheme with nonideal detectors modeled as unit-efficiency detectors with beam splitters placed in front of them; (b) equivalent model. Indicated are the photon annihilation operators associated with the different light modes involved.

from which one learns that the difference of the photocurrents in a homodyne detection scheme with unit-efficiency detectors ($\Theta=0$) is, in fact, a direct measure of the quadrature component of the field with respect to the local oscillator, as has been mentioned above. On the other hand, if just one attenuator is inserted into the incident signal beam [see Fig. 1(b)], the photon annihilation operator for the damped signal $\hat{a}'$ is connected with the photon annihilation operators for the incident field and the vacuum field entering the "unused" port of the beam splitter, $\hat{a}$ and $\hat{d}$, according to Eqs. (3), through the relation

$$\hat{a}'=\cos\Theta\,\hat{a}+\sin\Theta\,\hat{d} \ , \tag{8}$$

and Eqs. (2) with $\hat{a}$ replaced by $\hat{a}'$, describe the subsequent optical mixing with the local oscillator. For the difference of the two photocurrents we find now the expression

$$\Delta\hat{I}=\alpha_{\rm LO}(\cos\Theta\,\hat{a}+\sin\Theta\,\hat{d})+\text{H.c.}, \tag{9}$$

where again only the terms linear in $\alpha_{\rm LO}$ and $\alpha^*_{\rm LO}$ have been retained. Apart from the factor $\cos\Theta$ in front of the square bracket in Eq. (7) which is of no physical relevance, the two results (7) and (9) differ only in the fluctuation operators. However, since the latter act only on the vacuum state, and the combination $2^{-1/2}(\hat{c}_1-\hat{c}_2)$ corresponds to a unitary transformation, they in fact coincide in all their formal properties. Hence the two models for the description of homodyne measurements with the help of nonideal detectors are completely equivalent. This enables us to adopt the following procedure [see Fig. 1(b)]: In order to take properly into account detection efficiencies $\eta<1$, we formally attenuate the signal (by means of a fictitious beam splitter) before performing a conventional balanced homodyne measurement with unit-efficiency detectors [11].

III. MEASUREMENT OF THE Q FUNCTION

A. Beam splitting

It has been shown theoretically [3,5] that the Q function can be measured directly via beam splitting (provided the detectors have unit detection efficiency): A 50:50 beam splitter divides the signal into two coherent parts, and with the help of the balanced homodyne detection technique the two quadrature components of the electric-field strength, x_1' and p_2', are measured separately on the two beams [12]. The joint probability for those simultaneous measurements $w(x_1',p_2')$ turns out [3,5] to be just given by the Q function for the *initial* field. Formally, one can arrive at this result by first calculating the Wigner function for the whole field consisting of the split beam $P(x_1',x_2';p_1',p_2')$ and afterwards evaluating the marginal distribution with respect to x_1' and p_2' by averaging $P(x_1',x_2';p_1',p_2')$ over the remaining variables x_2' and p_1'. We wish to extend this scheme to allow for nonunit detection efficiencies.

Let us represent the state of the incident signal field by a Wigner function $P_1(x_1,p_1)$. The vacuum field entering

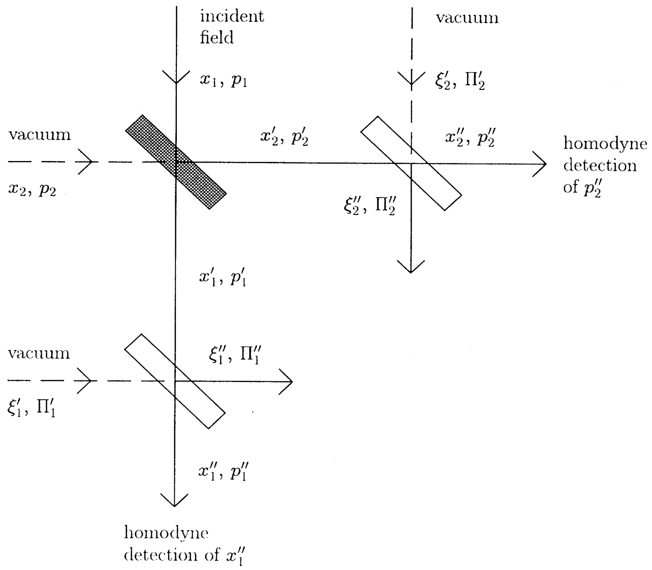

FIG. 2. Beam splitting followed by balanced homodyne detection with nonideal detectors in the equivalent model of Fig. 1(b). Indicated are the variables in the phase spaces associated with the different light modes involved.

the "unused" port of the beam splitter is characterized by the Wigner function

$$P_{\mathrm{vac}}(x_2,p_2)=\pi^{-1}\exp[-(x_2^2+p_2^2)] \ . \tag{10}$$

The action of a beam splitter with transmittivity $\cos^2\Theta$ is described by the following transformations [13]:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos\Theta & \sin\Theta \\ -\sin\Theta & \cos\Theta \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} , \quad \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} \cos\Theta & \sin\Theta \\ -\sin\Theta & \cos\Theta \end{bmatrix} \begin{bmatrix} p_1' \\ p_2' \end{bmatrix} , \tag{11}$$

where x_1', p_1' and x_2', p_2' are the quadrature components for the two outgoing beams. For the special case of a 50:50 beam splitter ($\Theta=\pi/4$), Eqs. (11) reduce to

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2^{-1/2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} , \quad \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = 2^{-1/2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} p_1' \\ p_2' \end{bmatrix} . \tag{12}$$

According to the equivalent model described in Sec. II, the nonunit efficiency of the detectors is taken properly into account by appropriately damping the fields *before* they enter the respective device for balanced homodyne detection (see Fig. 2). Hence the Wigner function of the total system (before detection) is given by

$$P_{\mathrm{out}}=P_1(x_1,p_1)P_{\mathrm{vac}}(x_2,p_2) \times P_{\mathrm{vac}}(\xi_1',\Pi_1')P_{\mathrm{vac}}(\xi_2',\Pi_2') \tag{13}$$

(for notation, see Fig. 2.). Here, the variables x_1, p_1, x_2, and p_2 are first to be substituted according to Eqs. (12), and afterwards the substitutions

$$\begin{bmatrix} x_k' \\ \xi_k' \end{bmatrix} = \begin{bmatrix} \cos\Theta & \sin\Theta \\ -\sin\Theta & \cos\Theta \end{bmatrix} \begin{bmatrix} x_k'' \\ \xi_k'' \end{bmatrix} , \quad \begin{bmatrix} p_k' \\ \Pi_k' \end{bmatrix} = \begin{bmatrix} \cos\Theta & \sin\Theta \\ -\sin\Theta & \cos\Theta \end{bmatrix} \begin{bmatrix} p_k'' \\ \Pi_k'' \end{bmatrix} (k=1,2) \tag{14}$$

have to be performed [see Eq. (11)]. Assuming that x_1'' and p_2'' are measured, we have to integrate over the remaining variables x_2'', ξ_1'', ξ_2'', p_1'', Π_1'', and Π_2'' in order to get the probability distribution $w(x_1'',p_2'')$ for this simultaneous measurement.

(i) Integration over x_2'', ξ_2'', p_1'', and Π_1'': We may interpret the appropriate equations in (14) as substitutions, and since the latter describe rotations, we may integrate over x_2', ξ_2', p_1', and Π_1' instead. Hence we may write

$$w(x_1'',p_2'')=\pi^{-3}\int P_1(x_1,p_1) \times \exp[-(x_2^2+p_2^2+\xi_1'^2+\Pi_1'^2+\xi_2'^2+\Pi_2'^2)]dx_2'd\xi_2'dp_1'd\Pi_1'd\xi_1''d\Pi_2'' \ . \tag{15}$$

(ii) Integration over ξ_1'', Π_2'', ξ_2', and Π_1': From Eqs. (12) and (14) we find the relations

$$x_1=\bar{x}+2^{-1/2}x_2' \ , \quad x_2=-\bar{x}+2^{-1/2}x_2' \ , \quad p_1=\bar{p}+2^{-1/2}p_1' \ , \quad p_2=\bar{p}-2^{-1/2}p_1' \ , \tag{16}$$

where the abbreviations

$$\bar{x}=2^{-1/2}(\cos\Theta\, x_1''+\sin\Theta\, \xi_1'') \ , \quad \bar{p}=2^{-1/2}(\cos\Theta\, p_2''+\sin\Theta\, \Pi_2'') \tag{17}$$

have been introduced. We integrate over $\bar{x}$ and $\bar{p}$ instead of ξ_1'' and Π_2''. Before doing so, we still have to express the variables ξ_1' and Π_2' occurring in Eq. (15) through $\bar{x}$ and $\bar{p}$. Utilizing the appropriate equations in (14), we obtain

$$\xi_1'=2^{1/2}\cot\Theta(\bar{x}-2^{-1/2}x_1''/\cos\Theta) \ , \quad \Pi_2'=2^{1/2}\cot\Theta(\bar{p}-2^{-1/2}p_2''/\cos\Theta) \ . \tag{18}$$

Noticing that the variables ξ_2' and Π_1' enter Eq. (15) only through the Gaussian $\exp[-(\xi_2'^2+\Pi_1'^2)]$, we can readily perform the corresponding integrations, and, by virtue of Eqs. (16)–(18), Eq. (15) reduces to

$$w(x_1'',p_2'')=2\pi^{-2}/\sin^2\Theta\int P_1(\tilde{x}+2^{-1/2}x_2',\tilde{p}+2^{-1/2}p_1')\exp[-(\tilde{x}-2^{-1/2}x_2')^2-(\tilde{p}-2^{-1/2}p_1')^2]$$
$$\times\exp\{-2\cot^2\Theta[(\tilde{x}-2^{-1/2}x_1''/\cos\Theta)^2+(\tilde{p}-2^{-1/2}p_2''/\cos\Theta)^2]\}dx_2'dp_1'd\tilde{x}\,d\tilde{p}\ . \tag{19}$$

Substituting here

$$x=\tilde{x}+2^{-1/2}x_2'\ ,\quad p=\tilde{p}+2^{-1/2}p_1' \tag{20}$$

gives us

$$w(x_1'',p_2'')=2\pi^{-2}/\sin^2\Theta\int P_1(x,p)\exp[-(x-2^{1/2}x_2')^2-(p-2^{1/2}p_1')^2]$$
$$\times\exp[-2\cot^2\Theta(x-2^{-1/2}x_2'-2^{-1/2}x_1''/\cos\Theta)^2]$$
$$\times\exp[-2\cot^2\Theta(p-2^{-1/2}p_1'-2^{-1/2}p_2''/\cos\Theta)^2]dx_2'dp_1'dx\,dp\ . \tag{21}$$

(iii) Integration over x_2' and p_1': Since only Gaussians are involved, those integrations can easily be done, and after a little algebra we arrive at the final result

$$w(x_1'',p_2'')=2\pi^{-1}(2-\eta)^{-1}\int P_1(x,p)\exp\left\{-\frac{\eta}{2-\eta}[(x-2^{1/2}\eta^{-1/2}x_1'')^2+(p-2^{1/2}\eta^{-1/2}p_2'')^2]\right\}dx\,dp\ , \tag{22}$$

where Eq. (4) has been observed. While for $\eta=1$ the right-hand side of Eq. (22) is, apart from a factor of 2, just the Q function for the initial field $Q_1(2^{1/2}x_1'',2^{1/2}p_2'')$ in accordance with Refs. [3,5], the Wigner function for the initial field $P_1(x,p)$ becomes more strongly smoothed for $\eta<1$, which is just what one expects. It is interesting to note that integrals of type (22)—the convolution of the Wigner function with a Gaussian—are special cases of the so-called s-parametrized quasiprobability distributions $W(x,p;s)$ introduced by Cahill and Glauber [1]. Quite generally, those distributions are connected through the following relation [14]:

$$W(x',p';s)=\pi^{-1}(t-s)^{-1}\int W(x,p;t)\exp\{-(t-s)^{-1}[(x'-x)^2+(p'-p)^2]\}dx\,dp\ . \tag{23}$$

Specializing to $t=0$, from Eq. (23) we obtain

$$W(x',p';s)=-\pi^{-1}s^{-1}\int P(x,p)\exp\{[(x'-x)^2+(p'-p)^2]/s\}dx\,dp\ , \tag{24}$$

where $P(x,p)=W(x,p;0)$ is the Wigner function. Hence our result (22) can be rewritten in the compact form

$$w(x_1'',p_2'')=2\eta^{-1}W_1(2^{1/2}\eta^{-1/2}x_1'',2^{1/2}\eta^{-1/2}p_2'';-(2-\eta)/\eta)\ , \tag{25}$$

where $W_1(x,p;s)$ denotes the s-parametrized quasiprobability distribution for the initial field.

B. Amplification

In the context of an operational definition of phase, it has been shown [4] that strong amplification with the help of either a laser amplifier with all atoms initially excited or an optical parametric amplifier [15] also enables one to measure directly the Q function of the initial field. Also in this case we will study the effect of nonunit-efficiency detectors on the measurements.

Let us first calculate the Wigner function for the amplified field. Let the initial field be injected into a parametric amplifier as a signal field, and the idler field be initially in the vacuum state. Hence, just as in the case of beam splitting, the total Wigner function before amplification is the product of the Wigner function for the signal $P_1(x_1,p_1)$ and the Wigner function (10). To describe amplification, the former relation (12) has to be replaced by [16]

$$\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}C & -S\\-S & C\end{pmatrix}\begin{pmatrix}x_1'\\x_2'\end{pmatrix}\ ,\quad \begin{pmatrix}p_1\\p_1\end{pmatrix}=\begin{pmatrix}C & S\\S & C\end{pmatrix}\begin{pmatrix}p_2'\\p_2'\end{pmatrix}\ . \tag{26}$$

Here, the following abbreviations have been introduced:

$$S=\sinh\kappa T\ ,\quad C=\cosh\kappa T\ , \tag{27}$$

where κ denotes the effective coupling constant and T the interaction time. Hence the Wigner function for the amplified signal field is given by

$$P_{\rm ampl}(x_1',p_1')=\pi^{-1}\int P_1(x_1,p_1)\exp[-(x_2^2+p_2^2)]dx_2'dp_2'\ . \tag{28}$$

It is advantageous to integrate over x_1 and p_1 instead of x_2' and p_2'. From Eqs. (26) we find

$$x_2=(x_1'-Cx_1)/S\ ,\quad p_2=(-p_1'+Cp_1)/S\ , \tag{29}$$

and hence Eq. (28) can be written as

$$P_{\rm ampl}(x_1',p_1')=\pi^{-1}S^{-2}\int P_1(x_1,p_1)\exp\{-C^2S^{-2}[(x_1-x_1'/C)^2+(p_1-p_1'/C)^2]\}dx_1dp_1 \tag{30}$$

or, according to Eq. (24),

$$P_{\rm ampl}(x_1',p_1')=C^{-2}W_1(x_1'/C,p_1'/C;-C^2/S^2)\ . \tag{31}$$

The amplified signal field will be taken as the input for the detection scheme studied in Sec. III A (see Fig. 2). Then we need only make use of the former result (22) in order to obtain the measured distribution function $w(x_1'',p_2'')$. Insertion of result (31) into Eq. (22), and substituting $x_1'/C=x,p_1'/C=p$, gives us

$$w(x_1'',p_2'')=2\pi^{-1}(2-\eta)^{-1}\int W_1(x,p;-C^2/S^2)\times\exp\left\{-\frac{\eta C^2}{2-\eta}[(x-2^{1/2}\eta^{-1/2}x_1''/C)^2+(p-2^{1/2}\eta^{-1/2}p_1''/C)^2]\right\}dx\,dp\ . \tag{32}$$

Utilizing the general formula (23), we arrive at the final result

$$w(x_1'',p_2'')=2\eta^{-1}C^{-2}W_1(2^{1/2}\eta^{-1/2}x_1''/C,2^{1/2}\eta^{-1/2}p_1''/C;-[1+2(1-\eta)\eta^{-1}C^{-2}])\ , \tag{33}$$

which clearly exhibits the deteriorating effect of nonunit-efficiency detectors. It becomes evident, however, from Eq. (33) that one can tolerate low detector efficiencies when there is strong enough amplification ($C\gg 1$).

Whereas the foregoing analysis applies to both laser (with completely inverted atoms) and parametric amplifiers, the latter device allows for a simplification of the measuring scheme: The two outputs (signal and idler) can be taken as well as the two inputs for the homodyne measurements as the split (amplified) signal beam so that beam splitting becomes, in fact, superfluous. Operated in this way, the parametric amplifier combines amplification and beam splitting. The theoretical treatment of this scheme follows closely the lines indicated in Sec. III A, with the difference, however, that Eqs. (12) characteristic of beam splitting have to be replaced by Eqs. (26) describing the action of a parametric amplifier. In this way, one finds the measured distribution function to be given by

$$w(x_1'',p_2'')=\pi^{-1}[1+(C^2-2)\eta]^{-1/2}[1+(C^2-1)\eta]^{-1/2}\times\int P_1(x,p)\exp\left\{-\frac{C^2\eta}{1+(C^2-2)\eta}(x-\eta^{-1/2}x_1''/C)^2\right\}\exp\left\{-\frac{(C^2-1)\eta}{1+(C^2-1)\eta}(p+\eta^{-1/2}p_2''/S)^2\right\}dx\,dp\ . \tag{34}$$

Obviously, the smoothing process is not symmetric with respect to x_1'' and p_2''. This is not surprising, since the intensity of the amplified idler is, in fact, lower than that of the amplified signal, since the former starts from the vacuum state. However, the aforementioned asymmetry virtually vanishes in case of strong amplification, $C\gg 1$. Then Eq. (34) reduces to the simpler form

$$\begin{aligned}w(x_1'',p_2'')&=\pi^{-1}(1+C^2\eta)^{-1}\int P_1(x,p)\exp\left\{-\frac{C^2\eta}{1+C^2\eta}[(x-\eta^{-1/2}x_1''/C)^2+(p+\eta^{-1/2}p_2''/C)^2]\right\}dx\,dp\\&=\eta^{-1}C^{-2}W_1(\eta^{-1/2}x_1''/C,-\eta^{-1/2}p_2''/C;-[1+\eta^{-1}C^{-2}])\ .\end{aligned} \tag{35}$$

As expected, this formula is similar to result (33). In particular, one learns from Eq. (35) that the deteriorating effect of low-efficiency detectors can be compensated for by strong amplification.

IV. EXPERIMENTAL DETERMINATION OF THE WIGNER FUNCTION

In a recent theoretical study [6], Vogel and Risken proposed a feasible tomographic scheme that allows one to reconstruct quasiprobability distributions from sets of data obtained by homodyne measurements. What one has to do is to carry out homodyne measurements on the field under investigation utilizing a strong reference field whose phase Θ is gradually varied from 0 to π. From the measured probability distributions $w(x_\Theta,\Theta)$ the corresponding quasiprobability function $W(x,p;s)$ can be determined through numerical inversion. In the physically relevant case of the Wigner function corresponding to $s=0$ the reconstruction is accomplished by the inverse Radon transformation, and very recently the Wigner functions for the vacuum state and a squeezed vacuum state were indeed determined experimentally [7] along the lines indicated by Vogel and Risken [6]. In the following, we will clarify the role low detection efficiency plays in this measuring scheme which has been termed optical homodyne tomography [7].

As was shown in Sec. II, nonunit detection efficiencies can properly be taken into account by attenuating the signal with the help of a fictitious beam splitter placed before the setup for homodyne detection [see Fig. 1(b)]. Hence, what is reconstructed in optical homodyne tomography is the Wigner function of the *attenuated* field. Its connection with the Wigner function of the initial field is readily established. Applying the general scheme for the description of beam splitting presented in Sec. III A, we obtain the total Wigner function of the field emerging from the beam splitter in the form

$$P_{\text{out}}(x_1',x_2';p_1',p_2')=P_1(x_1,p_1)P_{\text{vac}}(x_2,p_2) , \qquad (36)$$

where $P_1(x_1,p_1)$ and $P_{\text{vac}}(x_2,p_2)$ [given by Eq. (10)] are the Wigner functions of the incident beam and the vacuum field entering the "unused" port of the beam splitter, respectively. Here, the arguments on the right-hand side have to be substituted according to Eqs. (11). When the measurement is performed on the transmitted beam, we have to integrate over x_2' and p_2' in order to get the Wigner function $P_{\text{out}}(x_1',p_1')$ relevant for the experiment in question.

Considering the equations for x_1 and p_1 in Eqs. (11) as substitutions, we integrate over x_1 and p_1 rather than x_2' and p_2'. The variables x_2 and p_2 are then easily expressed through x_1 and p_1:

$$x_2=\cot\Theta(x_1-x_1'/\cos\Theta) , \quad p_2=\cot\Theta(p_1-p_1'/\cos\Theta) . \qquad (37)$$

Hence $P_{\text{out}}(x_1',p_1')$ is given by

$$P_{\text{out}}(x_1',p_1')=\int P_{\text{out}}(x_1',x_2';p_1',p_2')dx_2'dp_2'$$
$$=\pi^{-1}(1-\eta)^{-1}\int P_1(x_1,p_1)\exp\left\{-\frac{\eta}{1-\eta}[(x_1-\eta^{-1/2}x_1')^2+(p_1-\eta^{-1/2}p_1')^2]\right\}dx_1dp_1 , \qquad (38)$$

where Eq. (4) has been observed. The general formula (24) enables us to rewrite the result (38) in the simple form

$$P_{\text{out}}(x_1',p_1') =\eta^{-1}W_1(\eta^{-1/2}x_1',\eta^{-1/2}p_1';-[1-\eta]/\eta) , \qquad (39)$$

from which the deteriorating effect of low-efficiency detectors becomes obvious.

V. DISCUSSION

In the foregoing sections it has been shown that the use of nonunit-efficiency detectors results in an additional smoothing of the quasiprobability distributions measured or reconstructed from measurements. While this is certainly what one expects, the interesting point is that our analysis revealed that the smoothing process in question can properly be described in terms of the well-known s-parametrized quasiprobability functions: With their arguments appropriately scaled, those quasiprobabilities represent directly the respective measured distributions, whereby the actual value of the parameter s is determined by the detection efficiency η. The η dependence of s characterizes completely the deterioration of the measurement due to the employment of nonunit-efficiency detectors. Actually, the smoothing process has been shown to be a convolution with a Gaussian which is known when the detector efficiency is known. Hence, all the information contained in the true quasiprobability distribution is, in principle, preserved in the smoothed one. However, the deconvolution needed for reconstruction of the true distribution, since it involves the amplification (exponential increase) of noise in the smoothed distribution according to Eq. (23), can be performed satisfactorily only when the smoothed distribution is known in the whole (x,p) plane with virtually unlimited accuracy which is unattainable in actual measurements. So the effects of losses can be overcome only in principle, but in practice this will be extremely difficult to achieve. Let us now discuss our results in some detail.

A. Beam splitting

According to the result (25), the parameter s is given by

$$s=-(2-\eta)/\eta . \qquad (40)$$

It is only for $\eta=1$ that it equals -1, i.e., that one measures the Q function. For $\eta<1$ a smoothed Q function ($s<-1$) will be measured.

B. Amplification followed by beam splitting

Equation (33) gives us

$$s=-[1+2(1-\eta)\eta^{-1}C^{-2}] . \qquad (41)$$

First, one learns from Eq. (41) that for $\eta=1$ s takes the value -1, irrespective of whether the signal is amplified or not. This result is, in fact, not surprising since it is well known in the limiting cases of no amplification, $C=1$ [see Refs. [3,5] and Eq. (40)] and very strong amplification, $C\gg1$ (see Ref. [4]). (Note that in the latter case the field is amplified to a macroscopic level, where the noise introduced through beam splitting becomes negligible.) The physically relevant implication of Eq. (41), however, is that one can actually measure the Q function of the original field with low-efficiency detectors when strongly amplifying the field before detection. To be more quantitative, strong amplification means, according to Eq. (41), that the gain (with respect to intensity) C^2 should obey the inequality

$$2(1-\eta)/\eta\ll C^2 , \qquad (42)$$

which for $\eta=0.5$, e.g., requires in fact a moderate gain, $C^2\gg5$, to be fulfilled.

C. Parametric amplification

This scheme is of practical interest mainly in the case of strong amplification, $C \gg 1$. Then, according to Eq. (35), the parameter s takes the value

$$s = -(1+\eta^{-1}C^{-2}) . \quad (43)$$

As in the amplification scheme considered above, the Q function can be measured directly with the help of nonideal detectors provided that the gain is strong enough. The former inequality (42) now has to be replaced by

$$\eta^{-1} \ll C^2 , \quad (44)$$

which coincides with (42) for $\eta = 0.5$ and is weaker, by a factor of approximately 2, for $\eta \ll 1$.

Thus, from the experimental point of view, the amplification schemes prove to be superior compared to beam splitting of the original field which requires unit-efficiency detectors for measuring the Q function.

D. Reconstruction of the Wigner function

One learns from result (39) that optical homodyne tomography carried out with nonideal detectors does not allow in principle, reconstruction of the original Wigner function. What is actually reconstructed is the Wigner function of the *attenuated* field, i.e., a smoothed Wigner function corresponding to the s parameter

$$s = -(1-\eta)/\eta . \quad (45)$$

Putting $\eta = 0.5$, e.g., in Eq. (45), one recognizes that in this case one actually retrieves the Q function.

In this context it should be noticed that in general the inverse Radon transformation in the form of the standard filtered back-projection algorithm for parallel beam sampling geometry [17] yields not the true Wigner function even for $\eta = 1$, since the filtering removes high-frequency components. From the practical viewpoint, filtering sets the resolution with which the Wigner function can be determined from the experimental data.

In summary, we have shown that the recently discussed and partly already realized schemes for measuring the Q function and reconstructing the Wigner function for a given single-mode field allow to determine, in realistic experiments, only smoothed distributions, as a consequence of the use of nonunit-efficiency detectors. Those smoothed distributions could be identified with certain s-parametrized quasiprobability distributions, so that the deteriorating effect of low-efficiency detectors is fully characterized by the respective dependence of s on the detector efficiency [8].

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