Parametrization of radiative-recombination cross sections

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Using exact analytical expressions of the non relativistic dipole recombination cross sections for a pure Coulomb potential summed over the quantum numbers j and l, we numerically compute the recombination cross sections in the various n states for $1 \leq n \leq 500$ and the total cross section for all possible values of the incoming electron energy. We present a parametrization of the recombination cross sections in the various n states, of the total cross section, and of the energy emission cross section. A few low- and high-energy approximations are also derived and discussed.

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I. INTRODUCTION

The study of radiative recombination of elementary particles in the Coulomb field of a nucleus or an ion, particularly in the relatively unexplored (both theoretically and experimentally) very low energy region, is quite important in several different fields of physics: particle accelerators, plasma physics, astrophysics, both normal and exotic atoms, antimatter production, and laser-induced recombination.

As an example in the field of particle accelerators we consider the electron cooling of proton beams. When a proton captures an electron, a traveling neutral atom is formed; information on electron and proton beams can be obtained by observing such neutral atoms [1–5]. Let us remember that the cooling mechanism of protons by electrons consists of a redistribution of energy during the collision of the two gases; usually the energies of the incident particles are nonmonochromatic. Recently the subject of laser-induced recombination has been investigated both theoretically and experimentally with particular attention to the enhancement of recombination cross sections with moderately high n states [6–10], and also very recently the first measurements performed at Gesellschaft für Schwerionenforschung Darmstadt (GSI) of radiative recombination of very highly charged ions at very low energies have been published [11].

In plasma physics, when an electron emits a photon, with the heavy ion absorbing the momentum, the electron energy decreases and free-free or free-bound transitions can be observed; in the latter case an electron can be captured into any level of total quantum number n. The study of the recombination radiation is important in determining the rate at which positive ions, for instance, recapture electrons, although it is usually believed that at higher temperatures recombination radiation is negligible compared to bremsstrahlung [12–16].

In astrophysics, in addition to the application of recombination to opacity and to the traditional astrophysical problems, a few new applications have been proposed regarding recombination of primeval plasma in the presence of light gauginos and Higgsinos and recombination-induced stellar axion production [17–19].

In the study of exotic atoms, and particularly when studying the slowing-down, the Coulomb capture, and the electromagnetic cascade of negative mesonic and hadronic particles, there is great interest in the radiative-capture rate [20–24]. In addition, we wish to recall that radiative recombination is of great importance also in muon-catalyzed fusion [25] and antimatter production [26,27].

Using the nonrelativistic dipole cross section in a pure Coulomb potential $\sigma_{nlj}(v,\omega_n)$ for the capture of a negative particle into the atomic orbit (n,l,j) that we have calculated exactly [28,29], we may calculate measurable quantities like the rate coefficient:

$$\alpha = \int v f(\mathbf{v}) \sum_n \sum_{l,j} \sigma_{nlj}(\tilde{v},\omega_n) d^3v, \tag{1.1}$$

where $\hbar\omega_n$ is the energy of the photon emitted in the free-bound transition taking into account the recoil term, $\mathbf{v}$ is the velocity of the captured particles in the rest frame of the negative-particle gas (for instance the electron gas), $f(\mathbf{v})$ is the statistical distribution of velocities, $\tilde{\mathbf{v}}$ is the velocity referred to the center of mass of the captured and capturing particles: $\tilde{\mathbf{v}} = \mathbf{v}_p - \mathbf{v}_{\text{c.m.}}$ and $\mathbf{v}_p$ is the velocity of the positive particle (i.e. a proton) with respect to the rest frame of the negative particles (i.e., the electron gas). In the electron-proton atomic capture, the c.m. reference system coincides with the stationary proton system. Therefore, in this case, the velocity $\tilde{v}$ in Eq. (1.1) may be substituted by v:

$$\alpha = \langle v\sigma(v)\rangle. \qquad (1.2)$$

The hardest problem in the evaluation of α is to obtain a compact sum over n, l, and j of $\sigma_{nlj}(v,\omega_n)$; the difficulties involved in this particular topic were discussed in an early work by Stückelberg and Morse [30], by Stobbe [31], by Brussard and Van de Hulst [32]; recently the recombination cross sections and rate coefficients into specific energy levels have been studied by Omidvar and Guimaraes [33] and by Pajek and Schuch [34,35]. The total recombination cross section

$$\sigma(v) = \sum_{n,l,j} \sigma_{nlj}(v,\omega_n)$$

has been derived by Milstein [36], but his result is still an expression involving the principal part of an integral over the nonconfluent hypergeometric function; the total cross section is a partial sum because the sum is only over the discrete levels of the spectrum. We obtained directly [29] only a sum rule over the complete (discrete plus continuum) spectrum of levels. For the sum over l we wish to recall the works of Menzel and Pekeris [37], Seaton [38], and Burgess [39–42].

Usually the quantity

$$\sigma_n(v) = \sum_{l,j} \sigma_{lj}(v,\omega_n)$$

is taken to behave like n^{-3} at high energy. For low energies, $\epsilon \ll Z^2/n^2$ [$\epsilon = E_k/E_{1s}$; E_k is the incident kinetic energy in the c.m. system, E_{1s} is the ground-state energy defined in Eq. (2.2)], the cross section is taken to be proportional to n^{-1}; the dependence of the cross section on the principal quantum number n is n^{-1} (for $n \le n_{\max}$) and n^{-3} (for $n > n_{\max}$), with $n_{\max} = (Z^2/\epsilon)^{1/2}$. This gives, of course, assurance of convergence of the sum over n of σ_n, except for $E_k = 0$. For $\epsilon \gg 1$ the behavior of $\sigma_n(\epsilon)$ with respect to n is n^{-3}. This is also relevant in beam-foil spectroscopy when highly excited hydrogen atoms are formed. Different authors have determined the population of such atoms in states with principal quantum number n together with the scaling law with respect to n [43–45]. These high-energy experiments have established the validity of the n^{-3} scaling law. At low ϵ there is a very wide area in which $\sigma_n(\epsilon)$ behaves very differently from n^{-1} or n^{-3}. We have been able to calculate exactly the cross sections in that region, and we have also derived a parametrization of the cross sections for all n and ϵ, and a very precise parametrization of the total cross section which can be useful in the calculation of the rates.

Taking advantage of our exact calculation of $\sigma_{nlj}(\epsilon)$ we are able to show that the behavior of $\sigma_n(\epsilon)$ is $n^{-X(n,\epsilon)}$ with $X(n,\epsilon)$ being a family of simply behaved functions with respect to n for fixed values of ϵ. This scaling law is in our opinion easier to use than the formulas by Sobelman [46,47] containing the Gaunt factors $g_n(\epsilon)$, and looking at the behavior of $X(n,\epsilon)$ one gets immediately the behavior of the cross section.

In Sec. II of our work we report the exact values of $\sigma_n(\epsilon)$ we calculated and we give a parametrization of the cross section $\sigma_n(\epsilon)$. In Sec. III we report the exact values of the total cross section $\sigma(\epsilon)$ and give a parametrization of $\sigma(\epsilon)$. In Sec. IV we deal with the energy emitted in the recombination process and give a scaling law that allows one to compute easily the energy emitted. In Sec. V we give an analytical expression of $\sigma_{n,l}(\epsilon)$ in the low-energy limit, and report the details of the calculation in Appendix A. In Sec. VI we discuss some approximations useful to calculate the recombination rates in the low-energy cases. In Sec. VII we give an analytical expression of $\sigma_{n,l}(\epsilon)$ in the high-energy low-n limit and in the high-energy high-n limit; the details of the calculation are in Appendixes B and C, respectively. The conclusions are in Sec. VIII.

II. BEHAVIOR OF THE CROSS SECTION SUMMED OVER j AND l

It is well known that, in terms of the Gaunt factors $g_n(\epsilon)$, the cross section $\sigma_n(\epsilon)$ is

$$\sigma_n(\epsilon) = \frac{8\pi}{3\sqrt{3}} Z^4 \frac{(\hbar c)^2}{(n^{-2}+\epsilon)\epsilon} \left(\frac{e^2}{\hbar c}\right)^5 \frac{g_n(\epsilon)}{E_{1s}^2 n^3}, \qquad (2.1)$$

$$E_{1s} = \frac{1}{2}\left(\frac{Ze^2}{\hbar c}\right)^2 Mc^2, \qquad (2.2)$$

$$\epsilon = \frac{E_k}{E_{1s}}, \qquad (2.3)$$

where M is the reduced mass and E_k the c.m. kinetic energy. The numerical values of the Gaunt factors are usually obtained either from their curves, given by Karzas and Latter [48], or from their tables or their asymptotic expansions as given by Menzel and Pekeris [37], corrected by Burgess [39,40]. Recently other authors [49,50] claim to have obtained a better expansion valid at low ϵ and large n and claim also that the same expansion fits the low n Gaunt factors with an error of less than a few percent. We report here the analytical expression of the factor $g_1(\epsilon)$, which is also the only one of the Gaunt factors to have a simple exact analytical expression:

$$g_1(\epsilon) = 8\pi\sqrt{3}\frac{1}{\epsilon+1}\frac{\exp\left(-4\frac{\arctan\sqrt{\epsilon}}{\sqrt{\epsilon}}\right)}{1-\exp\left(-\frac{2\pi}{\sqrt{\epsilon}}\right)}. \qquad (2.4)$$

The quantity $g_n(\epsilon)$ is a very complicated function of ϵ. $g_n(\epsilon)$ as we will see later depends on nonconfluent hypergeometrical functions of a complex variable. Usually one takes $g_n(\epsilon) = 1$ with an error of (10–20)%. Approximated treatments are very good. However, we are able to calculate by means of the Jacobi polynomials the hypergeometrical functions of a complex variable [29] and therefore an exact numerical evaluation of $g_n(\epsilon)$ can be directly computed. To be more clear we think that, in the evaluation of the radial matrix elements, it is much better to use Gordon's integrals giving an analytical expression in terms of the nonconfluent hypergeometrical functions $F(z)$. To calculate numerically the radial integrals is not convenient, due to the oscillations of the functions to be integrated: in this case $\sigma_{nlj}(\epsilon)$ might be affected by a large error (sometimes 50%).

Let us find the behavior of

$$\sigma_n(\epsilon) = \sum_{l,j} \sigma_{nlj}(\epsilon)$$

as a function of n and ϵ. One can sum $\sigma_n(\epsilon)$ of Eq. (2.1) over n, taking $g_n(\epsilon) = 1$. However, this is not correct and this assumption can produce errors in the numerical evaluation of $\sigma_n(\epsilon)$ of about one order of magnitude for $\epsilon \leq 10^{-2}$. We wish to adopt an easy-to-use scaling law with respect to n for the quantity $\sigma_n(\epsilon)$, in order to carry out the sum over n of $\sigma_n(\epsilon)$, and to perform an analytical calculation of the recombination coefficients valid for any n and any reduced mass. Its formal expression is

$$\sigma_n(\epsilon) = \frac{\sigma_{1s}(\epsilon)}{n^{X(n,\epsilon)}}. \tag{2.5}$$

We can compute $\sigma_n(\epsilon)$ and therefore $X(n,\epsilon)$ for n as high as $n = 500$ in the case of ϵ of the order 10^{-4}, while we can reach at least $n = 200$ for any ϵ. The quantity $X(n,\epsilon)$ can be obtained exactly at $\epsilon = 0$ for any n using the exact relation:

$$X(n,\epsilon) = \frac{\ln\left[\frac{g_1(\epsilon)}{g_n(\epsilon)}\right]}{\ln n} + 1 + \frac{\ln\left[\frac{(1+\epsilon n^2)}{(1+\epsilon)}\right]}{\ln n} \tag{2.6}$$

because we know the exact value of $g_n(0)$ given by Costescu and co-workers [49,50]:

$$g_n(0) = 1 - 0.172\,825 n^{-\frac{2}{3}} - 0.016\,530 n^{-\frac{4}{3}} + 0.005\,714 n^{-2}. \tag{2.7}$$

We are very confident of the precision of our calculations because the method we used is based on the calculation of the Jacobi polynomials which are very stable; the fact that we stopped at $n = 500$ is entirely due to computing overflow, therefore with a bigger computer we can reach values of n much higher than $n = 500$. The function $X(n,\epsilon)$ is a smooth function of n and ϵ. For any value of ϵ, $X(n,\epsilon) \to 3$ for $n \to \infty$, except for $\epsilon = 0$ in which case $X(n,0) \to 1$ for $n \to \infty$. We think it is useful to report the plot of the function $X(n,\epsilon)$ as a function of n for different values of ϵ. In fact, knowing $\sigma_{1s}(\epsilon)$ [which comes immediately from Eqs. (2.1) and (2.4)] and $X(n,\epsilon)$, one can easily calculate $\sigma_n(\epsilon)$ for any n. In Fig. 1 we report the plot of $X(n,\epsilon)$ for several values of ϵ ranging from $\epsilon = 10^{-5}$ to $\epsilon = 10^3$ and up to $n = 50$. In Fig. 2 we plot $X(n,\epsilon)$ up to $n = 500$ for fewer values of ϵ. In Fig. 3 we plot $X(n,\epsilon)$ as a function of ϵ for several values of n from $n = 2$ to $n = 100$. In Fig. 4 we concentrate on the very low energy region; we have in particular that $X(n,\epsilon = 0) \leq 1$ for all n, and $X(n = \infty, \epsilon = 0) = 1$. Of particular importance is the curve with $n = 2$, for which we found a simple parametrization, which is given by

$$X(2,\epsilon) = 0.852\,716 + 2.147\,284\left(1 - \frac{1}{1.50\epsilon + 0.05\sqrt{\epsilon} + 1}\right). \tag{2.8}$$

It is clearly visible in Fig. 3 that two different regimes are present: at low ϵ, $X(n,\epsilon) \simeq 1$, while at high ϵ, $X(n,\epsilon) \simeq 3$. We found a precise parametrization of $X(n,\epsilon)$, which is given by

$$X(n,\epsilon) = X(2,\epsilon)\delta_{n,2} + \theta(n-3)\left[1 + \frac{1}{\ln n}\ln\left(G(n,\epsilon)\frac{1+\epsilon n^2}{1+\epsilon}\right)\right], \tag{2.9}$$

where $X(2,\epsilon)$ is given by Eq. (2.8), $\delta_{n,2}$ is the Kronecker delta, and

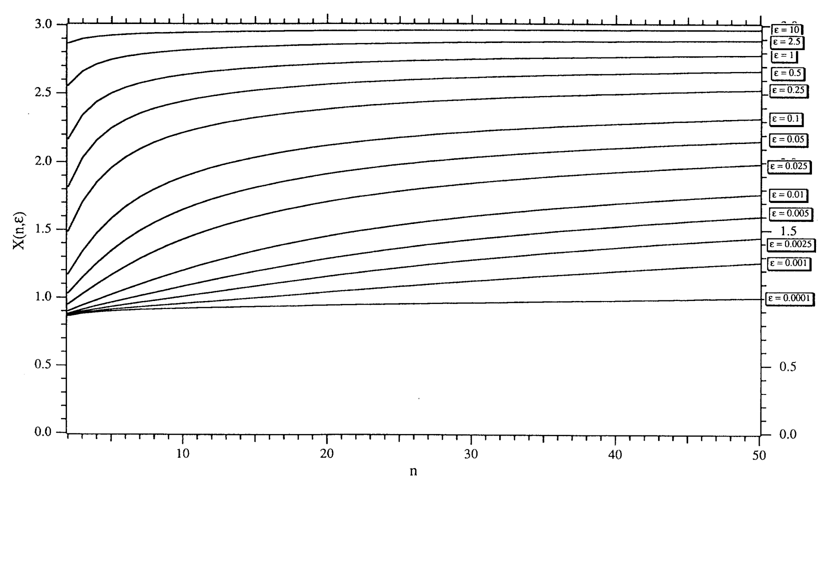

FIG. 1. The exact function $X(n,\epsilon)$ vs n from $n = 2$ to $n = 50$ for several values of ϵ ranging from 10^{-4} to 10. We want to stress that these are exact values of $X(n,\epsilon)$, evaluated from the exact values of the recombination cross sections $\sigma_{n,l}(\epsilon)$.

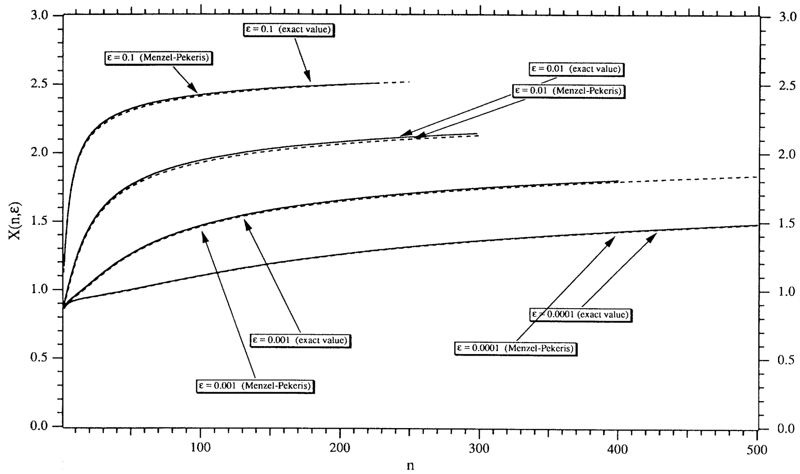

FIG. 2. The exact function $X(n,\epsilon)$ vs n up to $n = 500$. The curves corresponding to four different values of ϵ are shown. If we got computing overflow for a certain curve (i.e. $\epsilon = 10^{-2}$ and $\epsilon = 10^{-1}$), we indicate the value of n for which the overflow occurred. The exact values of $X(n,\epsilon)$ are compared to the Menzel-Pekeris values.

$$\theta(n-3) = \begin{cases} 0 & \text{if } n-3 < 0 \\ 1 & \text{if } n-3 \geq 0, \end{cases} \tag{2.10}$$

$$G(n,\epsilon) = \begin{cases} \left[1.1111 + 0.1920(\epsilon n)^{1/3} - 0.0551(\epsilon n)^{2/3}\right]^{-1} & \text{if } \epsilon n \leq 100; \\ 1 & \text{if } \epsilon n > 100. \end{cases} \tag{2.11}$$

In Figs. 5 and 6 we compare our parametrization of $X(n,\epsilon)$ to the exact values of $X(n,\epsilon)$, and can see that the parametrization (2.9) is in very good agreement with the exact values (the largest error we make is of the order of 1% for $\epsilon < 10^{-3}$ and n small; in all other cases the error is much smaller than 1%). We have also calculated $X(n,\epsilon)$ using the asymptotic expansion of the Gaunt factors given by Menzel and Pekeris, which leads to a more complicated form of $X(n,\epsilon)$ (see Fig. 2). The precision of our parametrization is very useful when calculating with a high level of accuracy the recombination rates in the various n states.

III. TOTAL CROSS SECTION

In many applications one has to know the total recombination cross section, which is defined as

$$\sigma(\epsilon) = \sum_{n=1}^{\infty} \frac{\sigma_{1s}(\epsilon)}{n^{X(n,\epsilon)}}. \tag{3.1}$$

Therefore we have to compute the function

$$s(\epsilon) = \sum_{n=1}^{\infty} \frac{1}{n^{X(n,\epsilon)}}. \tag{3.2}$$

In Fig. 7 we report the plot of the exact values of $s(\epsilon)$. We want to stress that, in computing the sum of the cross sections, we have used the exact values of $X(n,\epsilon)$

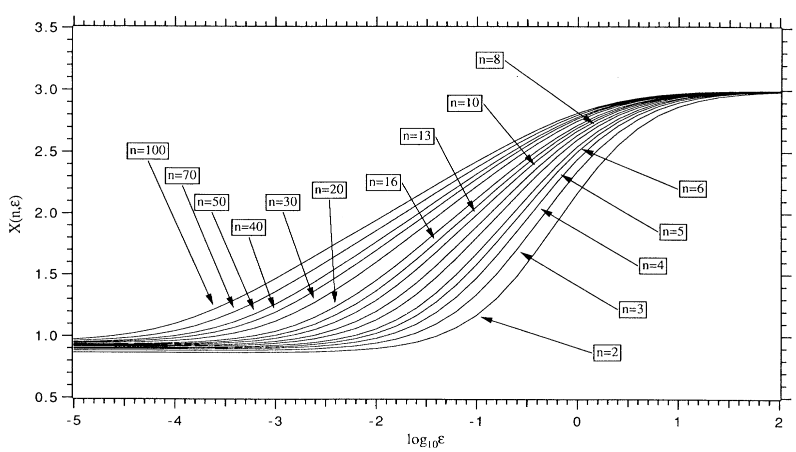

FIG. 3. The exact function $X(n,\epsilon)$ vs $\log_{10}\epsilon$ is plotted for several values of n. The two different regimes at low and high ϵ are clearly visible. It is important to point out that we do not report the curves with $n > 100$ because of the extensive computer time needed to calculate them, but a very good approximation can be obtained either using our parametrization of $X(n,\epsilon)$ [Eq. (2.9)], or using the parametrization of $X(n,\epsilon)$ with the Gaunt factors. We can say that for $n \to \infty$ $X(n,\epsilon)$ tends to a curve that is equal to 3 everywhere except at $\epsilon = 0$ where it equals 1, and therefore it has a discontinuity at $\epsilon = 0$.

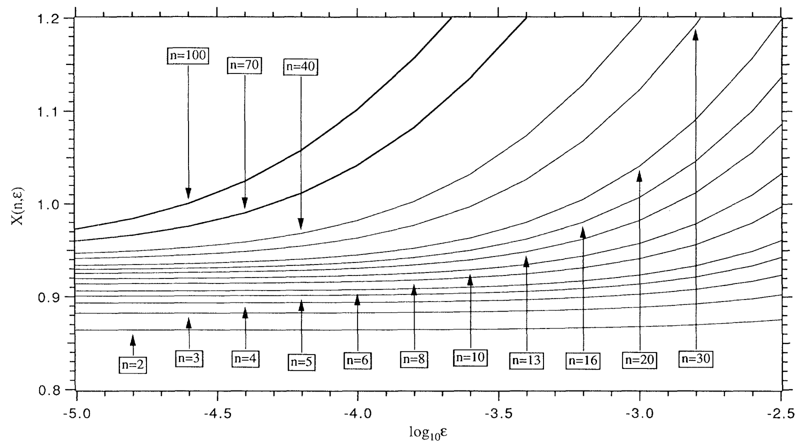

FIG. 4. The exact function $X(n,\epsilon)$ vs $\log_{10}\epsilon$ is plotted for several values of n at very low ϵ.

until we reached the value of n for which we got overflow, and from then on we used the value of $X(n,\epsilon)$ we got from the parametrization of the Gaunt factors by Menzel and Pekeris [37], which is very precise at high n. From the figure it is easy to see that for $\epsilon \to 0$, $s(\epsilon)$ diverges as $\ln(\epsilon)$. For $\epsilon \to \infty$, $s(\epsilon) \to \zeta(3)$, where $\zeta(s)$ is the Riemann zeta function. We have parametrized $s(\epsilon)$ with the following expansion:

$$s(\epsilon) = \begin{cases} -1.381\,33\log_{10}\epsilon + 0.480\,383 & \text{if } \epsilon \le 10^{-2} \\ 0.259\,387(\log_{10}\epsilon)^2 - 0.421437\log_{10}\epsilon + 1.396\,36 & \text{if } 10^{-2} < \epsilon \le 10 \\ 1.202\,056\,9 & \text{if } \epsilon > 10. \end{cases} \tag{3.3}$$

IV. RECOMBINATION ENERGY LOSS

Here we wish to mention that, knowing the effective cross sections for photorecombination and bremsstrahlung, it is possible to calculate the energy emitted by a negatively charged particle of reduced mass M and velocity v in the Coulomb field produced by a positive charge Ze. We may define as the emitted energy of the recombination process, per unit volume, per unit time, in the reference frame of stationary charge Ze, the following quantity:

$$Q_R = \int N_i N_e \sum_{n=1}^{\infty} \hbar\omega_n \sigma_n(v,\omega_n) v f(v) dv, \tag{4.1}$$

where N_i is the density of capturing particles, N_e is the density of captured particles. Assuming a Maxwellian distribution of velocities $f(v)$, Q_R may be written as the sum of two contributions, since $\hbar\omega_n = \frac{1}{2}Mv^2 + E_n$, and can be easily calculated by using the function $s(\epsilon)$ and an analog function $t(\epsilon)$ given by

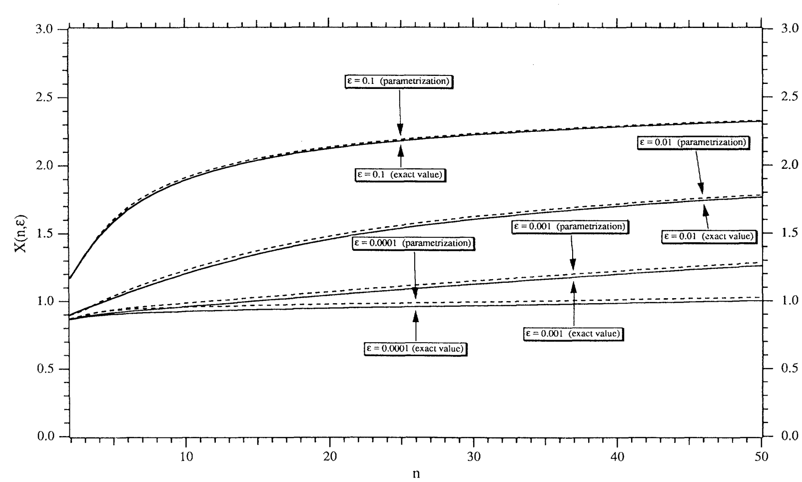

FIG. 5. Comparison between the exact values and the parametrization of $X(n,\epsilon)$ given in Eq. (2.9) for $n = 2$–50 at four different values of ϵ.

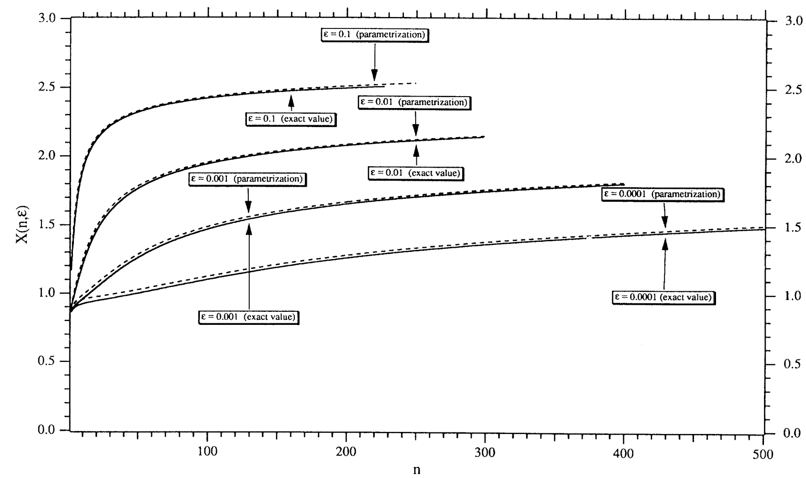

FIG. 6. Comparison between the exact values and the parametrization of $X(n,\epsilon)$ given in Eq. (2.9) for n = 2–500 at four different values of ϵ.

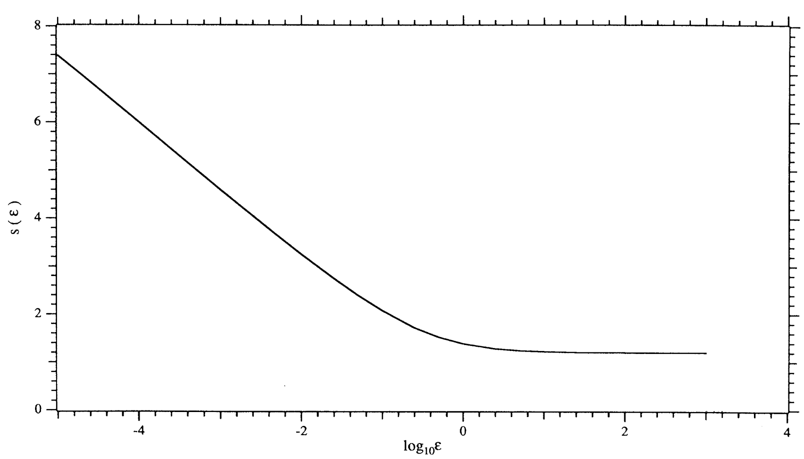

FIG. 7. The function $s(\epsilon)$ vs $\log_{10}\epsilon$. The parametrization (3.3) is also shown, but since it lies right on top of the exact curve it cannot be observed.

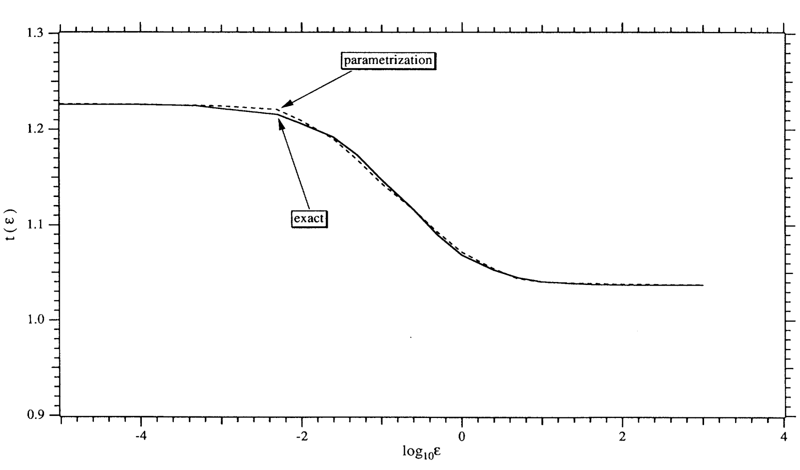

FIG. 8. The function $t(\epsilon)$ vs $\log_{10}\epsilon$. The parametrization (4.3) is also shown. The two different regimes at low ϵ and high ϵ are clearly shown.

$$t(\epsilon) = \sum_{n=1}^{\infty} \frac{1}{n^{X(n,\epsilon)+2}}. \tag{4.2}$$

In Fig. 8 we report the plot of the exact values of $t(\epsilon)$. We want to stress that, similarly to what we did in computing $s(\epsilon)$, we have used the exact values of $X(n,\epsilon)$ until we reached the value of n for which we got overflow, and from then on we used the value of $X(n,\epsilon)$ we got from the parametrization of Menzel-Pekeris. From the figure it is easy to see that for $\epsilon \to 0$, $t(\epsilon) \to 1.22$. For $\epsilon \to \infty$, $t(\epsilon) \to \zeta(5)$. We have parametrized $t(\epsilon)$ with the following expansion:

$$t(\epsilon) = \begin{cases} 1.220 & \text{if } \epsilon \le 10^{-3} \\ 0.008\,353\,81(\log_{10}\epsilon)^3 + 0.211\,32(\log_{10}\epsilon)^2 - 0.061\,266\,8\log_{10}\epsilon + 1.071\,83 & \text{if } 10^{-3} < \epsilon < 10 \\ 1.041 & \text{if } \epsilon \ge 10. \end{cases} \tag{4.3}$$

The energy emitted per unit volume and per second in the bremsstrahlung process may be obtained. Detailed study of the energy-loss balance of bremsstrahlung and captured emission and their comparison will be published elsewhere.

V. LOW-ENERGY APPROXIMATION

We report in Appendix A the rather complex analytical expression of $\sigma_{nlj}(v,\omega_n)$ we previously deduced exactly through the computation of nonconfluent hypergeometrical functions of a complex variable [29]. In this section we will write down a low-energy approximation (LEA), by means of which it is possible to verify the convergence of the sum over n of our $\sigma_n(\epsilon)$. The different factors entering $\sigma_{nlj}(\epsilon)$ have been derived in the limit $\epsilon \ll n^{-2}$ in Appendix A. We define some quantities we use in this section, M being the reduced mass of the recombining particles; the kinetic energy in the c.m. system, E_k, is

$$E_k = \frac{(\hbar k)^2}{2M},$$

$$E_n = -\frac{1}{2}\left(\frac{e^2}{\hbar c}\right)^2 \frac{Z^2 M c^2}{n^2},$$

$$q = ka\ , \quad a = \frac{\hbar^2}{e^2 M Z},$$

$$\alpha_n = \frac{Z}{a_0 n},$$

where a_0 is the Bohr radius of captured and capturing particle system. Using the LEA approximation we obtained the following result for the low-energy cross section:

$$\sigma_{n,l}^{\rm LEA}(\epsilon) = \frac{4\pi^2}{3}\frac{e^2}{\hbar c}\frac{Mc^2}{\hbar c}\frac{k_\gamma^3}{k^2}\frac{e^{-4n}(n+l)!}{(n-l-1)!}a^4 n^{2l+6}2^{4l+1} \times \frac{2^8(l+1)}{[(2l+1)!]^2}\bar{F}^2, \tag{5.1}$$

where in our approximation

$$k_\gamma = \frac{E_k - E_n}{\hbar c} = (q^2 n^2 + 1)\frac{E_n}{\hbar c} \simeq \frac{E_n}{\hbar c} \tag{5.2}$$

and the confluent hypergeometric function $\bar{F}$ is defined in Appendix A [Eq. (A23)]. The relation between k and ϵ is

$$k = \frac{\sqrt{\epsilon}}{\hbar c}\frac{e^2}{\hbar c} Z M c^2. \tag{5.3}$$

In this expression of the $\sigma_{n,l}^{\rm LEA}(\epsilon)$ we neglect the term that refers to the transitions $(l-1) \to l$ and we explain in Appendix A why we only need to consider the term proportional to $\bar{F}^2$ which refers to the transitions $(l+1) \to l$. A low-energy approximation has also been derived by Pajek and Schuch [35]; although their expression coincides with the one we derive in Appendix A [Eq. (A24)], they rather prefer to keep within the formal expression of the cross section all the contributions. A further comment on this point will be made in Appendix A.

Using the asymptotic calculus of hypergeometrical functions by Tricomi [51] it is possible to verify the convergence of

$$\sum_{n,l} \sigma_{n,l}^{\rm LEA}$$

for $n \to \infty$. It is possible to verify that the exact numerical value of

$$\sum_{n,l} \sigma_{n,l}$$

converges more rapidly than the LEA.

We now wish to mention that in the LEA, if one knows the cross section $\sigma_{n,l}$ for a particle x of rest mass $M(x)$ at energy $E_k(x)$, one can obtain at once the cross section $\sigma_{n,l}$ for another particle y of rest mass $M(y)$, at energy $E_k(y)$. In fact, the following relation holds:

$$\sigma_{n,l}^{\rm LEA}(M(x), E_k(x)) = \sigma_{n,l}^{\rm LEA}(M(y), E_k(y))\frac{M(y)E_k(y)}{M(x)E_k(x)}. \tag{5.4}$$

VI. APPROXIMATIONS FOR THE RADIATIVE-RECOMBINATION COEFFICIENT

Let us concentrate our attention on cases similar, for instance, to the electron cooling, where a cold gas of electrons is injected into the accelerator and travels at the same average velocity of the hot gas of protons (the formulas we will deduce can be used for any type of physical system). Let us take a reference frame which is stationary with the center of mass of the electron gas. Then in Eq. (1.1) we have

$$\tilde{\mathbf{v}} = \frac{M_e}{M_e + M_p}(\mathbf{v}_p - \mathbf{v}). \tag{6.1}$$

Let us define

$$\sum_{n,l,j} \sigma_{nlj}(|\tilde{v}|, \omega_n) = \tilde{\sigma}(|\tilde{\mathbf{v}}_\mathbf{p} - \mathbf{v}|); \tag{6.2}$$

then

$$\alpha \equiv \alpha(v_p) = \int v f(\mathbf{v}) \tilde{\sigma}(|\tilde{\mathbf{v}}_\mathbf{p} - \mathbf{v}|) d^3\mathbf{v}. \tag{6.3}$$

Now we calculate $\alpha(v_p)$ in the approximation $v < v_p$. We make a Taylor expansion of $\tilde{\sigma}(|\tilde{\mathbf{v}}_\mathbf{p} - \mathbf{v}|)$; after angular integration we have

$$\tilde{\sigma}(|\tilde{\mathbf{v}}_\mathbf{p} - \mathbf{v}|) = 4\pi\tilde{\sigma}(v_p) + \frac{4\pi}{3}\frac{v^2}{v_p}\tilde{\sigma}'(v_p) + \frac{4\pi}{3}v^2\tilde{\sigma}''(v_p). \tag{6.4}$$

A fraction of the electrons has energy larger than the proton energy; however, this fraction is absolutely negligible. In general:

$$\alpha(v_p) = 4\pi\tilde{\sigma}(v_p)\langle v\rangle + \frac{4\pi}{3}\left(\frac{\tilde{\sigma}'(v_p)}{v_p} + \tilde{\sigma}''(v_p)\right)\langle v^3\rangle. \tag{6.5}$$

A few words about the statistical distribution of velocities are appropriate. When the electrons are at $T = 0$ K, the distribution becomes very simple:

$$f(v) = c \quad \text{for} \quad v < v_F = \frac{\hbar k_F}{M}. \tag{6.6}$$

The constant c may be calculated after normalization and is given by

$$c = \frac{v}{(2\pi)^3}\left(\frac{\hbar}{M}\right)^3 \tag{6.7}$$

Then $\alpha(v_p)$ in the first approximation and for $v_p \to 0$ and $v_p < v$ is

$$\alpha(v_p) = 4\pi\int_0^{v_F} c v^3 \sigma(v) dv. \tag{6.8}$$

Therefore α gives a partial-sum rule up to $v = v_F$. Whereas radiative recombination generally plays a slight role in most laboratory plasmas, it is important in astrophysical nebulas and H II regions, in the ionosphere (tropical night glow) and in controlled-fusion plasmas [12,14,52–54]. In addition, we wish to make the following comment: if we assume a density of electrons as high as in a metal, about 10^{23} electrons/cm^3, we obtain $k_F \simeq 10^7$ cm^{-1}. Then the Fermi energy for an electron gas is

$$E_F = \frac{\hbar^2}{M}k_F^2 \gg E_{\text{ionization}}. \tag{6.9}$$

This means that in the evaluation of $\sigma(v)$ one has to be sure that not only the low-energy, but also the high-energy behavior is reproduced. The high-energy contribution to α is very important due to the v^3 factor, even if $\sigma(v)$ is much lower at high energy.

VII. HIGH-ENERGY APPROXIMATIONS

By high energy we mean $E_k \gg E_n$; $q^2n^2 \gg 1$. For n not too high we can make the further hypothesis $q^2 \gg 1$. In this high-energy low-n case we can apply the approximations shown in Appendix B to the exact expression of the recombination cross section (A1). We obtain

$$\sigma_{nl}(k) = \frac{4\pi}{3}\frac{e^2 m}{\hbar^2}\frac{k_\gamma^3}{k}l\frac{2^{4l+4}}{a^{2l+3}n^{2l+4}}\left[\frac{(l-1)!}{(2l-1)!}\right]^2 \times \frac{(n+l)!}{(n-l-1)!}k^{-2l-8} \tag{7.1}$$

with

$$k_\gamma = \frac{E_k + E_n}{\hbar c} \simeq \frac{E_k}{\hbar c}. \tag{7.2}$$

We can calculate another limiting case: the condition $q^2n^2 \gg 1$ is still valid, but we consider the condition $q^2 \ll 1$. In this high-energy high-n case we are interested in the radiative capture in the highest levels, near the continuum. The approximations valid in this case are shown in Appendix C. Using these approximations we obtain an approximated expression for $\sigma_{nl}(k)$ for high energy and high n:

$$\sigma_{nl}(k) = \frac{\pi^2}{3}\left(\frac{e^2}{\hbar c}\right)^3 (l+1)2^{4l+8}\frac{e^{-\frac{4}{q}\tan^{-1}qn}}{[(2l+1)!]^2}\frac{(n+l)!}{(n-l-1)!} \times g(q,l+1)\frac{a^2}{n^{2l+6}}\frac{1}{q^{2l+6}}. \tag{7.3}$$

The factor $g(q, l+1)$ can be further simplified in the hypothesis $ql \ll 1$, i.e., if we consider high-n low-l states:

$$g(q,l+1) = \prod_{s=1}^{l+1}\left(s^2 + \frac{1}{q^2}\right) \simeq 1/q^{2l+2}. \tag{7.4}$$

For $l = 0$ we get

$$\sigma_{n0}(k) = \frac{\pi^2}{3}\left(\frac{e^2}{\hbar c}\right)^3 2^8 \exp\left(-\frac{4}{q}\tan^{-1}qn\right)\frac{a^2}{n^5q^8}. \tag{7.5}$$

VIII. CONCLUSIONS

In this paper we have given the tools to easily compute some fundamental quantities in astrophysics, plasma physics, atomic physics, and accelerators like the recombination rates and the energy produced during recombination which we shall analyze in a later work. The tools we propose to use are as follows: the recombination cross section at the level n given as a scaling law with respect to the $n=1$ cross section $[\sigma_{1s}(\epsilon)]$, a scaling law for the sum over n of all the cross sections and also another scaling law that is useful in evaluating the energy produced during recombination. We give exact values of the above quantities, because of the approach used to calculate the hypergeometrical function of complex variables, and we also give the three parametrizations which are very useful in the evaluation of quantities averaged over statistical Maxwellian and non-Maxwellian distributions.

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APPENDIX A: LOW-ENERGY APPROXIMATION

In this appendix we report the detailed calculation of the LEA cross section. The analytical expression of the recombination cross section [29] is

$$\sigma_{n,l}(k)=\frac{4\pi}{3}\frac{e^2M}{\hbar^2}\frac{k_\gamma^3}{k}\left[q_{n,l}(k)\left|(\alpha_n-ik)^2F_1-(\alpha_n+ik)^2F_2\right|^2+p_{n,l}(k)\left|(\alpha_n-ik)^2F_3-(\alpha_n+ik)^2F_4\right|^2\right] \tag{A1}$$

with

$$q_{n,l}(k)=\frac{l2^{4l}k^{2l-2}}{[(2l-1)!]^2}\frac{e^{-\frac{4}{q}\tan^{-1}qn}}{a^{(2l+1)}n^{(2l+2)}}\frac{(n+l)!}{(n-l-1)!}e^{\frac{\pi}{q}}\left|\Gamma(l-i/q)\right|^2\frac{1}{(\alpha_n^2+k^2)^{2l+4}}, \tag{A2}$$

$$p_{n,l}(k)=\frac{(l+1)2^{4l+4}k^{2l}}{[(2l+1)!]^2}\frac{e^{-\frac{4}{q}\tan^{-1}qn}}{a^{(2l+3)}n^{(2l+4)}}\frac{(n+l)!}{(n-l-1)!}e^{\frac{\pi}{q}}\left|\Gamma(l+2-i/q)\right|^2\frac{1}{(\alpha_n^2+k^2)^{2l+6}}, \tag{A3}$$

and the $F_i (i=1,2,3,4)$ are nonconfluent hypergeometrical functions of complex variable defined as

$$F_1=F\left(-n+l-1,l+\frac{i}{q},2l,-\frac{4i\alpha_nk}{(\alpha_n-ik)^2}\right), \tag{A4}$$

$$F_2=F\left(-n+l+1,l+\frac{i}{q},2l,-\frac{4i\alpha_nk}{(\alpha_n-ik)^2}\right), \tag{A5}$$

$$F_3=F\left(-n+l+1,l+\frac{i}{q},2l+2,-\frac{4i\alpha_nk}{(\alpha_n-ik)^2}\right), \tag{A6}$$

$$F_4=F\left(-n+l+1,l+2+\frac{i}{q},2l+2,-\frac{4i\alpha_nk}{(\alpha_n-ik)^2}\right), \tag{A7}$$

and M, k_γ, k, a, α_n, q are defined in Sec. 5.

Low energy, for us, means

$$E_k\ll E_n=\frac{1}{n^2}\frac{1}{2}\left(\frac{e^2}{\hbar c}\right)^2(Z^2Mc^2)$$

and therefore $E_k/E_n=a^2k^2n^2=q^2n^2\ll 1$. The Γ function is given by

$$|\Gamma(l-i/q)|^2=\prod_{s=1}^{l-1}(s^2+1/q^2)\frac{i}{q}\Gamma\left(\frac{i}{q}\right)\Gamma\left(1-\frac{i}{q}\right) \tag{A8}$$

and then we find

$$e^{\frac{\pi}{q}}\left|\Gamma(l-i/q)\right|^2=\frac{2\pi}{q}\prod_{s=1}^{l-1}(s^2+1/q^2)\frac{1}{1-\exp(-2\pi/q)}. \tag{A9}$$

Since $s^2<n^2$ and $n^2\ll 1/q^2$; we can neglect s^2 compared to $1/q^2$; we can also neglect the exponential in the denominator of (A9) and then we get

$$e^{\frac{\pi}{q}}\left|\Gamma(l-i/q)\right|^2=2\pi q^{-2l+1} \tag{A10}$$

in the same way we get

$$e^{\frac{\pi}{q}}\left|\Gamma(l+2-i/q)\right|^2=2\pi q^{-2l-3}. \tag{A11}$$

We can also take, in Eqs. (A2) and (A3),

$$e^{-\frac{4}{q}\tan^{-1}qn} \sim e^{-4n},$$

$$(\alpha_n^2+n^2) \sim \alpha_n^2, \tag{A12}$$

and therefore, using Eqs. (A10), (A11), and (A12), we can write $q_{n,l}(k)$ and $p_{n,l}(k)$ as

$$q_{n,l}(k) = \frac{\pi l 2^{4l+1}e^{-4n}}{[(2l-1)!]^2}\frac{(n+l)!}{(n-l-1)!}a^8 n^{2l+6}\frac{1}{k}, \tag{A13}$$

$$p_{n,l}(k) = \frac{\pi(l+1)2^{4l+5}e^{-4n}}{[(2l+1)!]^2}\frac{(n+l)!}{(n-l-1)!}a^6 n^{2l+8}\frac{1}{k^3}. \tag{A14}$$

We now consider the nonconfluent hypergeometric function:

$$F\left(-n+l\pm 1, \lambda+\frac{i}{q}, \lambda', -\frac{4i\alpha_n k}{(\alpha_n-ik)^2}\right)$$

which is defined by the following series:

$$F = 1+\sum_{\nu=1}^{\infty}\frac{(-n+l\pm 1)_\nu}{(\lambda')_\nu}\left(\lambda+\frac{i}{q}\right)\cdots\left(\lambda+\frac{i}{q}+\nu-1\right) \times\frac{(-iq)^\nu}{\nu!}\frac{(4n)^\nu}{(1-iqn)^{2\nu}}, \tag{A15}$$

where

$$(\lambda')_\nu = \lambda'(\lambda'+1)\cdots(\lambda'+\nu-1)$$

and the same for $(-n+l\pm 1)_\nu$.

If we group together each one of the ν factors of the type $\lambda+\frac{i}{q}+r$ with one of the $(-iq)$, we get ν factors $[-(\lambda+r)iq+1]$ which, in the limit $q^2n^2 \ll 1$, are:

$$[-(\lambda)iq+1][-(\lambda+1)iq+1]\cdots[-(\lambda+\nu-1)iq+1]$$

$$\simeq 1 - iq\frac{(2\lambda+\nu-1)\nu}{2}. \tag{A16}$$

Keeping in mind that $(1-iqn)^{-2\nu} \simeq 1+2iqn\nu$, the approximated expression for (A15) is

$$\begin{aligned} F\left(-n+l\pm 1, \lambda+\frac{i}{q}, \lambda', -\frac{4i\alpha_n k}{(\alpha_n-ik)^2}\right) &\simeq 1+\sum_{\nu=1}^{\infty}\frac{(-n+l\pm 1)_\nu}{(\lambda')_\nu}\frac{(4n)^\nu}{\nu!}\left[1-iq\frac{(2\lambda+\nu-1)\nu}{2}+2iqn\nu\right] \\ &= F(-n+l\pm 1, \lambda', 4n) \\ &\quad +iq\sum_{\nu=1}^{\infty}\frac{(-n+l\pm 1)_\nu}{(\lambda')_\nu}\frac{(4n)^\nu}{(\nu-1)!}\left[-\frac{(2\lambda+\nu-1)}{2}+2n\right], \end{aligned} \tag{A17}$$

where

$$F(-n+l\pm 1, \lambda', 4n) = \sum_{\nu=0}^{\infty}\frac{(-n+l\pm 1)_\nu}{(\lambda')_\nu}\frac{(4n)^\nu}{\nu!} \tag{A18}$$

is a confluent hypergeometric function. We can therefore write the two parts of Eq. (A1) containing the differences of the hypergeometric functions as

$$(\alpha_n-ik)^2F_1-(\alpha_n+ik)^2F_2 = \alpha_n^2\left[(1-iqn)^2F_1-(1+iqn)^2F_2\right] \simeq \alpha_n^2\left(\bar{F}_1-\bar{F}_2\right), \tag{A19}$$

$$\begin{aligned} (\alpha_n-ik)^2F_3-(\alpha_n+ik)^2F_4 &\simeq -4i\alpha_n^2 qn\bar{F}+i\alpha_n^2 q\sum_{\nu=1}^{\infty}\frac{(-n+l+1)_\nu}{(2l+2)_\nu}\frac{(4n)^\nu}{(\nu-1)!} \\ &\quad\times\left[-\frac{(2l+\nu-1)}{2}+2n+\frac{(2l+4+\nu-1)}{2}-2n\right] \\ &= -4i\alpha_n^2 qn\bar{F}+2i\alpha_n^2 q\sum_{\nu=1}^{\infty}\frac{(-n+l+1)_\nu}{(2l+2)_\nu}\frac{(4n)^\nu}{(\nu-1)!} \\ &= -4i\alpha_n^2 qn\left[\bar{F}-\frac{(-n+l+1)}{(l+1)}F(-n+l+2, 2l+3, 4n)\right], \end{aligned} \tag{A20}$$

where we used the following notation for the confluent hypergeometric function:

$$\bar{F}_1 \equiv F(-n+l-1, 2l, 4n), \quad \text{(A21)}$$

$$\bar{F}_2 \equiv F(-n+l+1, 2l, 4n), \quad \text{(A22)}$$

$$\bar{F} \equiv F(-n+l+1, 2l+2, 4n). \quad \text{(A23)}$$

Finally, gathering Eqs. (A13), (A14), (A19), and (A20), we obtain the following expression for the LEA cross section (which is in agreement with the results of the derivation of Pajek and Schuch [35]):

$$\sigma_{n,l}^{\rm LEA}(\epsilon) = \frac{4\pi^2}{3}\frac{e^2}{\hbar c}\frac{Mc^2}{\hbar c}\frac{k_\gamma^3}{k^2}\frac{e^{-4n}(n+l)!}{(n-l-1)!}a^4 n^{2l+6}2^{4l+1} \times \left\{\frac{l}{n^4}\frac{(\bar{F}_1-\bar{F}_2)^2}{[(2l-1)!]^2} + \frac{2^8(l+1)}{[(2l+1)!]^2}\left[\bar{F} - \frac{(-n+l+1)}{(l+1)}F(-n+l+2, 2l+3, 4n)\right]^2\right\}. \quad \text{(A24)}$$

Several considerations must be made at this point: first of all the term proportional to $(\bar{F}_1 - \bar{F}_2)^2$ refers to the transitions $(l-1) \to l$, and is always very small (at the most a few percent) compared to the other term, therefore we conclude that the $(l-1) \to l$ transitions are suppressed compared to the $(l+1) \to l$ transitions and we can neglect this term. The expression of Eq. (5.1) is obtained from Eq. (A24) neglecting also the second term in the squared bracket. The reason is the following: Eq. (5.1) and Eq. (A24) being both approximations of the exact cross section (A1), they give very good estimates when q^2n^2 is much less than 1. If this is not the case , but still $q^2n^2 < 1$, we have verified the following: for $l = n-1$, the second term in the squared bracket is zero; for $l = 0$ and $n > 1$, Eq. (5.1) gives a larger value than the exact result, while Eq. (A24), even if it still gives a value larger than the exact one, is closer to the exact value; for $0 < l < n-1$ Eq. (A.24) underestimates the exact value more than it does Eq. (5.1). We may conclude that $\sum_l \sigma_{n,l}^{\rm LEA}$ is a better approximation of the exact value than the sum over l of the cross section of Eq. (A24). This is the reason why we may suggest, as an approximation to the cross section of Eq. (A1), the expression of Eq. (5.1) in addition to Eq. (A24).

APPENDIX B: HIGH-ENERGY, LOW-n APPROXIMATION

In this appendix we report the detailed calculation of the recombination cross section in the high-energy, low-n approximation. Since $E_k \gg E_n$, $q^2n^2 \gg 1$ and $q^2 \gg 1$, we have

$$e^{-\frac{4}{q}\tan^{-1} qn} \simeq 1. \quad \text{(B1)}$$

In this limit we also have $1 - \exp(-2\pi/q) \simeq -2\pi/q$, and therefore we can rewrite Eq. (A9) as

$$e^{\frac{\pi}{q}}\left|\Gamma(l - i/q)\right|^2 \simeq \prod_{s=1}^{l-1}(s^2 + 1/q^2) \simeq [(l-1)!]^2. \quad \text{(B2)}$$

Analogously we have

$$e^{\frac{\pi}{q}}\left|\Gamma(l+2 - i/q)\right|^2 \simeq [(l+1)!]^2. \quad \text{(B3)}$$

Furthermore,

$$\alpha_n^2 + k^2 \simeq k^2. \quad \text{(B4)}$$

Since

$$-\frac{4iqn}{(1-iqn)^2} \simeq \frac{4i}{qn} \ll 1 \quad \text{(B5)}$$

all the hypergeometrical functions (A4)–(A7) can be put equal to 1: $F_1 \simeq F_2 \simeq F_3 \simeq F_4 \simeq 1$, and we can write

$$|(\alpha_n - ik)^2F_1 - (\alpha_n + ik)^2F_2|^2 \simeq |(\alpha_n - ik)^2F_3 - (\alpha_n + ik)^2F_4|^2 \simeq 16\alpha_n^2k^2 \quad \text{(B6)}$$

substituting Eqs. (B1)–(B4) and (B6) into (A1), we obtain

$$\sigma_{nl}(k) = \frac{4\pi}{3}\frac{e^2m}{\hbar^2}\frac{k_\gamma^3}{k}\left\{l\frac{2^{4l+4}}{a^{2l+3}n^{2l+4}}\left[\frac{(l-1)!}{(2l-1)!}\right]^2\frac{(n+l)!}{(n-l-1)!}k^{-2l-8} + (l+1)\frac{2^{4l+8}}{a^{2l+5}n^{2l+6}}\left[\frac{(l+1)!}{(2l+1)!}\right]^2\frac{(n+l)!}{(n-l-1)!}k^{-2l-10}\right\}. \quad \text{(B7)}$$

Since $k^{-2l-10} \ll k^{-2l-8}$ we can neglect the second term in parentheses. This means that only $(l-1) \to l$ transitions are allowed in this limit and we obtain for the recombination cross section in the high-energy, low-n approximation the expression (7.1) in the main text.

APPENDIX C: HIGH-ENERGY, HIGH-n APPROXIMATION

In this appendix we report the detailed calculation of the recombination cross section in the high-energy, high-n approximation. Since $q^2 \ll 1$ and $n \gg 1$ we have

$$e^{\frac{\pi}{q}} \left|\Gamma(l - i/q)\right|^2 = \frac{2\pi}{q} g(q, l-1) \tag{C1}$$

and analogously

$$e^{\frac{\pi}{q}} \left|\Gamma(l + 2 - i/q)\right|^2 = \frac{2\pi}{q} g(q, l+1) \tag{C2}$$

with

$$g(q, \lambda) = \prod_{s=1}^{\lambda} (s^2 + 1/q^2). \tag{C3}$$

Substituting (B4), (C1), and (C2) in (A2) and (A3), we have for $q_{n,l}(k)$ and $p_{n,l}(k)$

$$q_{n,l}(k) = \pi l 2^{4l+1} \frac{e^{-\frac{4}{q}\tan^{-1} qn}}{[(2l-1)!]^2} \frac{(n+l)!}{(n-l-1)!} g(q, l-1) \times \frac{a^9}{n^{2l+2}} \frac{1}{q^{2l+11}}, \tag{C4}$$

$$p_{n,l}(k) = \pi (l+1) 2^{4l+5} \frac{e^{-\frac{4}{q}\tan^{-1} qn}}{[(2l+1)!]^2} \frac{(n+l)!}{(n-l-1)!} g(q, l+1) \times \frac{a^9}{n^{2l+4}} \frac{1}{q^{2l+13}}. \tag{C5}$$

We can neglect $q_{n,l}(k)$ compared to $p_{n,l}(k)$ and substituting Eq. (C5) into Eq. (A1) and using (B6) we obtain

$$\sigma_{nl}(k) = \frac{4\pi^2}{3} \frac{e^2 M}{\hbar^2} k_\gamma^3 (l+1) 2^{4l+9} \frac{e^{-\frac{4}{q}\tan^{-1} qn}}{[(2l+1)!]^2} \frac{(n+l)!}{(n-l-1)!} \times g(q, l+1) \frac{a^6}{n^{2l+6}} \frac{1}{q^{2l+12}}. \tag{C6}$$

Since in our approximation

$$k_\gamma = (q^2 n^2 + 1) \frac{E_n}{\hbar c} \simeq q^2 n^2 \frac{E_n}{\hbar c} \simeq q^2 \frac{M e^4}{2\hbar^3 c} \tag{C7}$$

we finally obtain the approximated expression of $\sigma_{nl}(k)$ for high-energy, high-n shown in the main text in Eq. (7.3).

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