Retarded electric and magnetic Casimir interaction of a polarizable system and a dielectric permeable wall

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Using an expansion of the electromagnetic field in terms of quantized Fresnel modes, we calculate the retarded Casimir interaction, in the asymptotic domain, of a polarizable system and a dielectric permeable wall, that is, a wall characterized by a dielectric constant ϵ_2 and a magnetic permeability μ_2. We obtain explicit analytic results for the atom-wall and for the electron-wall interactions. When magnetic effects are ignored, the results reduce to those obtained by Lifshitz [Sov. Phys. **2**, 73 (1956)] for the atom-wall interaction and by the present authors for the electron-wall interaction. The results simplify greatly for an ideal metallic wall and for a wall made of material such as liquid helium, for which the dielectric constant and the permeability are very close to unity. Other than in the determination of the amplitudes of the Fresnel modes, the calculations are entirely classical.

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I. INTRODUCTION AND NOTATION

In a recent work [1] we employed Fresnel modes [2], quantized [3], to calculate the retarded Casimir interaction of a polarizable system and a dielectric wall in the asymptotic domain. In that work the known results for the atom-wall interaction [4] were reproduced and apparently previously unrealized results were derived for the electron-wall interaction. However, in order to concentrate on matters of principle, we neglected the effects of the magnetic field and assumed $\mu_2 = 1$ for the magnetic permeability of the wall. Here, apart from the neglect of the spin of the electron, the full electric and magnetic Casimir interaction (in the asymptotic domain) will be treated. We shall employ the same notation as in Ref. [1] and draw heavily on the discussions in that work. In particular, it has been argued on physical grounds that only modes with frequencies $\omega \sim 0$ are important. This leads to an enormous simplification of the discussion compared to that given by Lifshitz [4] and in Ref. [1], but it is limited to the asymptotic domain. Thus, only the case of walls with real refractive index satisfying

$$n_{20} = (\epsilon_{20}\mu_{20})^{1/2} > 1 \tag{1.1}$$

need be considered. Here $\epsilon_{20} = \epsilon_2$ $(\omega = 0)$ and $\mu_{20} = \mu_2$ $(\omega = 0)$ are, respectively, the dielectric constant and the magnetic permeability of the wall at zero frequency. We always have $\epsilon_{20} > 1$. The restriction imposed by Eq. (1) is clearly satisfied for ferromagnetic materials, which have $\mu_{20} \gg 1$ and are the only materials for which μ_{20} plays a significant role. [It is also satisfied for paramagnetic materials, for which μ_{20} is very close to, but greater than, unity. Diamagnetic materials have $\mu_{20} < 1$, but μ_{20} is very close to unity, and even for this case Eq. (1) will almost always be satisfied.]

Consider two semi-infinite and homogeneous media with a plane interface, characterized by the permittivities and permeabilities ϵ_1, μ_1 and ϵ_2, μ_2. A Fresnel mode of class I consists of an incident plane wave (from medium 1 onto medium 2), a reflected plane wave, and a transmitted plane wave, all with a given frequency ω and a given linear polarization λ. We assume that

$$n_2(\omega) = [\epsilon_2(\omega)\,\mu_2(\omega)]^{1/2} > [\epsilon_1(\omega)\,\mu(\omega)]^{1/2} = n_1(\omega)\ .$$

There are two independent linear polarizations, namely $\mathbf{E}$ perpendicular $(\lambda = 1)$ and $\mathbf{E}$ parallel $(\lambda = 2)$ to the plane of scattering, where $\mathbf{E}$ denotes the electric field. (The $\lambda = 2$ case is depicted in Fig. 1 of Ref. [1].) The corresponding magnetic fields are

$$\mathbf{B} = (\sqrt{\epsilon\mu}/k)\mathbf{k}\times\mathbf{E}\ .$$

The mode $q \equiv (\mathbf{k}^i, \lambda)$, where $\mathbf{k}^i$ denotes the incident wave vector and $z = 0$ the plane interface, is defined by

$$\mathbf{f}_q(\mathbf{r}) = \begin{cases} N(\mathbf{E}^i e^{i\mathbf{k}^i\cdot\mathbf{r}} + \mathbf{E}^r e^{i\mathbf{k}^r\cdot\mathbf{r}})\ , & z < 0 \\ N\mathbf{E}^t e^{i\mathbf{k}^t\cdot\mathbf{r}}, & z > 0\ . \end{cases} \tag{1.2}$$

Here the superscripts i, r, and t refer to incident, reflected, and transmitted waves and N is a normalization constant. It can be shown by an analysis similar to that given in Appendix A of Ref. [1] that the choice

$$N^2|\mathbf{E}^i|^2 = \left[(2\pi)^3\epsilon_1\right]^{-1} \tag{1.3}$$

leads to the orthonormality relation

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$$\int \epsilon(\mathbf{r})\,\mathbf{f}_q(\mathbf{r})\cdot\mathbf{f}^*_{q'}(\mathbf{r})\,d\mathbf{r} = \delta_{qq} = \delta(\mathbf{k}^i - \mathbf{k}^{i'})\,\delta_{\lambda\lambda'} \qquad (1.4)$$

provided ϵ and μ are independent of the frequency ω. (They may depend, even continuously, on the location $\mathbf{r}$.) If the permittivities and permeabilities are frequency dependent, the different modes are no longer orthogonal but still form a basis for unique expansions of the fields [1]. The boundary conditions imposed on the components of the electromagnetic fields at the interface $z = 0$ lead to Snell's law

$$\mathbf{k}^i \equiv \mathbf{k} = (k_x, k_y, k_z), \quad \mathbf{k}^r = (k_x, k_y, -k_z), \mathbf{k}^t = (k_x, k_y, \tilde{k}_z), \qquad (1.5)$$

with $\tilde{k}_z$ given by Eq. (1.14) below, and to Fresnel's formulae [3] for the reflection and transmission coefficients $R \equiv E^r/E^i$ and $T \equiv E^t/E^i$. For $\mathbf{E}^i$ perpendicular to the plane of scattering these coefficients are given by

$$R^\perp = \frac{\mu_2 k_z - \mu_1 \tilde{k}_z}{\mu_2 k_z + \mu_1 \tilde{k}_z} \qquad (1.6)$$

and

$$T^\perp = \frac{2\mu_2 k_z}{\mu_2 k_z + \mu_1 \tilde{k}_z}\,, \qquad (1.7)$$

from which

$$R^\perp = T^\perp - 1 \qquad (1.8)$$

follows. For $\mathbf{E}^i$ parallel to the plane of scattering, one has

$$R^\parallel = \frac{\mu_1 n_2^2 k_z - \mu_2 n_1^2 \tilde{k}_z}{\mu_1 n_2^2 k_z + \mu_2 n_1^2 \tilde{k}_z} \qquad (1.9)$$

and

$$T^\parallel = \frac{2\mu_2 n_1 n_2 k_z}{\mu_1 n_2^2 k_z + \mu_2 n_1^2 \tilde{k}_z}\,, \qquad (1.10)$$

and hence

$$R^\parallel = 1 - T^\parallel (n_1 \tilde{k}_z / n_2 k_z)\,. \qquad (1.11)$$

In both cases the intensities R^2 and T^2 are related by

$$R^2 + T^2 (\mu_1 \tilde{k}_z / \mu_2 k_z) = 1\,. \qquad (1.12)$$

The components of the wave vectors in Eq. (1.5) are all real, k_z and $\tilde{k}_z$ are non-negative, and, with ω and $n^2 = \epsilon\mu$ denoting the frequency and index of refraction, they satisfy the dispersion relations

$$k_x^2 + k_y^2 + k_z^2 = k^2 = n_1^2 \omega^2/c^2, \qquad (1.13)$$

$$k_x^2 + k_y^2 + \tilde{k}_z^2 = n_2^2 \omega^2/c^2\,. \qquad (1.14)$$

A second class of Fresnel modes (class II) consists of plane waves incident from medium 2 on to medium 1. We must here distinguish between $\theta < \theta_c$ and $\theta > \theta_c$, where θ_c is the critical angle for total reflection, given by $\sin\theta_c = n_1/n_2$. (Recall that we are assuming $n_2 > n_1$.) For $\theta < \theta_c$, the reflected and transmitted waves are plane waves of the same general form as those for class I. For $\theta > \theta_c$, however, the transmitted wave in the region $z < 0$ is replaced by a decaying wave. Thus, for class II modes with $\theta > \theta_c$ we have

$$\mathbf{k}^i \equiv \mathbf{k} = (k_x, k_y, -k_z), \quad \mathbf{k}^r = (k_x, k_y, +k_z), \quad \mathbf{k}^t = (k_x, k_y, -iK_z), \qquad (1.15)$$

where the signs have been chosen to make k_z and K_z nonnegative. These wave vectors satisfy the dispersion relations

$$k_x^2 + k_y^2 + k_z^2 = n_2^2\,\omega^2/c^2, \qquad (1.16)$$

$$k_x^2 + k_y^2 - K_z^2 = n_1^2\,\omega^2/c^2\,. \qquad (1.17)$$

It can be shown that the orthogonality relation Eq. (1.4) remains valid even for decaying modes, provided ϵ_1 is replaced by ϵ_2 in expression (1.3) for the normalization condition.

II. THE INTERACTION OF A POLARIZABLE SYSTEM AND A DIELECTRIC PERMEABLE WALL

Let a polarizable system be placed in medium 1, which we now assume to be a vacuum with $\epsilon_1 = \mu_1 = 1$, at a distance ℓ from a wall (medium 2) characterized by ϵ_2 and μ_2. We assume that in the absence of the wall, the polarizable system placed at $(0, 0, z = -\ell)$ would be in a spherically symmetric state. The introduction of the wall changes the electric and magnetic fields at the location of the system, and the system becomes polarized by the vacuum fluctuating fields. The interaction energy of the system with given components of the fields $\mathbf{E}_q$ and $\mathbf{B}_q$ [defined in Eqs. (2.3) and (2.4) below] is

$$\langle 0 \mid [-\tfrac{1}{2}\alpha_1(\omega)\,\mathbf{E}_q^2 - \tfrac{1}{2}\beta_1(\omega)\,\mathbf{B}_q^2] \mid 0\rangle, \qquad (2.1)$$

where $\alpha_1(\omega)$ and $\beta_1(\omega)$ are the dynamic electric and magnetic dipole polarizabilities of the system. Only that part of the interaction energy which depends on the distance ℓ of the system from the wall need concern us — the rest, namely the part which does not depend on ℓ, merely contributes to the self-energy of the system-wall pair at infinite separation. In writing Eq. (2.1) the dipole approximation, in which the fields are assumed to stay constant over the system, has been made. For a given distance ℓ, this approximation breaks down at sufficiently high frequencies ω. To avoid spurious divergencies, we shall employ, whenever necessary, a cutoff factor in the integrals over ω.

We turn now to the quantized electromagnetic field. In the Coulomb gauge, in a box of volume τ, the expansion of the field in terms of Fresnel modes $\mathbf{f}_q(\mathbf{r})$ is [2]

$$\mathbf{A} = \sum_q (2\pi\hbar c^2/\omega\tau)^{1/2}\,(a_q \mathbf{f}_q + a_q^\dagger \mathbf{f}_q^*), \qquad (2.2)$$

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = i\sum_q (2\pi\hbar\omega/\tau)^{1/2}\,(a_q\mathbf{f}_q - a_q^\dagger \mathbf{f}_q^*) \equiv \sum_q \mathbf{E}_q, \tag{2.3}$$

$$\begin{aligned}\mathbf{B} &= \boldsymbol{\nabla}\times\mathbf{A}\\ &= \sum_q (2\pi\hbar c^2/\omega\tau)^{1/2}\,(a_q\boldsymbol{\nabla}\times\mathbf{f}_q + a_q^\dagger\boldsymbol{\nabla}\times\mathbf{f}_q^*)\\ &\equiv \sum_q \mathbf{B}_q\ .\end{aligned} \tag{2.4}$$

Here a_q and $a_q^\dagger$ are the usual destruction and creation operators, satisfying

$$[a_q, a_{q'}] = [a_q^\dagger, a_{q'}^\dagger] = 0\ , \qquad [a_q, a_{q'}^\dagger] = \delta_{qq'}\ . \tag{2.5}$$

With $|0\rangle$ denoting the vacuum state for the photons, use of the commutation relations (2.5), the normalization condition (1.3), and the relation

$$\int \frac{1}{\mu}(\boldsymbol{\nabla}\times\mathbf{f}_q)\cdot(\boldsymbol{\nabla}\times\mathbf{f}_q^*)\,d\mathbf{r} = \int \epsilon\frac{\omega^2}{c^2}\mathbf{f}_q\cdot\mathbf{f}_q^*\,d\mathbf{r} \tag{2.6}$$

[which follows from the definition (1.2) of Fresnel's modes] yields

$$\int\left\langle 0\left|\frac{\epsilon\mathbf{E}^2}{8\pi} + \frac{1}{\mu}\frac{\mathbf{B}^2}{8\pi}\right|0\right\rangle d\mathbf{r} = \sum_q\left(\frac{\hbar\omega}{4} + \frac{\hbar\omega}{4}\right) = \sum_q\frac{\hbar\omega}{2}. \tag{2.7}$$

This is, indeed, the desired result for the expectation value of the ground-state energy of the electromagnetic field; each mode of frequency ω carries, at it should, the energy $\hbar\omega/2$. Since

$$\langle 0 \mid \mathbf{E}^2 \mid 0\rangle = \left\langle 0\left|\sum_q \mathbf{E}_q^2\right|0\right\rangle = \sum_q 2\pi\hbar\omega\,\mathbf{f}_q\cdot\mathbf{f}_q^* \tag{2.8}$$

and

$$\begin{aligned}\langle 0|\mathbf{B}^2|0\rangle &= \left\langle 0\left|\sum_q \mathbf{B}_q^2\right|0\right\rangle\\ &= \sum_q (2\pi\hbar c^2/\omega)\,(\boldsymbol{\nabla}\times\mathbf{f}_q)\cdot(\nabla\times\mathbf{f}_q^*)\ ,\end{aligned} \tag{2.9}$$

Eq. (2.1) yields for the total energy due to class I modes

$$\begin{aligned}\mathcal{V}^{\mathrm{I}} &= \mathcal{V}^{\mathrm{I}}_{\mathrm{el}} + \mathcal{V}^{\mathrm{I}}_{\mathrm{mag}}\\ &= -\pi\sum_q \alpha_1(\omega)\hbar\omega\,\mathbf{f}_q\cdot\mathbf{f}_q^*\\ &\quad -\pi\sum_q [\beta_1(\omega)\hbar c^2/\omega]\,(\boldsymbol{\nabla}\times\mathbf{f}_q)\cdot(\boldsymbol{\nabla}\times\mathbf{f}_q^*)\ .\end{aligned} \tag{2.10}$$

Here $\mathbf{f}_q(\mathbf{r})$ is to be evaluated at the system's location, $\mathbf{r} = (0,0,-\ell)$. As mentioned above, the energy $\mathcal{V}^{\mathrm{I}}$ is given by the sum of two terms: a term depending on the distance ℓ, which we will denote by V^{I}, and a self-energy term. For example, writing explicitly the expression for $\mathcal{V}^{\mathrm{I}}_{\mathrm{el}}$ — recall that k_z is non-negative —

$$\begin{aligned}\mathcal{V}^{\mathrm{I}}_{\mathrm{el}} &= -\pi\sum_q \alpha_1(\omega)\hbar\omega\mathbf{f}_q(0,0,-\ell)\cdot\mathbf{f}_q^*(0,0,-\ell)\\ &= -\pi\sum_{\lambda=1,2}\int_{-\infty}^{\infty}dk_x\int_{-\infty}^{\infty}dk_y\int_0^{\infty}dk_z\,\alpha_1(\omega)\hbar\omega N^2\\ &\quad\times|\mathbf{E}^i\,e^{-ik_z\ell} + \mathbf{E}^r e^{ik_z\ell}|^2,\end{aligned} \tag{2.11}$$

we see that only cross terms of the form $\mathbf{E}^i\cdot\mathbf{E}^{r^*}$ and $\mathbf{E}^{i^*}\cdot\mathbf{E}^r$ contribute to the interaction energy $V^{\mathrm{I}}_{\mathrm{el}}$. The direct terms $\mathbf{E}^i\cdot\mathbf{E}^{i^*}$ and $\mathbf{E}^r\cdot\mathbf{E}^{r^*}$ do not depend on ℓ and, hence, express self-energy. Employing polar coordinates with $k = \omega/c$ (since $n_1 = 1$) and $k_z = k\cos\theta = (\omega/c)p$ — we use $\theta_i \equiv \theta$ and $p \equiv \cos\theta$ — the contribution of class I modes to V^{I} is

$$\begin{aligned}V^{\mathrm{I}} &= V^{\mathrm{I}}_{\mathrm{el}} + V^{\mathrm{I}}_{\mathrm{mag}}\\ &= -\frac{4\pi^2\hbar}{c^3}\sum_{\lambda=1,2}\int d\omega\ \mathrm{Re}\int_0^1 dp\,N^2\,e^{2i\omega\ell p/c}\\ &\quad\times[\omega^3\alpha_1(\omega)(\mathbf{E}^{i^*}\cdot\mathbf{E}^r)\\ &\quad + c^2\omega\,\beta_1(\omega)(\mathbf{k}\times\mathbf{E}^{i^*})\cdot(\mathbf{k}^r\times\mathbf{E}^r)].\end{aligned} \tag{2.12}$$

As argued before [1], the range of contributing ω's in these integrals is $0 \le \omega \le \bar{\omega}$ where $\bar{\omega} \ll \omega_0$ and where ω_0 is the smallest frequency of significance in the absorption spectra of either the wall or the system.

To perform the sum over the polarizations λ in Eq. (2.12) we note that for $\mathbf{E}^i$ perpendicular to the plane of scattering

$$\mathbf{E}^{i^*}\cdot\mathbf{E}^r = |\mathbf{E}^i|^2 R^{\perp} \tag{2.13}$$

and

$$(\mathbf{k}\times\mathbf{E}^{i^*})\cdot(\mathbf{k}^r\times\mathbf{E}^r) = |\mathbf{E}^i|^2 k^2 R^{\perp}(1 - 2\cos^2\theta) \tag{2.14}$$

whereas, for $\mathbf{E}^i$ parallel to the plane of scattering,

$$\mathbf{E}^{i^*}\cdot\mathbf{E}^r = |\mathbf{E}^i|^2\,R^{\parallel}(1 - 2\cos^2\theta) \tag{2.15}$$

and

$$(\mathbf{k}\times\mathbf{E}^{i^*})\cdot(\mathbf{k}^r\times\mathbf{E}^r) = |\mathbf{E}^i|^2 k^2 R^{\parallel}\ .$$

We use these results together with the normalization condition (1.3) and express the reflection coefficients $R^{\perp}$ and $R^{\parallel}$ [Eqs. (1.6) and (1.9)] in terms of the variables p and

$$s = (\epsilon_2\mu_2 - 1 + p^2)^{1/2} \tag{2.16}$$

instead of $k_z = kp$ and $\tilde{k}_z = ks$, that is, we write

$$R^{\perp} = \frac{\mu_2 p - s}{\mu_2 p + s} \tag{2.17}$$

and

$$R^{\parallel} = \frac{\epsilon_2 p - s}{\epsilon_2 p + s}\ . \tag{2.18}$$

We thereby obtain

$$V^{\mathrm{I}} = -\frac{\hbar}{2\pi c^3}\int_0^\infty d\omega\,\omega^3\,\mathrm{Re}\int_1^0 dp\,e^{2i\omega\ell p/c} \times[\alpha_1(\omega)H(p,\epsilon_2,\mu_2)+\beta_1(\omega)\,H(p,\mu_2,\epsilon_2)]\,, \tag{2.19}$$

where

$$H(p,x,y) \equiv \frac{s-yp}{s+yp}+(1-2p^2)\frac{s-xp}{s+xp}\,. \tag{2.20}$$

We turn now to the contribution of class II modes, those with waves incident from $z > 0$. Since only modes with low frequencies contribute, $n_2(\omega) > 1$ and there will be a critical angle for total reflection, given by $\sin\theta_c = 1/n_2$. For $\theta < \theta_c$, the transmitted wave contributes only to the self-energy of the system-wall pair, since

$$|\mathbf{E}^t\,e^{ik_z\ell}|^2$$

is independent of ℓ. On the other hand, for $\theta > \theta_c$, the transmitted wave

$$\mathbf{E}^t\,e^{i(k_x x+k_y y-iK_z z)}\,, \qquad K_z \geq 0$$

decays for $z < 0$. Its absolute value squared at the system's position $z = -\ell$, namely $|\mathbf{E}^t|^2\,e^{-2K_z\ell}$, does depend on ℓ and therefore contributes to the system-wall interaction energy. Using polar coordinates with $k = n_2(\omega/c)$, the contribution of class II modes is

$$\begin{aligned} V^{\mathrm{II}} &= V^{\mathrm{II}}_{\mathrm{el}}+V^{\mathrm{II}}_{\mathrm{mag}} \\ &= -2\pi^2\hbar\sum_{\lambda=1,2}\int_0^\infty d\omega\,n_2^3\int_0^{P_c}dp\,N^2\,e^{-2K_z\ell} \\ &\quad\times\left(\frac{\omega^3}{c^3}\alpha_1(\omega)|\mathbf{E}^t|^2+\frac{\omega}{c}\beta_1(\omega)|\mathbf{k}^t\times\mathbf{E}^t|^2\right)\,, \end{aligned} \tag{2.21}$$

where

$$P_c = (1-\sin^2\theta_c)^{1/2} = \left[(n_2^2-1)/n_2^2\right]^{1/2}\,. \tag{2.22}$$

Using Eqs. (1.17), (1.18), and

$$k_z^2 = n_2^2(\omega^2/c^2)P^2 \tag{2.23}$$

to express K_z^2 in terms of P, we find

$$K_z^2 = (\omega^2/c^2)[n_2^2(1-P^2)-1]\,. \tag{2.24}$$

In performing the sum over the polarizations λ in Eq. (2.21), two points must be kept in mind. First, since the incident waves in class II modes fall from medium 2 onto medium 1 (the vacuum), the roles of the two media in the expressions given for the reflection and transmission coefficients must be interchanged. Second, if $\mathbf{a}$ is a vector with complex component, we of course have $|\mathbf{a}|^2 = |a_x|^2+|a_y|^2+|a_z|^2$. We then find, for $\mathbf{E}^i$ perpendicular to the scattering plane,

$$|\mathbf{E}^t|^2 = |\mathbf{E}^i|^2|T^\perp|^2 \tag{2.25}$$

and

$$|\mathbf{k}^t\times\mathbf{E}^t|^2 = (\omega^2/c^2)|\mathbf{E}^i|^2|T^\perp|^2\left[|\cos\theta_t|^2+\sin^2\theta_t\right]\,, \tag{2.26}$$

where θ_t is, formally, the angle of the transmitted wave. Since

$$\sin^2\theta_t = n_2^2\sin^2\theta \geq 1 \quad \text{for} \quad \theta \geq \theta_c\,, \tag{2.27}$$

we have

$$|\cos\theta_t|^2 = |1-\sin^2\theta_t| = \sin^2\theta_t-1 = n_2^2(1-p^2)-1 \tag{2.28}$$

and

$$|\mathbf{k}^t\times\mathbf{E}^t|^2 = (\omega^2/c^2)|\mathbf{E}^i|^2|T^\perp|^2\left[(n_2^2-1)+n_2^2(1-2p^2)\right]\,. \tag{2.29}$$

From Eq. (1.7) we have

$$T^\perp = \frac{2k_z}{k_z-i\mu_2K_z}\,, \tag{2.30}$$

so that, using Eqs. (2.23) and (2.24), we obtain

$$|T^\perp|^2 = \frac{4\epsilon_2p^2}{\mu_2(\epsilon_2\mu_2-1)-(\mu_2^2-1)\epsilon_2p^2}\,. \tag{2.31}$$

In the second case, that is, for $\mathbf{E}^i$ parallel to the scattering plane, we have

$$\begin{aligned} |\mathbf{E}^t|^2 &= |\mathbf{E}^i|^2|T^\parallel|^2\left[|\cos\theta_t|^2+\sin^2\theta_t\right] \\ &= |\mathbf{E}^i|^2|T^\parallel|^2\left[(n_2^2-1)+n_2^2(1-2p^2)\right] \end{aligned} \tag{2.32}$$

and

$$|\mathbf{k}^t\times\mathbf{E}^t|^2 = (\omega^2/c^2)|\mathbf{E}^i|^2|T^\parallel|^2\,. \tag{2.33}$$

By Eq. (1.10)

$$T^\parallel = \frac{2n_2k_z}{\mu_2k_z-in_2^2K_z}\,, \tag{2.34}$$

which leads, using Eqs. (2.23) and (2.24), to

$$|T^\parallel|^2 = \frac{4\epsilon_2p^2}{\epsilon_2(\epsilon_2\mu_2-1)-(\epsilon_2^2-1)\mu_2p^2}\,. \tag{2.35}$$

Inserting these results in Eq. (2.21) for V^{II} and using the normalization condition (1.3) with $\epsilon_1 \to \epsilon_2$ leads to

$$\begin{aligned} V^{\mathrm{II}} = &-\frac{\hbar}{4\pi c^3}\int_0^\infty d\omega\,\omega^3\,n_2^3\int_0^{P_c}dp\,e^{-2K_z\ell} \\ &\times\{\alpha_1(\omega)[G(p,\epsilon_2,\mu_2) \\ &+F(p,\mu_2,\epsilon_2)\,G(p,\mu_2,\epsilon_2)]+\beta_1(\omega)[\epsilon_2\leftrightarrow\mu_2]\}. \end{aligned} \tag{2.36}$$

In Eq. (2.36), we have

$$\begin{aligned} F(p,\epsilon_2,\mu_2) &= F(p,\mu_2,\epsilon_2) \\ &= (n_2^2-1)+n_2^2(1-2p^2)\,, \end{aligned} \tag{2.37}$$

$$G(p,\epsilon_2,\mu_2) \equiv 4p^2/\left[\mu_2(\epsilon_2\mu_2 - 1) - (\mu_2^2 - 1)\epsilon_2 p^2\right] = |T^{\perp}|^2/\epsilon_2, \quad (2.38)$$

and $G(p,\mu_2,\epsilon_2)$ is given by Eq. (2.38) with ϵ_2 and μ_2 interchanged. Note that

$$G(p,\mu_2,\epsilon_2) = |T^{\parallel}|^2/\epsilon_2 \;. \quad (2.39)$$

Changing the variable of integration from p to $P = (c/\omega)K_z$ we obtain, using Eq. (2.24),

$$V^{\mathrm{II}} = -\frac{\hbar}{2\pi c^3}\int d\omega\,\omega^3\left[\alpha_1(\omega)\,Q_{\mathrm{el}}(\omega) + \beta_1(\omega)\,Q_{\mathrm{mag}}(\omega)\right], \quad (2.40)$$

where

$$Q_{\mathrm{el}}(\omega) = \int_0^{\Delta} dP\,2P(n_2^2 - 1 - P^2)^{1/2}\,e^{-2\omega\ell P/c} \times\left[h(\mu_2,\epsilon_2,P) + (1+2P^2)\,h(\epsilon_2,\mu_2,P)\right] \;, \quad (2.41)$$

$$Q_{\mathrm{mag}}(\omega) = Q_{\mathrm{el}}(\omega;\,\epsilon_2 \leftrightarrow u_2) \;, \quad (2.42)$$

$$h(\epsilon_2,\mu_2,P) = \epsilon_2/[(n_2^2-1) + (\epsilon_2^2 - 1)P^2], \quad (2.43)$$

and

$$\Delta = (n_2^2 - 1)^{1/2} \;. \quad (2.44)$$

Now, employing the definitions (2.20) and (2.16), the functions $Q_{\mathrm{el}}(\omega)$ and $Q_{\mathrm{mag}}(\omega)$ can be written as

$$Q_{\mathrm{el}}(\omega) = \int_0^{\Delta} dP\,e^{-2\omega\ell P/c}\mathrm{Re}[iH(P,\epsilon_2,\mu_2)], \quad (2.45)$$

$$Q_{\mathrm{mag}}(\omega) \equiv Q_{\mathrm{el}}(\omega;\,\epsilon_2 \leftrightarrow u_2) \;. \quad (2.46)$$

Defining

$$J_{\mathrm{el}}(C,\omega) \equiv \mathrm{Re}\int_C dp\,e^{2i\omega\ell p/c}\,H(p,\epsilon_2,\mu_2) \quad (2.47)$$

and

$$J_{\mathrm{mag}}(C,\omega) \equiv J_{\mathrm{el}}(C,\omega;\epsilon_2 \leftrightarrow u_2) \;, \quad (2.48)$$

where C is an arbitrary contour in the complex p plane, we have

$$Q_{\mathrm{el}}(\omega) = J_{\mathrm{el}}(C_{0,i\Delta},\omega) \quad (2.49)$$

and

$$Q_{\mathrm{mag}}(\omega) = J_{\mathrm{mag}}(C_{0,i\Delta},\omega), \quad (2.50)$$

where the contour $C_{0,i\Delta}$ runs from zero to $i\Delta$ along the imaginary p axis. Since, by Eqs. (2.20) and (2.16),

$$\mathrm{Re}\,[iH(p,\epsilon_2,\mu_2)] = 0 \quad \text{for} \quad p > \Delta \;, \quad (2.51)$$

the contour $C_{0,i\Delta}$ can be extended to $C_{0,i\infty}$, which runs along the imaginary p axis from 0 to $i\infty$. Using these results expressions (2.19) and (2.40) can be combined to give

$$V = -\frac{\hbar}{2\pi c^3}\int d\omega\,\omega^3\left[\alpha_1(\omega)\,J_{\mathrm{el}}(C_{1,0,i\infty},\omega) + \beta_1(\omega)\,J_{\mathrm{mag}}(C_{1,0,i\infty},\omega)\right] \;, \quad (2.52)$$

where the contour $C_{1,0,i\infty}$ runs from 1 to zero along the real p axis and then from zero to $i\infty$ along the imaginary p axis. In Secs. III and IV, the general expression (2.52) for the interaction of a polarizable system and a wall will be specialized to the atom–dielectric-wall case and to the electron–dielectric-wall case, whereby considerable simplifications can be achieved.

III. THE ATOM-WALL INTERACTION

As mentioned above, only low frequencies $0 \le \omega \le \bar{\omega}$ contribute significantly to the integral in Eq. (2.52). We can, therefore, replace $\epsilon_2(\omega)$ and $\mu_2(\omega)$ appearing in the definitions (2.47) and (2.48) of J_{el} and J_{mag} by $\epsilon_{20} \equiv \epsilon_2$ $(\omega = 0)$ and $\mu_{20} \equiv \mu_2$ $(\omega = 0)$, thereby reducing the ω dependence of the J's to the exponential $e^{2i\omega\ell p/c}$. Similarly, the polarizabilities $\alpha_1(\omega)$ and $\beta_1(\omega)$ in Eq. (2.52) will be replaced by their static values $\alpha_{10} \equiv \alpha_1(\omega = 0)$ and $\beta_{10} \equiv \beta_1(\omega = 0)$. Having done that, we can extend the upper limit of the integral over ω to ∞. We now change the order of integration in Eq. (2.52). Introducing a cutoff factor $e^{-\gamma\omega}$ to take care of spurious divergencies arising from the dipole approximation, we find

$$\lim_{\gamma\to 0}\int_0^{\infty} e^{-\gamma\omega}\,e^{2i\omega\ell p/c}\,\omega^3\,d\omega = 6\left(\frac{c}{2\ell p}\right)^4 \;. \quad (3.1)$$

Inserting this result in Eq. (2.52) and denoting the atom-wall interaction by the subscript AtD, we find

$$V_{\mathrm{At}D} = -\frac{3\hbar c}{16\pi\ell^4}\mathrm{Re}\left(\int_1^0 dp + \int_0^{i\infty} dp\right) \times\frac{\alpha_{10}H(p,\epsilon_{20},\mu_{20}) + \beta_{10}H(p,\mu_{20},\epsilon_{20})}{p^4}. \quad (3.2)$$

Now consider the integrals

$$\int_{C_0} dp\,H(p,\epsilon_{20},\mu_{20})/p^4, \qquad \int_{C_0} dp\,H(p,\mu_{20},\epsilon_{20})/p^4 \;, \quad (3.3)$$

where C_0 is a closed contour running along the real axis from $P > 0$ (eventually we will let $P \to \infty$) to 0, then along the imaginary axis to iP, and then, along the quarter circle with radius P, back to P. We now assume that $\mu_{20} - 1 > 0$; since $\epsilon_{20} - 1 > 0$, it is easy to check that neither $H(p,\epsilon_{20},\mu_{20})$ nor $H(p,\mu_{20},\epsilon_{20})$ can have poles in the upper right quadrant of the complex p plane. As pointed out by Lifshitz [4], the singularity of the integrands at $p = 0$ need not concern us; the contribution from $p = 0$ to $V_{\mathrm{At}D}$ is independent of ℓ and therefore adds to the self-energies of each of the two bodies but not to their interaction energy. We therefore conclude, by Cauchy's theorem, that both integrals in (3.3) vanish. Since $H(P,\epsilon_{20},\mu_{20})$ is proportional to P^2 for $P \to \infty$, the integration along the arc iP to P contributes nothing, as $P \to \infty$, and we are led to

$$\mathrm{Re}\left(\int_1^0 dp\cdots + \int_0^{i\infty} dp\ldots\right) = \mathrm{Re}\int_1^\infty dp\cdots = \int_1^\infty dp\cdots \ , \quad (3.4)$$

where the last step follows from the fact that the integrand in Eq. (3.2) is real along the real p axis. We thus derive the following simplified result for the atom–dielectric-wall interaction in the asymptotic domain

$$V_{\mathrm{At}D} = -\frac{3\hbar c}{16\pi\ell^4} \times \int_1^\infty \frac{dp\,[\alpha_{10}\,H(p,\epsilon_{20},\mu_{20}) + \beta_{10}H(p,\mu_{20},\epsilon_{20}]}{p^4} \ . \quad (3.5)$$

Setting $\beta_{10} = 0$ and $\mu_{20} = 1$ in the last equation we recover Lifshitz's result [4]. Since Eq. (3.5) involves only one-dimensional integration, it is easy to calculate the interaction $V_{\mathrm{At}D}$ numerically as a function of the two parameters involved, namely ϵ_{20} and μ_{20}. It is also possible to evaluate the integrals analytically. Using Eq. (2.20) and the Appendix, we find

$$\begin{aligned} V_{\mathrm{At}D} = \frac{3\hbar c}{16\pi\ell^4}\{\alpha_{10}\,[\tfrac{4}{3} - 2\mu_{20}I^{(4)}(1,\mu_{20}) \\ -2\epsilon_{20}I^{(4)}(1,\epsilon_{20}) \\ +4\epsilon_{20}I^{(2)}(1,\epsilon_{20})] + \beta_{10}[\epsilon_{20} \leftrightarrow \mu_{20}]\} \ . \end{aligned} \quad (3.6)$$

Setting $\beta_{10} = 0$ and $\mu_{20} = 1$, Eq. (3.6) reduces, as it should, to the result derived by Dzyaloshinskii, Lifshitz, and Pitaevskii et al [5].

As a further check, we could use the analytic properties of the functions $I_b^{(9)}(1)$, defined in the Appendix, to derive from Eq. (3.6) some known special results (e.g., the atom–ideal-wall interaction). It turns out to be simpler to derive these special results directly from Eq. (3.5). Thus, using $\epsilon_{20} \to \infty$ for the case of the ideal conducting wall, we find from Eqs. (2.16) and (2.20) that as $\epsilon_{20} \to \infty$

$$\begin{aligned} H(p,\epsilon_{20},\mu_{20}) &\to 2p^2 \ , \\ H(p,\mu_{20},\epsilon_{20}) &\to -2p^2 \ , \end{aligned} \quad (3.7)$$

and, hence, by Eq. (3.5),

$$V_{\mathrm{At}M} = \frac{-3\hbar c}{8\pi\ell^4}(\alpha_{10} - \beta_{10}) \ , \quad (3.8)$$

where the subscript M indicates that the wall is an ideal (metallic) wall. This result was derived by Boyer [6]. The original derivation, with $\beta_{10} = 0$, is due to Casimir [7]. For an ideal ferromagnet we let $\mu_{20} \to \infty$. The signs in Eqs. (3.7) and (3.8) are then reversed, that is, as $\mu_{20} \to \infty$,

$$\begin{aligned} H(p,\epsilon_{20},\mu_{20}) &\to -2p^2 \ , \\ H(p,\mu_{20},\epsilon_{20}) &\to 2p^2 \ , \end{aligned} \quad (3.7')$$

and, with the subscript F denoting a ferromagnet,

$$V_{\mathrm{At}F} = \frac{3\hbar c}{8\pi\ell^4}(\alpha_{10} - \beta_{10}) \ . \quad (3.8')$$

As can be checked, Eqs. (3.8) and (3.8)$'$ follow also from the general expression (3.6) by use of the asymptotic expressions (A19)–(A23).

The case $\epsilon_{20} \approx 1$ and $\mu_{20} \approx 1$ is also of interest. From Eq. (2.16) we obtain

$$\begin{aligned} \frac{s_0 - \epsilon_{20}p}{s_0 + \epsilon_{20}p} &\sim \frac{n_{20}^2 - 1}{4p^2} - \frac{\epsilon_{20} - 1}{2} \\ \frac{s_0 - \mu_{20}p}{s_0 + \mu_{20}p} &\sim \frac{n_{20}^2 - 1}{4p^2} - \frac{\mu_{20} - 1}{2} \ . \end{aligned} \quad (3.9)$$

Inserting this approximation into Eq. (2.20) and performing the integration in Eq. (3.5), we find

$$\begin{aligned} V_{\mathrm{At}D}(\epsilon_{20} \approx 1,\ \mu_{20} \approx 1) \\ \approx \frac{\hbar c}{32\pi\ell^4}\{\alpha_{10}\,[\tfrac{2}{5}(n_{20} - 1) - 5(\epsilon_{20} - 1) + (\mu_{20} - 1)] \\ +\beta_{10}\,[\tfrac{2}{5}(n_{20} - 1) - 5(\mu_{20} - 1) + (\epsilon_{20} - 1)]\} \ . \end{aligned} \quad (3.10)$$

As a special case, consider a wall made up of a dilute gas of atoms. We then have

$$\begin{aligned} \epsilon_{20} - 1 &= 4\pi N_{2\mathrm{At}}\alpha_{20} \ll 1, \\ \mu_{20} - 1 &= 4\pi N_{2\mathrm{At}}\beta_{20} \ll 1 \ , \end{aligned} \quad (3.11)$$

where $N_{2\mathrm{At}}$ is the number density of atoms in the wall. Furthermore, the atom-wall interaction in this case is given by the sum of the interactions of the atom in the vacuum with all the atoms of the wall. Assuming the atom-atom interaction at a distance r to be given by a power law

$$V_{\mathrm{AtAt}}(r) = -C_q/r^q \ , \quad (3.12)$$

we find by integration [8]

$$V_{\mathrm{At}D}(\epsilon_{20} \approx 1,\ \mu_{20} \approx 1) = -2\pi\, N_{2\mathrm{At}}C_q/[(q-2)(q-3)\ell^{q-3}]. \quad (3.13)$$

Comparing Eq. (3.10) with (3.13) and using (1.1) and (3.11), we obtain $q = 7$ for the atom-atom power law and

$$C_7 = \frac{\hbar c}{4\pi}\,[23(\alpha_{10}\alpha_{20} + \beta_{10}\beta_{20}) - 7(\alpha_{10}\beta_{20} + \alpha_{20}\beta_{10})] \quad (3.14)$$

for the coefficient in Eq. (3.12). These results have been known for a long time [9,6].

IV. THE ELECTRON-WALL INTERACTION

Our analysis of $V_{\mathrm{El}D}$ will differ from that of $V_{\mathrm{At}D}$ in two respects. First, we will neglect the magnetic properties of the electron, that is, we ignore the spin and set $\beta_1 = 0$. (We do not ignore the permittivity of the wall.) Second, while the static limit α_{10} of the polarizability $\alpha_1(\omega)$ exists for an atom, it does not for a free electron, for which

$$\alpha_1(\omega) = -e^2/m\omega^2 \ . \tag{4.1}$$

Using the low-frequency limit ϵ_{20} and μ_{20} for the wall and inserting expression (4.1) for $\alpha_1(\omega)$ into Eq. (2.52), we could try (as in the atomic case) to integrate first over ω, that is, to employ

$$\lim_{\gamma\to 0} \int_0^\infty e^{-\gamma\omega}\, e^{2i\omega\ell p/c}\, \omega\, d\omega = -\left(\frac{c}{2\ell p}\right)^2 \tag{4.2}$$

to obtain

$$\begin{aligned} V_{\mathrm{E1}D} &= -\frac{\hbar e^2}{8\pi\, mc\ell^2} \operatorname{Re}\left(\int_1^0 dp + \int_0^{i\infty} dp\right) \\ &\quad \times \frac{H(p,\epsilon_{20},\mu_{20})}{p^2} \ . \end{aligned} \tag{4.3}$$

While it is still true that $H(p,\epsilon_{20},\mu_{20})$ has no poles in the upper right quadrant of the complex p plane, it is no longer true that the integral along the quarter circle from iP to P vanishes. On the contrary, since

$$H(P,\epsilon_{20},1) \sim 2\frac{\epsilon_{20}-1}{\epsilon_{20}+1}P^2 \equiv BP^2 \tag{4.4}$$

as $P \sim \infty$, this integral diverges. We must, therefore, extract from H its asymptotic part given by Eq. (4.4), before the procedure leading to Eq. (3.5) can be applied. Writing $H(p,\epsilon_{20},\mu_{20})$ in the form

$$H(p,\epsilon_{20},\mu_{20}) = Bp^2 + \overline{H}(p,\epsilon_{20},\mu_{20}) \ , \tag{4.5}$$

we can calculate explicitly the contribution V_B from the asymptotic part, namely

$$V_B = \frac{\hbar e^2}{8\pi\, mc\ell^2}\, 2\, \frac{\epsilon_{20}-1}{\epsilon_{20}+1} \ , \tag{4.6}$$

where Eqs. (4.4) and (4.3) have been employed. [The second integral in Eq. (4.3) does not contribute, since the integrand there is purely imaginary.] Since $\overline{H}(p,\epsilon_{20},\mu_{20})$ defined by Eq. (4.5) behaves as a constant for $p \to \infty$, the procedures which lead to Eq. (3.5) can be applied to $\overline{H}$ and one obtains

$$V_{\mathrm{E1}D} = \frac{\hbar e^2}{8\pi\, mc\ell^2}\left[2\frac{\epsilon_{20}-1}{\epsilon_{20}+1} - \int_1^\infty \frac{dp}{p^2}\overline{H}(p,\epsilon_{20},\mu_{20})\right] \ . \tag{4.7}$$

Finally, using the definitions (4.4), (4.5), and (2.20), we obtain

$$\begin{aligned} V_{\mathrm{E1}D} &= \frac{\hbar e^2}{8\pi\, mc\ell^2}\left[-4 - 2\mu_{20} I^{(2)}(1,\mu_{20})\right. \\ &\quad \left. -2\epsilon_{20} I^{(2)}(1,\epsilon_{20}) + 4\epsilon_{20} I^{(0)}(1,\epsilon_{20})\right] \ , \end{aligned} \tag{4.8}$$

where the I's are given in the Appendix.

We now evaluate $V_{\mathrm{E1}D}$ for some known particular cases. For the electron–ideal wall interaction ($\epsilon_{20} = \infty$) we use

$$\begin{aligned} H(p,\epsilon_{20} = \infty,\mu_{20}) &= 2p^2 \ , \\ \overline{H}(p,\epsilon_{20} = \infty,\mu_{20}) &= 2p^2 - 2p^2 = 0 \ , \end{aligned} \tag{4.9}$$

to obtain from (4.7)

$$V_{\mathrm{E1}M} = \frac{\hbar e^2}{4\pi\, mc\ell^2}, \tag{4.10}$$

a known result [10,11].

A similar procedure, starting from Eq. (4.7), does not work for the electron-ferromagnetic wall ($\mu_{20} \to \infty$). The failure can be traced to the lack of commutativity of some limiting processes. Thus, we have

$$\lim_{\mu_{20}\to\infty} \lim_{p\sim\infty} H(p,\epsilon_{20},\mu_{20}) \sim 2\frac{\epsilon_{20}-1}{\epsilon_{20}+1}p^2,$$

but

$$\lim_{p\sim\infty} \lim_{\mu_{20}\to\infty} H(p,\epsilon_{20},\mu_{20}) \sim -2p^2 \ .$$

[The problem did not arise in the electron–metallic-wall case since $H(p,\epsilon_{20},\mu_{20})$ approaches $2p^2$ for $\epsilon_{20} \to \infty$ and $p \sim \infty$, for either order of the limiting processes.] One has to fall back upon Eq. (4.3) from which one easily obtains both Eq. (4.10), in the limit $\epsilon_{20} \to \infty$, and

$$V_{\mathrm{E1}F} = -\frac{\hbar e^2}{4\pi\, mc\ell^2} \tag{4.10$'$}$$

in the limit $\mu_{20} \to \infty$. Incidentally, the same difficulty arises when one tries to apply the asymptotic behavior (A19)–(A23) to the general expression (4.8). It works for the electron–metallic-wall case but fails for the electron–ferromagnetic-wall case due to the divergence noted in Eq. (A24).

For $\epsilon_{20} \approx 1$ and $\mu_{20} \approx 1$, we have by Eqs. (2.20), (3.9), (4.4), and (4.5)

$$\begin{aligned} &\overline{H}(p,\epsilon_{20} \approx 1,\ \mu_{20} \approx 1) \\ &\qquad \approx [(\epsilon_{20}-1) + (\mu_{20}-1)]\left(\tfrac{1}{2p^2} - 1\right) . \end{aligned} \tag{4.11}$$

Inserting this result into Eq. (4.7), we obtain

$$\begin{aligned} &V_{\mathrm{E1}D}(\epsilon_{20} \approx 1,\ \mu_{20} \approx 1) \\ &\qquad \approx \frac{\hbar e^2}{48\pi\, mc\ell^2}\left[11(\epsilon_{20}-1) + 5(\mu_{20}-1)\right] . \end{aligned} \tag{4.12}$$

In particular, for a wall made up of a dilute gas of atoms, assuming a power law for the electron-atom interaction,

$$V_{\mathrm{E1At}}(r) = C_q r^q \ , \tag{4.13}$$

we obtain, using (3.11), (3.13), and (4.12),

$$q = 5, \quad c_5 = \frac{\hbar \bar{e}^2}{4\pi\, mc}(11\alpha_{20} + 5\beta_{20}) \ . \tag{4.14}$$

The α and β terms were obtained previously for the interaction of a charged point particle and a neutral system [12], while the α term was obtained previously for the interaction of a charged point particle and a charged system [11].

V. DISCUSSION

For the very limited case of asymptotic separations, the results obtained encompass essentially all of the results obtained previously for the interaction of an electron or of an atom with a dielectric permeable wall, of two atoms, and of an electron and an ion or atom, and are somewhat more general than those results. The asymptotic domain is a relatively simple one, by virtue of the fact that the various frequency-dependent polarizabilities reduce to their zero- or very-low frequency values. A manifestation of that simplicity is that apart from the determination of the amplitudes of the Fresnel modes, the calculation is entirely classical.

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APPENDIX: EVALUATION OF SOME INTEGRALS

Only definite integrals are required for our purposes, but it will be useful to begin by considering some associated indefinite integrals. We introduce

$$J^{(q)}(P,b) = \int^P \frac{dp}{p^q}\,\frac{s-bp}{s+bp} \tag{A1}$$

where

$$s = (a^2+p^2)^{1/2} , \quad a^2 = n_{20}^2 - 1 > 0 , \quad b = \epsilon_{20} \text{ or } \mu_{20} ,$$

and $q = 4, 2,$ or 0. We rewrite Eq. (A1) as

$$\begin{aligned} J^{(q)}(P,b) &= \int^P \frac{dp}{p^q}\left(1 - \frac{2bp}{s+bp}\right) \\ &= -\frac{1}{q-1}\,\frac{1}{P^{q-1}} - 2b\,I^{(q)}(P,b) , \end{aligned} \tag{A2}$$

where

$$I^{(q)}(P,b) = \int^P \frac{dp}{p^{q-1}}\,\frac{1}{s+bp} , \quad q = 4,2,0 . \tag{A3}$$

The transformation

$$t = s + p = (a^2+p^2)^{1/2} + p \tag{A4}$$

brings the integrand in Eq. (A3) to a rational form suitable for quadrature. One obtains

$$\begin{aligned} &I^{(4)}(P,b) \\ &= -\frac{1}{2P^2}\,\frac{1}{s+P} + \frac{2b-1}{2a^2}\,\frac{1}{P} + \frac{2b^2-1}{2a^3}\ln A(P) \\ &\quad -\frac{b(b^2-1)^{1/2}}{a^3}\ln B(P) , \end{aligned} \tag{A5}$$

where

$$A(P) = \frac{s+P-a}{s+P+a} \tag{A6}$$

and

$$B(P) = \frac{s+P-[(b-1)/(b+1)]^{1/2}a}{s+P+[(b-1)/(b+1)]^{1/2}a} . \tag{A7}$$

Similarly, we find

$$I^{(2)}(P,b) = \frac{1}{a}\left[\ln A(P) - \frac{b}{(b^2-1)^{1/2}}\ln B(P)\right] \tag{A8}$$

and

$$\begin{aligned} I^{(0)}(P,b) &= \frac{1}{2(b+1)}\left[s+P - \frac{b+1}{b-1}\,\frac{a^2}{s+P}\right. \\ &\quad \left. -\frac{2ab}{(b-1)(b^2-1)^{1/2}}\ln B(P)\right] . \end{aligned} \tag{A9}$$

For further reference we note that

$$\begin{aligned} &I^{(4)}(P,b) \to 0, I^{(2)}(P,b) \to 0, \\ &I^{(0)}(P,b) \sim P/(b+1) \end{aligned} \tag{A10}$$

as $P \to \infty$. Using

$$a^2 = \epsilon_{20}\mu_{20} - 1 = n_{20}^2 - 1 , \tag{A11}$$

we find

$$\begin{aligned} I^{(4)}(P=1,b) &= \frac{2b-n_{20}}{2(n_{20}^2-1)} - \frac{2b^2-1}{2(n_{20}^2-1)^{3/2}}\ln C \\ &\quad -\frac{b(b^2-1)^{1/2}}{(n_{20}^2-1)^{3/2}}\ln D , \end{aligned} \tag{A12}$$

where

$$C = n_{20} + (n_{20}^2-1)^{1/2} \tag{A13}$$

and

$$D = (n_{20}+b) \,/\, \left[(bn_{20}+1) + (b^2-1)^{1/2}(n_{20}^2-1)^{1/2}\right] . \tag{A14}$$

Similarly, we find

$$\begin{aligned} &I^{(2)}(P=1,b) \\ &= \frac{1}{(n_{20}^2-1)^{1/2}}\left[-\ln C - \frac{b}{(b^2-1)^{1/2}}\ln D\right] \end{aligned} \tag{A15}$$

and

$$\begin{aligned} I^{(0)}(P=1,b) = \frac{1}{b^2-1}\Bigg[&(b-n_{20}) \\ &-\frac{b(n_{20}^2-1)^{1/2}}{(b^2-1)^{1/2}}\ln D\Bigg] . \end{aligned} \tag{A16}$$

To facilitate comparison with other works, we will need the values of $I^{(q)}(P=1,b=1)$ for $q=4$ and 2. These are given by

$$I^{(4)}(P=1,b=1) = \frac{1}{2(n_{20}^2-1)}\left[2-n_{20}-\frac{1}{(n_{20}^2-1)^{1/2}}\ln C\right] \quad \text{(A17)}$$

and, using Eq. (A15) and l'Hôspital's rule,

$$I^{(2)}(P=1,b=1) = \frac{1}{(n_{20}^2-1)^{1/2}}\left[-\ln C+\left(\frac{n_{20}-1}{n_{20}+1}\right)^{1/2}\right] . \quad \text{(A18)}$$

Finally, it is useful to have expressions for the asymptotic behavior of the functions $I^{(q)}(1,b)$ in the limit $b\to\infty$ or $n_{20}\to\infty$. For $b\to\infty$ we expand in powers of the parameter (n_{20}/b) [recall that (n_{20}/b) is proportional to either $\epsilon_{20}^{-1/2}$ or $\mu_{20}^{-1/2}$]. Keeping powers up to and including third order, we find, as $b\to\infty$,

$$bI^{(4)}(1,b)\to-\frac{1}{3} , \quad \text{(A19)}$$

$$bI^{(2)}(1,b)\to-1 , \quad \text{(A20)}$$

$$bI^{(0)}(1,b)\to 1 . \quad \text{(A21)}$$

The limits, as $n_{20}\to\infty$, are much easier to find. We obtain

$$I^{(4)}(1,b) \underset{n_{20}\to\infty}{\longrightarrow} 0 \quad \text{(A22)}$$

and

$$I^{(2)}(1,b)\overset{n_{20}\to\infty}{\longrightarrow}0. \quad \text{(A23)}$$

Note, however, that

$$I^{(0)}(1,b)\propto n_{20} , \quad n_{20}\sim\infty . \quad \text{(A24)}$$

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