# Retarded Casimir interaction in the asymptotic domain of an electron and a dielectric wall

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(Received 6 April 1993)

A number of different approaches have been used to determine the interactions between a variety of pairs of polarizable systems asymptotically far apart; these are examples of retardation or Casimir interactions. The pairs include two dielectric walls, and an atom and a dielectric wall, but not an electron and a dielectric wall. (The asymptotic value of the interaction between an electron and an ideal conducting wall has been evaluated.) We here apply a method not previously used in the evaluation of retardation potentials, quantized Fresnel modes, to determine the interaction in the asymptotic domain of a polarizable system and a dielectric wall. We thereby reproduce the known result for an atom and a dielectric wall and obtain a previously unobtained result, that for an electron and a dielectric wall. The result simplifies greatly for an ideal metallic wall and for a wall made from a material such as liquid helium for which the dielectric constant is very close to unity. We also discuss the question of a connection between the electron–dielectric-wall interaction and Lifshitz's force per unit area between two dielectric walls. The determination of the amplitudes of the Fresnel modes, by quantization, is the only nonclassical element in the calculation.

PACS number(s): 31.30.Jv, 12.20.Ds, 77.90.+k, 41.20.−q

## I. INTRODUCTION

There are a number of pairs of systems for which the asymptotic value of the interaction is known. These interactions represent retardation or Casimir interactions. The pairs include two ideal walls [1] with a vacuum between them, and, more generally, two dielectric walls with a vacuum [2] or a dielectric medium [3,4] between them. The walls are semi-infinite, plane, and parallel; ideal walls have infinite conductivity. The pairs also include an atom and an ideal wall [5] and, more generally, an atom and a dielectric wall [2], two atoms [6], an electron and a neutral system [7], an electron and an ion [8–10], and an electron and an ideal wall [11,9,12]. In many of these cases more is known than just the leading asymptotic form.

A pair for which the asymptotic form of the interaction does not seem to have been determined is an electron and a dielectric wall. We will use a method not previously applied to the study of Casimir interactions, namely that of quantized Fresnel modes. The method is of interest in its own right and, as discussed in Sec. VI, the electron–dielectric-wall interaction may possibly offer an alternative approach for confirming a Casimir effect.

We turn now to the important question of how to describe a wall. A useful description is given by its frequency-dependent dielectric "constant." In Lifshitz's approach [2], the electromagnetic modes are calculated by means of the fluctuation-dissipation theorem and, as indicated by the word dissipation, retention of the frequency-dependent imaginary (absorptive) component of the electric permittivity — often referred to as the dielectric "constant" — is essential. In our approach, the quantized electromagnetic Fresnel modes are introduced by hand, so to speak, and, rather surprisingly, the imaginary parts must be excluded; $\epsilon_1 = \epsilon_1(\omega)$ and $\epsilon_2 = \epsilon_2(\omega)$ will refer to the *real* parts of the electric permittivities of the two walls. In the present context in which only asymptotically large values of the separation $\ell$ are considered, a crude physical justification for using the real parts can be based on the fact that only very low frequencies are relevant — a point to be much discussed below — and that $\epsilon(\omega)$ becomes real as $\omega$ approaches zero. For some justification for using the real parts for arbitrary values of $\omega$, see Ginzburg [13], especially p. 338, and references therein. See also Milonni and Shih [14]. They give a much more formal but rather wide-ranging discussion of the subject, showing why various seemingly disparate approaches lead to the same result. They too give references to earlier papers on the question of the retention or elimination of the imaginary parts.

We will make contact between our results for the electron–dielectric-wall interaction and results deduced from the expression derived by Lifshitz for the force per unit area between dielectric walls. Indeed, we will show, with perhaps some ambiguity, how one can derive the electron-wall interaction from Lifshitz's expression.

Lifshitz ignored all magnetic effects — he set the magnetic permeability equal to unity and he ignored magnetic fields generated by oscillating electric multipole moments — and in the present article we will do the same.

We emphasize the fact that *all of our considerations are limited to separations $\ell$ which are asymptotically*

 

*large.* Nevertheless, we use the equal sign when $\ell$ is the only parameter which is asymptotically large. This enables us to make clear, by the use of the asymptotic symbol $\sim$, when some other parameter is asymptotically large.

### Preliminary remarks on the electron-wall interaction

Our primary goal will be to obtain the interaction $V_{\mathrm{El}D}$ of an electron and a dielectric wall, where the electron is at a distance $\ell$ from the wall. (We use the subscripts El, At, $D$, and $M$ to refer to electron, atom, dielectric wall, and metallic wall, respectively.) Our determination of $V_{\mathrm{El}D}$ will involve a fair amount of notation, formalism, and algebra, and one can arrive at the final result with little physical understanding. It is therefore useful to note that a wealth of information concerning $V_{\mathrm{El}D}$ follows on general grounds [15], so much so that one can readily derive a simple useful approximation to $V_{\mathrm{El}D}$. We give a brief review of the argument. We begin by determining the dependence of $V_{\mathrm{El}D}$ on dimensional quantities $e$, $m$, $c$, $\hbar$, and $\ell$. To do so we use a slight adaptation of a method discussed previously [10] in the determination of some Casimir interactions which did not involve dielectric walls. We extract some of the essential physics and then use dimensional analysis. Dimensional analysis alone does not always suffice. We then go on to obtain the approximate dependence of $V_{\mathrm{El}D}$ on the electric permittivity $\epsilon_2(\omega)$ of the wall.

The essential physics includes the recognition that the wall is completely characterized by its electric permittivity $\epsilon_2(\omega)$, and the electron, which can be treated as free for present purposes, is completely characterized by its dynamic electric dipole polarizability $\alpha_1(\omega) = -e^2/m\omega^2$. At the asymptotically large values of $\ell$ under consideration, only low-frequency components of the vacuum fluctuation electric field are significant, and $\epsilon_2(\omega)$ can be replaced by the real number $\epsilon_2(\omega = 0) \equiv \epsilon_{20}$. Furthermore, $V_{\mathrm{El}D}$ is the integral over all electric field modes of $-(1/2)\alpha_1(\omega)\mathbf{E}^2(\omega)$, where $\mathbf{E}(\omega)$ is one such mode — a more precise definition of $\mathbf{E}(\omega)$ will be given later — so that $V_{\mathrm{El}D}$ is proportional to $e^2/m$. Since $\mathbf{E}^2(\omega)$ is proportional to the energy density of a particular mode, $\mathbf{E}^2(\omega)$, and therefore $V_{\mathrm{El}D}$, are proportional to $\hbar$. The only other relevant ingredients are $\ell$ and $c$, and dimensional analysis leads to

$$V_{\mathrm{El}D} = \frac{e^2\hbar}{mc\ell^2}\, g_{\mathrm{El}D}(\epsilon_{20}) \ . \tag{1.1}$$

This differs from the form arrived at in the analysis of $V_{\mathrm{El}M}$ in Ref. [10] in the multiple of $(e^2\hbar/mc\ell^2)$ is not an arbitrary real constant but $g_{\mathrm{El}D}(\epsilon_{20})$, which at this stage is an arbitrary real function. We can impose two conditions on $g_{\mathrm{El}D}(\epsilon_{20})$. First, the interaction of an atom and an ideal (metallic) conducting wall ($\epsilon_{20} = \infty$) is known to be

$$V_{\mathrm{El}M} = \frac{1}{4\pi}\,\frac{e^2\hbar}{mc\ell^2} \ , \tag{1.2}$$

so that

$$g_{\mathrm{El}D}(\infty) = \frac{1}{4\pi} \ . \tag{1.3}$$

Second, we (formally) consider a wall consisting of a dilute gas of atoms. We then have $\epsilon_{20} \approx 1$, or, more precisely,

$$\epsilon_{20} \approx 1 + 4\pi\, N_{2\mathrm{At}}\alpha_{20} \ , \tag{1.4}$$

where $N_{2At}$ is the number of atoms per cubic centimeter and $\alpha_{20}$ is the zero-frequency electric dipole polarizability of one of those atoms. (We note that $\epsilon_{20} > 1$.) $V_{\mathrm{El}D}(\epsilon_{20} \approx 1)$ is the additive sum of the *retarded* interactions of the electron with the (independent) atoms in the wall. That retarded interaction, for a separation $r$, is [7,8]

$$V_{\mathrm{El}\,\mathrm{At}}(r) = \frac{11}{4\pi}\,\frac{\alpha_{20}e^2\hbar}{mcr^5} \ . \tag{1.5}$$

The sum (really an integral) is

$$\begin{aligned} V_{\mathrm{El}D}(\epsilon_{20} \approx 1) &\approx \frac{11}{12}\,\frac{e^2\hbar}{mc\ell^2}\, N_{2\mathrm{At}}\alpha_{20} \\ &\approx \frac{11}{48\pi}\,\frac{e^2\hbar}{mc\ell^2}(\epsilon_{20} - 1) \ , \end{aligned} \tag{1.6}$$

so that

$$g_{\mathrm{El}D}(\epsilon_{20} \approx 1) \approx \frac{11}{48\pi}(\epsilon_{20} - 1) \ . \tag{1.7}$$

[Since for our purposes walls are completely characterized by $\epsilon_{20}$, Eq. (1.6) is valid not only for dilute walls but for walls, such as liquid-helium walls, for which $\epsilon_{20} \approx 1$.]

Perhaps the simplest form for $g_{\mathrm{El}D}(\epsilon_{20})$ which satisfies Eqs. (1.3) and (1.7) is the approximation (denoted by a prime)

$$g'_{\mathrm{El}D}(\epsilon_{20}) = \frac{1}{4\pi}\,\frac{\epsilon_{20} - 1}{\epsilon_{20} + (1/11)} \ , \tag{1.8}$$

which is roughly reliable over the entire range $1 \le \epsilon_{20} \le \infty$, and which leads to the approximation [15]

$$V'_{\mathrm{El}D}(\epsilon_{20}) = \frac{1}{4\pi}\,\frac{e^2\hbar}{mc\ell^2}\,\frac{\epsilon_{20} - 1}{\epsilon_{20} + (1/11)} \ . \tag{1.9}$$

Analogous arguments [15] provide approximations to $V_{\mathrm{At}D}$ and $V_{DD}$.

## II. NOTATION AND QUANTIZED FRESNEL MODES

Two semi-infinite materials, characterized by dielectric constants $\epsilon_1(\mathbf{r})$ and $\epsilon_2(\mathbf{r})$, respectively, have a plane interface. A plane electromagnetic wave is incident at an angle $\theta_i$ to the normal to the wall from medium 1. See Fig. 1. The superscripts $i$, $r$, and $t$ refer to incident, reflected, and transmitted waves. The reflection and transmission

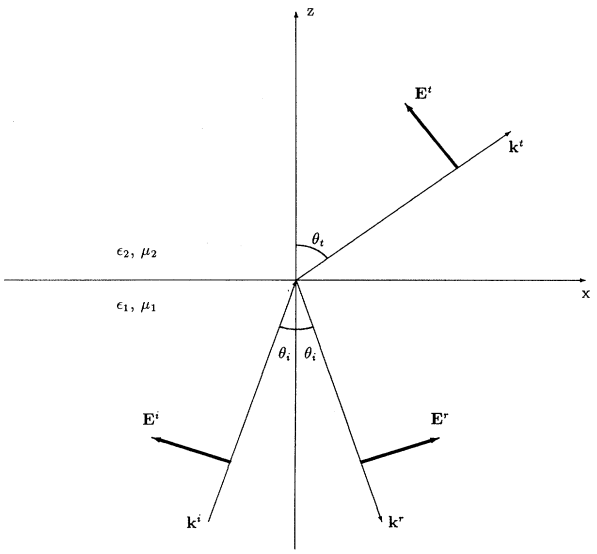


FIG. 1. Fresnel mode, class I. The polarization depicted is that for **E** parallel to the plane of scattering.

coefficients are denoted by $R = E^r/E^i$ and $T = E^t/E^i$. In particular, we use $R^\perp$ and $T^\perp$ for $\mathbf{E}^i$ perpendicular to the scattering plane and $R^\parallel$ and $T^\parallel$ for $\mathbf{E}^i$ parallel to (in) the scattering plane. The symbol $q \equiv (\mathbf{k}, \lambda)$ for a mode represents the wave vector $\mathbf{k}$ and the index of polarization $\lambda$. If $\epsilon_1(\mathbf{r}) = \epsilon_1$ and $\epsilon_2(\mathbf{r}) = \epsilon_2$, where $\epsilon_1$ and $\epsilon_2$ are constants and $\epsilon_2 > \epsilon_1$, the Fresnel modes for waves incident from the half-space $z < 0$, defined as class I modes, are given by [16]

$$\begin{aligned} \mathbf{f}_q(\mathbf{r}) &= N^{\mathrm{I}}(\mathbf{E}^i e^{i\mathbf{k}\cdot\mathbf{r}} + \mathbf{E}^r e^{i\mathbf{k}^r\cdot\mathbf{r}}) , \qquad z < 0 \\ &= N^{\mathrm{I}}\mathbf{E}^t e^{i\mathbf{k}^t\cdot\mathbf{r}} , \qquad z > 0 . \end{aligned} \tag{2.1}$$

The $q$ dependence of the fields will generally be suppressed. $N^{\mathrm{I}}$ is a normalization constant, defined below, and the components of the different wave vectors are given by

$$\begin{aligned} \mathbf{k}^i \equiv \mathbf{k} = (k_x, k_y, k_z) , \quad \mathbf{k}^r = (k_x, k_y, -k_z) , \\ \mathbf{k}^t = (k_x, k_y, \tilde{k}_z) . \end{aligned} \tag{2.2}$$

All of the components are real, and $k_z$ and $\tilde{k}_z$ are nonnegative. With $\omega$ the frequency, the wave vectors satisfy the dispersion relations

$$k_x^2 + k_y^2 + k_z^2 = \mathbf{k}^2 = \epsilon_1\omega^2/c^2 , \tag{2.3}$$

$$k_x^2 + k_y^2 + \tilde{k}_z^2 = \epsilon_2\omega^2/c^2 . \tag{2.4}$$

A second set of Fresnel modes (class II) represents waves incident from $z > 0$. For an angle of incidence $\theta$ less than the critical angle $\theta_c$, defined by $\sin\theta_c = n_1/n_2$ where $n_1$ and $n_2$ are the indices of refraction, the class II modes are obtained from the class I modes by interchanging the constants $\epsilon_1$ and $\epsilon_2$ and replacing $z$ by $-z$. These modes are given by

$$\mathbf{f}_q(\mathbf{r}) = \begin{cases} N^{\mathrm{II}}(\mathbf{E}^i e^{i\mathbf{k}\cdot\mathbf{r}} + \mathbf{E}^r e^{i\mathbf{k}^r\cdot\mathbf{r}}) , & z > 0 \\ N^{\mathrm{II}}\mathbf{E}^t e^{i\mathbf{k}^t\cdot\mathbf{r}}, & z < 0. \end{cases} \tag{2.5}$$

To avoid a very cumbersome notation we have used the same notation as for class I modes, but note that some of the symbols have different meanings for the two classes; $\omega$, however, is the same. The components of the different class II wave vectors for $\theta > \theta_c$ are given by

$$\begin{aligned} \mathbf{k}^i &\equiv \mathbf{k} = (k_x, k_y, -k_z) , \quad \mathbf{k}^r = (k_x, k_y, +k_z) , \\ \mathbf{k}^t &= (k_x, k_y, -iK_z) , \end{aligned} \tag{2.6}$$

where $k_x, k_y, k_z$, and $K_z$ are real and $k_z$ and $K_z$ are nonnegative. The dispersion relations are now

$$k_x^2 + k_y^2 + k_z^2 = \epsilon_2\omega^2/c^2, \tag{2.7}$$

$$k_x^2 + k_y^2 - K_z^2 = \epsilon_1\omega^2/c^2 . \tag{2.8}$$

Since we are here concerned with total internal reflection — frustrated refraction is the more appropriate terminology — the $\mathbf{f}_q$'s are decaying functions in the region $z < 0$, the reason the third component of $\mathbf{k}^t$ was taken to be $-iK_z$. The normalization constant $N^{\mathrm{II}}$ will be defined below.

In the above analysis it was assumed that $\epsilon_1$ and $\epsilon_2$ were different from one another, but independent of $\mathbf{r}$; it was not necessary to assume that they were independent of frequency. Let us now consider the opposite situation, in which $\epsilon_1$ and $\epsilon_2$ can depend upon $\mathbf{r}$ but are independent of $\omega$ [16].

The $\mathbf{f}_q(\mathbf{r})$ then obey the eigenmode equation

$$\frac{1}{\epsilon(\mathbf{r})}\boldsymbol{\nabla}\times[\boldsymbol{\nabla}\times\mathbf{f}_q(\mathbf{r})] = \frac{\omega^2}{c^2}\mathbf{f}_q(\mathbf{r}) \tag{2.9}$$

and the transversality condition

$$\boldsymbol{\nabla}\cdot[\epsilon(\mathbf{r})\,\mathbf{f}_q(\mathbf{r})] = 0 ; \tag{2.10}$$

this condition demands that the $z$ component of $\epsilon(\mathbf{r})\,\mathbf{f}_q(\mathbf{r})$ be continuous at $z = 0$. In addition, there are the usual continuity conditions on components of the electric and magnetic fields.

Since the operator in the eigenmode equation, Eq. (2.9), is not Hermitian, the $\mathbf{f}_q(\mathbf{r})$ are not orthogonal. However, the vector functions

$$\mathbf{g}_q(\mathbf{r}) \equiv [\epsilon(\mathbf{r})]^{1/2}\,\mathbf{f}_q(\mathbf{r}) \tag{2.11}$$

satisfy

$$\frac{1}{\sqrt{\epsilon(\mathbf{r})}}\boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\times\frac{\mathbf{g}_q(\mathbf{r})}{\sqrt{\epsilon(\mathbf{r})}}\right) = \frac{\omega^2}{c^2}\,\mathbf{g}_q(\mathbf{r}) , \tag{2.12}$$

that is, the $\mathbf{g}_q(\mathbf{r})$ are eigenvectors of a Hermitian operator, and it follows [16], with the appropriate choice of $N$, that

$$\begin{aligned} \int \mathbf{g}_q(\mathbf{r})\cdot\mathbf{g}_{q'}^*(\mathbf{r})\,d\mathbf{r} &= \int \epsilon(\mathbf{r})\,\mathbf{f}_q(\mathbf{r})\cdot\mathbf{f}_{q'}^*(\mathbf{r})\,d\mathbf{r} \\ &= \delta_{qq'} = \delta(\mathbf{k}-\mathbf{k}')\delta_{\lambda\lambda'} . \end{aligned} \tag{2.13}$$

Equation (2.13) is valid whether or not each of the two

media is uniform, but in the problems we will consider they will be uniform. As shown in Appendix A, the proper choice for that case is

$$|\mathbf{E}^i|^2(N^{\mathrm{I}})^2 = [(2\pi)^3\epsilon_1]^{-1} \tag{2.14a}$$

and

$$|\mathbf{E}^i|^2(N^{\mathrm{II}})^2 = [(2\pi)^3\epsilon_2]^{-1}. \tag{2.14b}$$

[As a partial check on Eqs. (2.14), note that they are correct for $\epsilon_1 = \epsilon_2$, for which $\mathbf{f}(\mathbf{r})$ is a plane wave throughout space.] For $\epsilon_1$ and $\epsilon_2$ functions of $\mathbf{r}$ but not of $\omega$, Glauber and Lewenstein prove not only that the $\mathbf{g}_q$'s are orthogonal but demonstrate explicitly that the $\mathbf{g}_q$'s are complete [16]. They do not, however, consider the case for which $\epsilon_1$ and $\epsilon_2$ are frequency dependent, and allowing for $\omega$ dependence is essential for our purposes. If we do so, however, the operator in the eigenvalue equation, Eq. (2.12), is $\omega$ dependent, that is, depends upon the eigenvalue, and the $\mathbf{g}_q$'s need not be orthogonal. Indeed, using the explicit forms of the $\mathbf{g}_q$'s, one can check that they are not. Fortunately, all that will be needed for our purposes is for the $\mathbf{g}_q$'s to be complete and for the expansion of an arbitrary function in the $\mathbf{g}_q$'s to be unique, and that is the case. (Between them, class I and class II modes are represented by six waves — all plane waves, or five plane waves and one decaying mode. The saving feature is that between them they are represented by only one plane wave moving toward the interface in the region $z > 0$ and one in the region $z < 0$.)

In the Coulomb gauge, normalizing in a box of volume $\tau$, the quantized electromagnetic field is given in terms of the $\mathbf{f}_q$'s by

$$\mathbf{A} = \sum_q (2\pi\hbar c^2/\omega\tau)^{1/2}(a_q\mathbf{f}_q + a_q^\dagger\mathbf{f}_q^*)i, \tag{2.15}$$

$$\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} = i\sum_q (2\pi\hbar\omega/\tau)^{1/2}(a_q\mathbf{f}_q - a_q^\dagger\mathbf{f}_q^*) \equiv \sum_q \mathbf{E}_q, \tag{2.16}$$

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A} = \sum_q \left(\frac{2\pi\hbar c^2}{\omega\tau}\right)^{1/2}(a_q\boldsymbol{\nabla}\times\mathbf{f}_q + a_q^\dagger\boldsymbol{\nabla}\times\mathbf{f}_q^*)\ . \tag{2.17}$$

The $a_q$ and $a_q^\dagger$ are the usual destruction and creation operators, satisfying

$$[a_q, a_{q'}] = [a_q^\dagger, a_{q'}^\dagger] = 0\ , \quad [a_q, a_{q'}^\dagger] = \delta_{qq'}\ . \tag{2.18}$$

In going from discrete to continuous eigenvalues, $\sum_q$ is to be replaced by

$$\sum_{\lambda=1,2}\int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \int dk_z\,\tau\ ,$$

with the limits on $k_z$ given by 0 and $\infty$ for class I and $-\infty$ to 0 for class II modes; there is no factor of $1/(2\pi)^3$, it having been absorbed in the $(N)^2$. With $|0\rangle$ representing the vacuum state for the photons, use of the normalization condition given by Eq. (2.14) leads to the desired result,

$$\int\left\langle 0\left|\frac{\epsilon(\mathbf{r})\,\mathbf{E}^2(\mathbf{r})}{8\pi} + \frac{\mathbf{B}^2(\mathbf{r})}{8\pi}\right|0\right\rangle d\mathbf{r} = \sum_q\left(\frac{\hbar\omega}{4}+\frac{\hbar\omega}{4}\right) = \sum_q\frac{\hbar\omega}{2}\ ; \tag{2.19}$$

the energy of a mode of frequency $\omega$ is, as it should be, $\hbar\omega/2$.

## III. THE INTERACTION OF A POLARIZABLE SYSTEM AND A WALL

In this section we will derive explicit, if formal, expressions for the interaction $V$ of a polarizable system and a wall. (We have used the subscripts El, At, $D$, and $M$ to denote electron, atom, dielectric wall, and metallic wall, respectively. It will suffice for this section to use the simple notation $V$ for the polarizable-system–wall interaction for it is the only interaction considered in this section; it will be useful to do so since we will have to append subscripts and superscripts on $V$ — specifying the class, and the domain of $\omega$, under consideration — in the course of its derivation.)

In Secs. IV and V these expressions will be specialized to the interaction of an atom with a wall and an electron with a wall, whereby considerable simplification can be achieved. We assume that $\epsilon_1 = 1$ and that, in the absence of the wall, the polarizable system would be in a spherically symmetric state. With the system placed at a distance $\ell$ from the wall, we consider its interaction $\mathcal{V}$ (to be distinguished from $V$ — see later) with the vacuum fluctuations of the electric field; the wall manifests itself by its influence on the electric field. We will make the dipole approximation, in which the electric field is assumed to be constant over the system. This approximation is better the greater the value of $\ell$. (As will be shown below, contributions to the interaction from modes with $\omega \gtrsim c/\ell$ tend to cancel, and for $\omega \lesssim c/\ell$ and $\ell$ large, the variation of $\mathbf{E}_q$ — the Fourier component of the field at the location of the system — will be small over the dimension of the system. It is essential, however, to recognize that the dipole approximation, for $\ell$ fixed, is incorrect for $\omega$ sufficiently large and often leads to infinities in the estimation of interaction energies. We will, whenever necessary, compensate for the incorrect treatment of high frequencies by introducing a high-frequency cutoff factor.) The interaction of the system with the component $\mathbf{E}_q$, which is independent of $\lambda$ for the spherically symmetric system under consideration, can then be taken to be

$$\langle 0 \mid -\tfrac{1}{2}\alpha_1(\omega)\,\mathbf{E}_q^2 \mid 0\rangle\ .$$

Using Eq. (2.16), the total interaction is then

$$-\left\langle 0\left|\sum_q \frac{1}{2}\alpha_1(\omega)\mathbf{E}_q^2\right|0\right\rangle = -\sum_q [\pi\hbar\omega\alpha_1(\omega)/\tau]\mathbf{f}_q(\mathbf{r})\cdot\mathbf{f}_q^*(\mathbf{r}) \ , \quad (3.1)$$

to be evaluated at the location $\mathbf{r}$ of the system. Here $\alpha_1(\omega)$ is the (real part of) the dynamic electric dipole polarizability of the system. In determining the modes in Sec. II, we assumed that $\epsilon_2 > \epsilon_1$ or, in the present context, that $\epsilon_2(\omega) > 1$. There are ranges of values of $\omega$ for which this is true, but there are also ranges for which $\epsilon_2(\omega) < 1$, and it will be necessary to evaluate the contributions from $\epsilon_2(\omega) > 1$ and $\epsilon_2(\omega) < 1$ separately.

### A. $\epsilon_2(\omega) > 1$

We use the subscripts $>$ and $<$ to denote contributions associated with $\epsilon_2(\omega) > 1$ and $\epsilon_2(\omega) < 1$, respectively. Summing over all modes, the contribution of class I modes to $\mathcal{V}$, with $\omega$'s restricted to those for which $\epsilon_2(\omega) > 1$, is

$$\mathcal{V}_>^{\rm I} = -\frac{1}{2}\sum_{\lambda=1,2}\int_{-\infty}^{\infty}dk_x\int_{-\infty}^{\infty}dk_y\int_0^{\infty}dk_z\alpha_1(\omega)\,\mathbf{E}^2(\omega) \ , \quad (3.2)$$

where

$$\omega^2 = c^2(k_x^2+k_y^2+k_z^2) \ . \quad (3.3)$$

With $(0,0,-\ell)$ representing the cartesian coordinates of the system, this becomes

$$\mathcal{V}_>^{\rm I} = -\pi\sum_q \alpha_1(\omega)\hbar\omega\,\mathbf{f}_q(0,0,-\ell)\cdot\mathbf{f}_q^*(0,0,-\ell)$$
$$= -\pi\sum_{\lambda=1,2}\int_{-\infty}^{\infty}dk_x\int_{-\infty}^{\infty}dk_y\int_0^{\infty}dk_z\,\alpha_1(\omega)\hbar\omega(N^{\rm I})^2$$
$$\times|\mathbf{E}^i e^{-ik_z\ell}+\mathbf{E}^r e^{ik_z\ell}|^2 \ . \quad (3.4)$$

Note that $\mathbf{E}^t$ does not contribute, since it does not exist in the region in which the system resides. The contributions of the terms $\mathbf{E}^i\cdot\mathbf{E}^{i^*}$ and $\mathbf{E}^r\cdot\mathbf{E}^{r^*}$ can be neglected. They are independent of $\ell$ and therefore add to the self-energy of the pair consisting of a free system at infinity and a wall, but not to the system-wall interaction. Only the cross terms $\mathbf{E}^i\cdot\mathbf{E}^{r^*}$ and $\mathbf{E}^{i^*}\cdot\mathbf{E}^r$ contribute to the interaction. Denoting the part of the interaction which depends on $\ell$ by $V_>^{\rm I}$, not $\mathcal{V}_>^{\rm I}$, and using polar coordinates, we arrive at

$$V_>^{\rm I} = -\frac{4\pi^2\hbar}{c^3}\sum_{\lambda=1,2}\int_> d\omega\,\omega^3\,\alpha_1(\omega)$$
$$\times{\rm Re}\int_0^1 dp(N^{\rm I})^2\mathbf{E}^{i^*}\cdot\mathbf{E}^r\,e^{2i\omega\ell p/c} \ , \quad (3.5)$$

where $k_z = k\cos\theta \equiv kp$ and, since $\epsilon_1 = 1$, $k = \omega/c$. Here and henceforth we will use $\theta$ for the angle of incidence $\theta_i$. In Eqs. (3.2) and (3.4) it is understood that the only $\omega$'s which contribute are those for which $\epsilon_2(\omega) > 1$; this is made explicit in Eq. (3.5) by the presence of the subscript $>$ on the integral sign.

To perform the sum over the polarizations $\lambda$, we note that for $\mathbf{E}^i$ perpendicular to the scattering plane,

$$\mathbf{E}^{i^*}\cdot\mathbf{E}^r = |\mathbf{E}^i|^2R^\perp,$$

where [17]

$$R^\perp = \frac{k_z-\tilde{k}_z}{k_z+\tilde{k}_z} = \frac{p-s_2}{p+s_2}, \quad (3.6)$$

$$k_z = kp \ , \quad \tilde{k}_z = ks_2 \ , \quad s_2 = (\epsilon_2-1+p^2)^{1/2} = (n_2^2-\sin^2\theta)^{1/2} \ ; \quad (3.7)$$

$n_2(\omega)$ is the frequency-dependent index of refraction of the wall. For $\mathbf{E}^i$ parallel to the scattering plane, we have (see Fig. 1),

$$\begin{aligned}\mathbf{E}^i &= (-E^i\cos\theta\,,\,0\,,\,E^i\sin\theta),\\ \mathbf{E}^r &= (E^iR^\parallel\cos\theta\,,\,0\,,\,E^iR^\parallel\sin\theta),\\ \mathbf{E}^t &= (-E^iT^\parallel\cos\theta\,,\,0\,,\,E^iT^\parallel\sin\theta) \ .\end{aligned} \quad (3.8)$$

We then find

$$\mathbf{E}^{i^*}\cdot\mathbf{E}^r = -|\mathbf{E}^i|^2R^\parallel\cos(2\theta) = |\mathbf{E}^i|^2R^\parallel(1-2p^2) \ ,$$

where

$$R^\parallel = \frac{\epsilon_2k_z-\tilde{k}_z}{\epsilon_2k_z+\tilde{k}_z} = \frac{\epsilon_2p-s_2}{\epsilon_2p+s_2} \ . \quad (3.9)$$

(For later purposes, we note that for $\epsilon_2-1<0$ and for $\theta > \theta_c$, where $\theta_c$ is the critical angle for total reflection, both $R^\perp$ and $R^\parallel$ become complex, with absolute value equal to 1.) We then find, using the normalization Eq. (2.14a),

$$V_>^{\rm I} = -\frac{\hbar}{2\pi c^3}\int_> d\omega\,\omega^3\alpha_1(\omega)$$
$$\times{\rm Re}\int_1^0 dp\,e^{2i\omega\ell p/c}\,H(p,\epsilon_2) \ , \quad (3.10)$$

where

$$H(p,\epsilon_2) = \frac{s_2-p}{s_2+p}+(1-2p^2)\frac{s_2-\epsilon_2p}{s_2+\epsilon_2p} \ . \quad (3.11)$$

Defining

$$J(C,\omega) \equiv {\rm Re}\int_C e^{2i\omega\ell p/c}\,H(p,\epsilon_2)\,dp \ , \quad (3.12)$$

where $C$ is an arbitrary contour in the complex $p$ plane, and

$$M_>(C) \equiv -\frac{\hbar}{2\pi c^3}\int_> d\omega\,\omega^3\alpha_1(\omega)\,J(C,\omega) \ , \quad (3.13)$$

we can write

$$V_>^{\rm I} = M_>(1 \text{ to } 0) = M_>(C_{1,0}) \ , \tag{3.14}$$

where $C_{1,0}$ is the contour from 1 to 0 along the real $p$ axis.

We turn now to the contribution of class II modes, those with waves incident from $z > 0$. Since we are here considering $\epsilon_2(\omega) > 1$, there will be a critical angle $\theta_c$ for total reflection, given by $\sin\theta_c = 1/n_2$. For $\theta < \theta_c$, the transmitted wave contributes only to the self-energy of the pair system-wall, since

$$|\mathbf{E}^t\, e^{i(k_x x + k_y y - k_z z)}|^2$$

is independent of $\ell$. On the other hand, for angles of incidence $\theta > \theta_c$, the transmitted wave

$$\mathbf{E}^t\, e^{i(k_x x + k_y y)}\, e^{K_z z} \ , \quad K_z \geq 0 \ , \quad z \leq 0$$

decays with $z$. Its absolute value squared at the system's position $z = -\ell$, namely $|\mathbf{E}^t|^2 \exp(-2K_z\ell)$, does depend on $\ell$ and therefore contributes to the system-wall interaction energy. Using polar coordinates with $k = \sqrt{\epsilon_2}(\omega/c)$, the contribution from class II modes is

$$V_>^{\rm II} = -\frac{2\pi^2\hbar}{c^3} \sum_{\lambda=1,2} \int_> d\omega\, \omega^3 \alpha_1(\omega) \times \epsilon_2^{3/2} \int_0^{p_c} dp (N^{(\rm II)})^2\, |\mathbf{E}^t|^2\, e^{-2K_z\ell} \ , \tag{3.15}$$

where

$$p_c = (1 - \sin^2\theta_c)^{1/2} = [(\epsilon_2 - 1)/\epsilon_2]^{1/2} \ . \tag{3.16}$$

Using Eqs. (2.7), (2.8), and

$$k_z^2 = \epsilon_2(\omega^2/c^2)p^2 \ , \tag{3.17}$$

we find

$$K_z^2 = (\omega^2/c^2)[\epsilon_2(1 - p^2) - 1] \ . \tag{3.18}$$

We now sum over the polarizations. For $\mathbf{E}^i$ perpendicular to the plane of scattering

$$|\mathbf{E}^t|^2 = |\mathbf{E}^i|^2 |T^\perp|^2 \ ,$$

where

$$T^\perp = \frac{2k_z}{k_z - iK_z} \ , \tag{3.19}$$

so that, using Eqs. (3.17) and (3.18), we find

$$|T^\perp|^2 = \frac{4k_z^2}{k_z^2 + K_z^2} = 4\frac{\epsilon_2}{\epsilon_2 - 1}p^2 \ . \tag{3.20}$$

For $\mathbf{E}^{\rm I}$ in the scattering plane, $|\mathbf{E}^t|^2$ is not equal to $|\mathbf{E}^i|^2|T^\parallel|^2$. Rather, we have

$$\begin{aligned} |\mathbf{E}^t|^2 &= \left(|E_x^t|^2 + |E_y^t|^2\right) + |E_z^t|^2 \\ &= |\mathbf{E}_i|^2 |T^\parallel|^2 \left(|\cos\theta_t|^2 + \sin^2\theta_t\right) \ , \end{aligned}$$

where $\theta_t$ is, formally, the angle of the transmitted wave. Since

$$\sin^2\theta_t = \epsilon_2 \sin^2\theta \geq 1 \quad \text{for} \quad \theta \geq \theta_c \ ,$$

we have

$$|\cos\theta_t|^2 = |1 - \sin^2\theta_t| = \sin^2\theta_t - 1 = \epsilon_2(1 - p^2) - 1$$

and

$$|\mathbf{E}^t|^2 = |\mathbf{E}^i|^2 |T^\parallel|^2 \left[(\epsilon_2 - 1) + \epsilon_2(1 - 2p^2)\right] \ . \tag{3.21}$$

$T^\parallel$ is given by

$$T^\parallel = \frac{2\sqrt{\epsilon_2}k_z}{k_z - i\epsilon_2 K_z} \tag{3.22}$$

so that

$$|T^\parallel|^2 = \frac{4\epsilon_2}{\epsilon_2 - 1} \frac{p^2}{\epsilon_2 - (\epsilon_2 + 1)p^2} \ . \tag{3.23}$$

Inserting these results in Eq. (3.15) for $V_>$, and using the normalization condition Eq. (2.14b), we have

$$\begin{aligned} V_>^{\rm II} = -\frac{2\pi^2\hbar}{c^3} \int_> d\omega\, \omega^3 \alpha_1(\omega) \epsilon_2^{3/2} \\ \times \int_0^{p_c} dp \frac{1}{8\pi^3\epsilon_2} \frac{4\epsilon_2}{\epsilon_2 - 1} p^2 \\ \times \left[1 + \frac{\epsilon_2 - 1 + \epsilon_2(1 - 2p^2)}{\epsilon_2 - (\epsilon_2 + 1)p^2}\right] e^{-2K_z\ell} \ . \end{aligned}$$

Changing the variable of integration from $p$ to $P = (c/\omega)K_z$, with $K_z$ as a function of $p$ given by Eq. (3.18), and introducing

$$\Delta \equiv (\epsilon_2 - 1)^{1/2} = \epsilon_2^{1/2} p_c \ , \tag{3.24}$$

the interaction takes the form

$$V_>^{\rm II} = -\frac{\hbar}{2\pi c^3} \int_> d\omega\, \omega^3 \alpha_1(\omega)\, Q(\omega) \ , \tag{3.25}$$

where

$$Q(\omega) \equiv \int_0^\Delta dP\, e^{-2P\omega\ell/c} \left\{ \frac{2P(\epsilon_2 - 1 - P^2)^{1/2}}{\epsilon_2 - 1} \times \left[1 + \frac{\epsilon_2(1 + 2P^2)}{1 + (\epsilon_2 + 1)P^2}\right] \right\} \ . \tag{3.26}$$

Since $0 \leq P \leq \Delta$ in Eq. (3.26), it follows from Eq. (3.11) that

$$Q(\omega) = \int_0^\Delta dP\, e^{-2P\omega\ell/c}\, \mathrm{Re}\left[iH(iP, \epsilon_2)\right] \ .$$

By Eq. (3.11), we can allow $P$ to range from 0 to $\infty$, the contribution from $\Delta$ to $\infty$ vanishing since $P \geq \Delta$ there. Comparison with Eq. (3.12) then gives

$$Q(\omega) = J(C_{0,i\infty}, \omega) \ , \tag{3.27}$$

where $C_{0,i\infty}$ is the contour along the imaginary $p$ axis

running from 0 to $i\infty$. Equations (3.13), (3.25), and (3.27) give

$$V_>^{\rm II} = M_>(0 \text{ to } i\infty) = M_>(C_{0,i\infty}) \ , \tag{3.28}$$

and therefore

$$V_> = V_>^{\rm I} + V_>^{\rm II} = M_>\left(C_{1,0,i\infty}\right) \ ; \tag{3.29}$$

the contour $C_{1,0,i\infty}$ runs along the real $p$ axis from 1 to 0 and then along the imaginary $p$ axis from 0 to $i\infty$.

### B. B. $\epsilon_2(\omega) < 1$

Our description in Sec. II of class I and II modes is not applicable for $\epsilon_2(\omega) < 1$. Thus, for $\epsilon_2 < 1$, there is no total reflection of class II modes and hence no contribution to $V_<$ of the type $V_>^{\rm II}$. On the other hand, for angles of incidence $\theta > \theta_c$, where $\cos\theta_c = (1-\epsilon_2)^{1/2} \equiv \delta$, there is total reflection of class I modes, leading to a new type of contribution to $V_<$. The contributions of class I modes from the two regions $\theta < \theta_c$ and $\theta > \theta_c$ add up to

$$\begin{aligned} V_< = \frac{\hbar}{2\pi c^3}\int_< d\omega\,\omega^3\alpha_1(\omega)\Bigg\{ &\mathrm{Re}\int_\delta^1 dp\, e^{2i\omega\ell p/c}\, H(p,\epsilon_2) \\ &+\mathrm{Re}\int_0^\delta dp\, e^{2i\omega\ell p/c} \\ &\times\left[\frac{i\sigma - p}{i\sigma + p} + (1-2p^2)\frac{i\sigma - \epsilon_2 p}{i\sigma + \epsilon_2 p}\right]\Bigg\} \ , \end{aligned} \tag{3.30}$$

where

$$\sigma = \left[(1-\epsilon_2) - p^2\right]^{1/2} = -is_2 \ ;$$

$s_2$ and $H(p,\epsilon_2)$ are defined by Eqs. (3.7) and (3.11), respectively. By adding and subtracting

$$\mathrm{Re}\int_0^\delta dp\, e^{2i\omega\ell p/c}\, H(p,\epsilon_2)$$

to the integrand of Eq. (3.30), it follows after some algebra that $V_<$ can be written as

$$V_< = -\frac{\hbar}{2\pi c^3}\int_< d\omega\,\omega^3\alpha_1(\omega) J(C_{1,0,i\infty},\omega) \ . \tag{3.31}$$

Thus, $V_<$ is identical in form to $V_>$ of Eq. (3.29). We arrive finally at

$$\begin{aligned} V &= V_> + V_< \\ &= -\frac{\hbar}{2\pi c^3}\int_0^\infty d\omega\,\omega^3\alpha_1(\omega) \\ &\quad\times\mathrm{Re}\left(\int_1^0 dp + \int_0^{i\infty} dp\right) e^{2i\omega\ell p/c}\, H(p,\epsilon_2(\omega)) \ , \end{aligned} \tag{3.32}$$

where the integral over $\omega$ is now over the full range.

Equation (3.32) can be used as it is for certain limiting cases. In particular, it is simple to apply it directly to the case of a system and an ideal metallic wall, for which $\epsilon_2 = \infty$. For $\epsilon_2 \neq \infty$, it is inconvenient for numerical integration, since it is oscillatory for $0 \le p \le 1$, and complex. We will show, using a modification of a method used by Lifshitz, that $V$ can be reduced to a very much simpler form for $V_{{\rm At}D}$ (a form obtained by Lifshitz [2]) *and* for $V_{{\rm El}D}$. We will return to Eq. (3.32) in Secs. IV and V and rederive the simpler forms for $V_{{\rm At}D}$ and $V_{{\rm El}D}$ by using a low-frequency approximation which bypasses many of the analytic difficulties encountered in the Lifshitz approach (but is applicable only in the asymptotic domain).

## IV. THE ATOM-WALL INTERACTION

We seek alternative forms for $V$ of Eq. (3.32) which are neither oscillatory nor complex. (We will find such forms; further, the double integral will be reduced to a single integral.) To obtain such forms, it will be necessary to treat the atom-wall interaction $V_{{\rm At}D}$ and the electron-wall interaction $V_{{\rm El}D}$ separately, the reason being the different behavior of $\alpha_1(\omega)$. $V_{{\rm At}D}$ is well known and we are interested in obtaining $V_{{\rm El}D}$, but we will nevertheless begin by considering $V_{{\rm At}D}$. We do so because an understanding of $V_{{\rm At}D}$ provides a guide in the procedure to be used in obtaining $V_{{\rm El}D}$.

$V_{{\rm At}D}$ is given by Eq. (3.32), with $\alpha_1(\omega)$ taken to be that of an atom. Note that the $p$ and $\omega$ integrations are coupled through the exponential factor and through $\epsilon_2(\omega)$ in $H(p,\epsilon_2)$. We now use a result obtained by Lifshitz in his determination of the force per unit area between two homogeneous walls at a separation $\ell$, where the walls are characterized by $\bar\epsilon_1(\omega)$ and $\bar\epsilon_2(\omega)$. We use $\bar\epsilon(\omega)$ to represent the complex electric permittivity, $\epsilon(\omega)$ having been preempted in our notation by the real component of the electric permittivity. Lifshitz showed that

$$\begin{aligned} \mathrm{Re}\int_0^\infty \omega^3\, d\omega\,\alpha_1(\omega)\left(\int_1^0 dp + \int_0^{i\infty} dp\right) &\\ \times G\left(p,\bar\epsilon_1(\omega),\bar\epsilon_2(\omega),\ell\right) &\\ = \int_0^\infty \alpha_1(i\xi)\xi^3\, d\xi \int_1^\infty dp\, G\left(p,\epsilon_1(i\xi),\epsilon_2(i\xi),\ell\right) \ . \end{aligned} \tag{4.1}$$

[We need not use a bar on the real function $\epsilon(i\xi)$. The fact that the result can be written in a form which depends only upon $\epsilon_1(i\xi)$ and $\epsilon_2(i\xi)$ strongly suggests, in line with comments by Ginzburg in Ref. [13], that one can use the study of transparent media to obtain information on absorptive media.] The function $G$ is rather complicated. The point of present interest arises on allowing wall number 1 to consist of a dilute gas of atoms, that is, setting

$$\epsilon_1(\omega) - 1 = 4\pi N_{1\rm At}\alpha_1(\omega) \ . \tag{4.2}$$

The force per unit area can then be taken to originate in the forces of atoms in wall 1 treated independently of one another with wall 2, from which one readily derives $V_{{\rm At}D}$. (A similar approach was used in the discussion

in Sec. I.) $V_{AtD}$ is then found to be identical in form to the $V_{AtD}$ of Eq. (3.32), but with $\epsilon_2(\omega)$ replaced by $\bar{\epsilon}_2(\omega)$. By Eq. (4.1), in which $\omega \to i\xi$ but the range of $\xi$ is the same as that of $\omega$ and the range of $p$ has been changed, it follows that $V_{AtD}$ of Eq. (3.32) can also be written as (the real quantity)

$$V_{AtD} = -\frac{\hbar}{2\pi c^3}\int_0^\infty d\xi\,\xi^3\alpha_1(i\xi) \times \int_1^\infty dp\, e^{-2\xi\ell p/c}\, H[p,\epsilon_2(i\xi)]\ , \qquad (4.3)$$

where $H(p,\epsilon_2)$ is defined by Eq. (3.11). We conclude from the exponential factor that the only significant values of $\xi$ are those for which $\xi \lesssim c/(2\ell p)$or, since $p \geq 1$, $\xi \lesssim c/2\ell$. With $\omega_0$ the smallest frequency of significance in the absorption spectrum of either the wall or the atom, it follows since $c/2\ell \ll \omega_0$ that $\xi \ll \omega_0$. We can therefore replace $\alpha_1(i\xi)$ by $\alpha_1(0) \equiv \alpha_{10}$ and $\epsilon_2(i\xi)$ by $\epsilon_2(0) \equiv \epsilon_{20}$. We can now interchange the order of integration and use

$$\int_0^\infty d\xi\,\xi^3\, e^{-2\xi\ell p/c} = 6\left(\frac{c}{2\ell p}\right)^4\ , \qquad (4.4)$$

to find the relatively simple known form

$$V_{AtD} = -\frac{3\hbar c}{16\pi\ell^4}\,\alpha_{10}\int_1^\infty \frac{H(p,\epsilon_{20})}{p^4}\,dp\ . \qquad (4.5)$$

Analytic results for $V_{AtD}$ are given in Ref. [3]. With $g_{AtD}(\epsilon_{20})$ defined by a comparison of Eq. (4.5) and the analog of (1.1), we find, using the properties of $H(p,\epsilon_{20})$ listed in Eqs. (B9), (B10), and (B12), that $g_{AtD}(\epsilon_{20})$ satisfies equations [15] which are the analogs of Eqs. (1.3) and (1.7) for $g_{ElD}(\epsilon_{20})$.

By virtue of the rather arbitrary and complicated forms which $\epsilon_1(\omega)$ and $\epsilon_2(\omega)$ can assume, the change in the contours in $\omega$ and $p$ space *cannot* be performed sequentially; one cannot show in Eq. (4.1), and thereby in going from Eq. (3.32) applied to the atomic case to Eq. (4.3), that, for all $\omega$, the change in the contour in $p$ space will not cross a pole. Lifshitz, using the theory of functions of two complex variables, found a way of simultaneously displacing both paths, that in $\omega$ space and that in $p$ space, without passing through a pole of the integrand. If all one were interested in were $V_{AtD}$, one could use his technique to proceed directly from Eq. (3.32) applied to the atomic case to Eq. (4.3); the analysis of the pole structure would be much simpler because the integrand defined by Eq. (3.32) is much simpler than $G[p,\epsilon_1(\omega),\epsilon_2(\omega),\ell]$.

We note that $V_{AtD}$ of Eq. (4.5) does not depend upon the functions $\epsilon_2(\omega)$ and $\alpha_1(\omega)$ but only upon $\epsilon_{20}$ and $\alpha_{10}$. *Ex post facto*, therefore, we recognize that $V_{AtD}$ depends only upon asymptotically low frequencies. In fact, this is intuitively more or less clear; one would expect the significant frequencies to satisfy $\omega \lesssim c/2\ell$. One can go somewhat beyond intuition. Returning to Eq. (3.32), we recognize that the exponential leads to $\omega \lesssim c/(2\ell p)$. In Eq. (3.32), however, as opposed to Eq. (4.5), $p$ can assume arbitrarily small values. We can, though, exclude small values of $p$ on physical grounds. Thus, the interaction of two atoms at large distances is the Casimir-Polder $1/r^7$ potential. The atoms in the wall do not act independently of one another, but one nevertheless expects that the atoms in the wall which contribute significantly are those near the surface and close to the line which passes through the atom in free space and is perpendicular to the surface. Now an electromagnetic wave traveling at an angle $\theta$ to the normal to the surface generates an interaction between the free atom and atoms very near the surface, at a distance $\ell/\cos\theta = \ell/p$ from the free atom. We therefore have every reason to believe that the significant contribution comes from the smaller distances associated with $p \approx 1$, and therefore from $\omega \lesssim c/2\ell$.

Having concluded from Eq. (3.32) — *without* depending upon Eq. (4.5) — that the relevant domain of $\omega$ is $\omega \lesssim c/2\ell$, we can replace $\epsilon_2$ by $\epsilon_{20}$ and $\alpha_1$ by $\alpha_{10}$ in Eq. (3.32). The derivation of Eq. (4.5) is then an order of magnitude simpler than that presented above, since it relies on the theory of functions not of two complex variables but of one complex variable. Thus, with $H(p,\epsilon_{20})$ independent of $\omega$, we use

$$\lim_{\gamma\to 0}\int_0^\infty e^{-\gamma\omega}\, e^{2i\omega\ell p/c}\,\omega^3\,d\omega = 6\left(\frac{c}{2\ell p}\right)^4\ , \qquad (4.6)$$

the same value as that obtained in Eq. (4.4). [In line with our remarks in Sec. III on the dipole approximation, we have introduced a convergence factor. We could, alternatively, have used the fact that $\alpha_1(\omega)$ is proportional to $1/\omega^2$ for $\omega \sim \infty$ to obtain the same result.] We now have

$$V_{AtD} = -\frac{\hbar}{2\pi c^3}\times 6\left(\frac{c}{2\ell}\right)^4\alpha_{10} \times \mathrm{Re}\left(\int_1^0 dp + \int_0^{i\infty} dp\right)\frac{H(p,\epsilon_{20})}{p^4}\ . \qquad (4.7)$$

[As pointed out by Lifshitz [2] in his analysis of $V_{DD}$, the singularity of the integrand at $p = 0$ need not concern us. By Eq. (3.32), the contribution to $V$ from $p = 0$ is independent of $\ell$ and thus adds to the self-energy of each of the two bodies but not to their interaction energy.] Now consider the integral

$$\int_{C(\text{closed})} dp\frac{H(p,\epsilon_{20})}{p^4}\ ,$$

where the closed contour runs from 1 to 0 to $iP$ to $P$ to 1, where the real number $P$ is later made to approach infinity, and where the contour from $iP$ to $P$ is a quarter circle of radius $P$. Since $H(p,\epsilon_{20})$ has no poles within $C$ (closed) — see Appendix B — the integral vanishes; since further $H(P,\epsilon_{20})$ is proportional to $P^2$ for $|P| \to \infty$, the integral along the arc from $iP$ to $P$ vanishes as $P \to \infty$, and we conclude that we can replace the integral from 1 to 0 and 0 to $i\infty$ as in Eq. (4.7) by the (real) integral from 1 to $\infty$. We have thereby rederived Eq. (4.5). This gives us confidence in the result for $V_{ElD}$ derived in

the following section by a similar but slightly simplified approach.

Note that once one assumes that only low frequencies are relevant, the derivation of Eq. (3.32) is greatly simplified. Taking the relevant frequencies to be $0 < \omega < \bar{\omega}$, where $\bar{\omega} \ll \omega_0$, one has $\epsilon_2(\omega) - 1 \geq 0$; one need never consider $\epsilon_2(\omega) - 1 \leq 0$. It follows that $V$ is given by $V_>$, but with the range not that for which $\epsilon_2(\omega) - 1 \geq 0$, but $0 \leq \omega \leq \bar{\omega}$. Since only low frequencies are relevant, we can in fact extend the range to $0 \leq \omega \leq \infty$ and arrive again at Eq. (3.32) for $V$.

## V. THE ELECTRON-WALL INTERACTION

The analysis of $V_{\mathrm{El}D}$ requires a somewhat more delicate treatment than that for $V_{\mathrm{At}D}$. The difference in the two treatments originates in the difference in $\alpha_1(\omega)$. While $\alpha_{10}$ exists for an atom, it does not for a free electron, for which

$$\alpha_1(\omega) = -e^2/m\omega^2 \ . \tag{5.1}$$

(In determining the retarded interaction $V_{\mathrm{El}D}$ to leading order, we can neglect the interaction $-e^2/4\ell$ of the electron with its image, and treat the electron as free.) From Eq. (3.32), we then have

$$V_{\mathrm{El}D} = \frac{\hbar e^2}{2\pi mc^3} \int_0^\infty d\omega\, \omega \mathrm{Re}\left( \int_1^0 dp + \int_0^{i\infty} dp \right) e^{2i\omega\ell p/c} H(p, \epsilon_2(\omega)) \ . \tag{5.2}$$

*If* one could obtain $V_{\mathrm{El}D}$ in a simplified form by using Eq. (4.3) for $V_{\mathrm{At}D}$ with $\alpha_1(i\xi)$ set equal to $+e^2/m\xi^2$, obtained from Eq. (5.1), and (as we did in the atomic case) with $\epsilon_2(i\xi)$ replace by $\epsilon_{20}$, we *would* have

$$V_{\mathrm{El}D} = -\frac{\hbar e^2}{2\pi\, mc^3} \int_1^\infty dp\, H(p, \epsilon_{20}) \int_0^\infty d\xi\, \xi\, e^{-2\xi\ell p/c} = -\frac{\hbar e^2}{2\pi\, mc^3} \left(\frac{c}{2\ell}\right)^2 \int_1^\infty dp \frac{H(p, \epsilon_{20})}{p^2} \ .$$

However, $H(p, \epsilon_{20})$ is proportional to $p^2$ for $p \to \infty$, and the last integral diverges. (In the atomic case, the denominator was $p^4$, not $p^2$, and the integral converged.)

Having recognized that the difficulty arises because of the asymptotic form of $H(p, \epsilon_2)$ as $p \to \infty$, it would be natural to attempt to extract the asymptotic form and integrate it out in the form of Eq. (5.2) for $V_{\mathrm{El}D}$. It seems difficult to do so, however, because of the complicated behavior of $H(p, \epsilon_2)$, for a general $\epsilon_2(\omega)$, as $p \to \infty$. The problem can be avoided by using the argument presented in Sec. IV that only asymptotically low frequencies are relevant. $\epsilon_2$ can then be replaced by $\epsilon_{20}$ in Eq. (5.2) and, as opposed to $H(p, \epsilon_2)$, $H(p, \epsilon_{20})$ has a simple asymptotic form for $|p| \to \infty$, namely

$$H(p, \epsilon_{20}) \sim \frac{\epsilon_{20} - 1}{\epsilon_{20} + 1} 2p^2 \equiv Bp^2 \ , \qquad |p| \to \infty \ . \tag{5.3}$$

We write

$$H(p, \epsilon_{20}) = Bp^2 + \overline{H}(p, \epsilon_{20}) \ ; \tag{5.4}$$

$\overline{H}(p, \epsilon_{20})$, defined by this equation, behaves as a constant as $|p| \to \infty$. The integral over $\omega$ in $V_{\mathrm{El}D}$ of Eq. (5.2) is

$$S(p) = \lim_{\gamma \to 0} \int_0^\infty e^{-\gamma\omega} e^{2i\omega\ell p/c}\, \omega\, d\omega = -\left(\frac{c}{2\ell p}\right)^2 \ .$$

[The introduction of a convergence factor can in this case be justified by the fact that the dipole approximation was used, or by recognizing, before setting $\epsilon_2 = \epsilon_{20}$, that $\epsilon_2(\omega) - 1 \sim -\omega_{\mathrm{pl}}^2/\omega^2$ for $\omega \sim \infty$, where $\omega_{\mathrm{pl}}$ is the plasma frequency. The value of $\omega_{\mathrm{pl}}$ is known but irrelevant; what is relevant is that the $1/\omega^2$ factor, at large $\omega$, generates convergence.] We therefore have the one-dimensional integral

$$V_{\mathrm{El}D} = -\left(\frac{\hbar e^2}{8\pi\, mc\ell^2}\right) \mathrm{Re}\left( \int_1^0 dp + \int_0^{i\infty} dp \right) \frac{Bp^2 + \overline{H}(p, \epsilon_{20})}{p^2} \ . \tag{5.5}$$

Denote the term which contains $B$ by $V_B$. The integral from 0 to $i\infty$ is imaginary and therefore makes no contribution, and with $B$ defined by Eq. (5.3),

$$V_B = 2\frac{\epsilon_{20} - 1}{\epsilon_{20} + 1} \frac{\hbar e^2}{8\pi\, mc\ell^2} \ . \tag{5.6}$$

[Note that the ratio of $V_B$ to $V_{\mathrm{El}D}$, for an ideal conductor, is $(\epsilon_{20} - 1)/(\epsilon_{20} + 1)$. This result also follows on noting that by Eqs. (B13) and (B10), $H(p \to \infty, \epsilon_{20})$ and $H(p, \epsilon_{20} = \infty)$ have the same ratio.]

We turn now to the $\overline{H}$ term in Eq. (5.5). $\overline{H}(p, \epsilon_{20})$ has the same analytic structure, for finite values of $p$, as $H(p, \epsilon_{20})$, and $\overline{H}(p, \epsilon_{20})/p^2$ has the same asymptotic form as $H(p, \epsilon_{20})/p^4$. The analysis which allowed us to replace the range 1 to 0 to $i\infty$ by 1 to $\infty$ in the simplified treatment of $V_{\mathrm{At}D}$ can therefore be applied here, and we arrive at our final result,

$$V_{\mathrm{El}D} = \frac{\hbar e^2}{8\pi\, mc\ell^2} \left( 2\frac{\epsilon_{20} - 1}{\epsilon_{20} + 1} - \int_1^\infty dp \frac{\overline{H}(p, \epsilon_{20})}{p^2} \right) \ . \tag{5.7}$$

The integral in Eq. (5.7) can be evaluated analytically, but we will not give the result. First, it is a function of only the one variable $\epsilon_{20}$, so that numerical integration for a fixed value of $\epsilon_{20}$ is trivial; second, we will in the following paper give the analytic result for the more general case of two parameters ($\epsilon_2$ and $\mu_2$) where we do not assume the magnetic permeability $\mu_2$ to be unity.

A comparison of Eq. (5.7) with Eq. (1.1) gives $g_{\mathrm{El}D}(\epsilon_{20})$. One can readily check that the conditions imposed on $g_{\mathrm{El}D}(\epsilon_{20})$ by Eqs. (1.3) and (1.7) are satisfied.

### Does $V_{\mathrm{El}D}$ follow from $V_{DD}$?

With $\alpha_1(\omega)$ interpreted as an atomic polarizability, Eq. (3.32) gives $V_{\mathrm{At}D}$; the result is identical to that de-

duced by starting with the force per unit area $(F/A)_{DD}$ between two walls and taking one of the walls to consist of a dilute gas of atoms. Formally, one can then obtain $V_{\mathrm{El}D}$ by reinterpreting $\alpha_1(\omega)$ as the polarizability of a free electron. There may well be a justification for doing so, but we do not know how to provide it. Thus, in our derivation of Eq. (3.32), the starting point was the integral over all of the electromagnetic modes of $-(1/2)\alpha_1(\omega)\mathbf{E}^2(\omega)$, where $\alpha_1(\omega)$ was the polarizability of a system, which could be an atom or an electron. The Lifshitz result [2] for $(F/A)_{DD}$ characterizes a wall by its $\epsilon(\omega)$. One can conceive of one wall as a dilute gas of atoms and thereby derive $V_{\mathrm{At}D}$. One cannot readily conceive of a wall as a dilute gas of electrons. [It may be possible to obtain $V_{\mathrm{El}D}$ from $(F/A)_{DD}$ by conceiving of the wall as having an enormously *high* density of electrons and nuclei, since in the high-density limit the electrons become free.]

To understand our misgivings about a derivation of $V_{\mathrm{El}D}$ through a reinterpretation of the $\alpha_1(\omega)$, starting with $(F/A)_{DD}$, we recall that Lifshitz begins with a derivation of the electromagnetic field at the surface of the walls. (As a side comment, we note that his derivation is based on fluctuation theory. One could also obtain the field by using quantized Fresnel modes appropriate to three regions, in analogy to the quantized Fresnel modes we used in our two-region analysis of the interaction of a polarizable system and a wall. An analysis along these lines of the interaction of two finite or semi-infinite *one*-dimensional "walls" has been given [18].) Lifshitz then used the Maxwell stress tensor to determine $(F/A)_{DD}$. Let us write the $V_{\mathrm{At}D}$ derived from this in the schematic form

$$V_{\mathrm{At}D} = \int Q(\omega,\ell)\alpha_1(\omega)\,d\omega\ .$$

Lifshitz's form of $Q(\omega,\ell)$ is identical to the result that we obtain and it suggests that one can obtain $V_{\mathrm{el}}(\ell)$ by setting $\alpha_1(\omega) = -e^2/m\omega^2$, but we know of no *a priori* reason why one might not have obtained $V_{\mathrm{At}D}$ in the form

$$V_{\mathrm{At}D} = \int Q'(\omega,\ell)\alpha_1(\omega)\,d\omega\ , \tag{5.8}$$

where

$$\int [Q'(\omega,\ell) - Q(\omega,\ell)]\,\alpha_1(\omega)\,d\omega = 0\ . \tag{5.9}$$

A reinterpretation of $\alpha_1(\omega)$ in the $V_{\mathrm{At}D}$ of Eq. (5.8) as $(-e^2/m\omega^2)$ would then give an incorrect result for $V_{\mathrm{El}D}$.

[One could, of course, use the Lifshitz approach to evaluate the field at any point between the walls, and then, starting with $-(1/2)\alpha(\omega)\mathbf{E}^2(\omega)$, as we did, be guaranteed that the result would be valid for an atom or an electron.]

We can give a much more concrete example in which one cannot take an expression for $V_{\mathrm{At}D}$ which is a function of the atomic $\alpha_1(\omega)$ and derive $V_{\mathrm{El}D}$ by replacing the atomic $\alpha_1(\omega)$ by the electronic $\alpha_1(\omega)$. Equation (3.32), valid for $V_{\mathrm{At}D}$ with the proper interpretation of $\alpha_1(\omega)$, does lead to $V_{\mathrm{El}D}$ with a reinterpretation of $\alpha_1(\omega)$, but, as we have seen, Eq. (4.3) does *not* lead to $V_{\mathrm{El}D}$ on reinterpretation of $\alpha_1(\omega)$. [Indeed, as noted above, with $\alpha_1(\omega)$ reinterpreted the integral in Eq. (4.3) is not even finite.]

## VI. DISCUSSION

There have been many experimental efforts designed to confirm a Casimir interaction. Perhaps the most promising are those of Hessels *et al.* [19] on energy-level shifts of Rydberg helium atoms, with one electron in a $1s$ state and one in an $n\ell$ with $n$ and $\ell$ much greater than one. This system has the advantage that, being rather simple, theoretical estimates of the shifts can be made with great precision. It has the disadvantage that one must very accurately estimate the many corrections which are much larger than the Casimir correction.

Another approach, that of Sukenik *et al.* [20], consists of the deflection of a neutral atom initially moving parallel to a surface. This approach has the disadvantage, theoretically, of involving rather complicated properties of matter, but at distances of the order of 100 Å the atom-wall Casimir interaction is dominated by low-frequency components, and the interaction of a surface with such components is reasonably well understood. The advantage of the system is that the measurement, the deflection, is, in its entirety, at large separation, a Casimir effect; one does not have any number of other effects to contend with.

The present analysis suggests that one might consider the possibility of studying the deflection of electrons moving roughly parallel to a surface. Relative to studies of energy-level shifts in Rydberg helium atoms, it has about the same advantages and disadvantages as studies of the deflection of atoms. Relative to atom-deflection studies, electron-deflection studies have the advantages that an electron is simpler than an atom, and that, for a fixed separation, low frequencies play a more dominant role. [One replaces $\alpha_1(0)$ for the deflection of an atom by $-e^2/m\omega^2$ for the deflection of an electron.] Finally, the deflecting potential falls off less rapidly with separation, varying as $1/\ell^2$ rather than as $1/\ell^4$. Electron-deflection studies, however, have the disadvantage that the dominant effect is not the retardation $1/\ell^2$ interaction but the Coulombic $1/\ell$ interaction.

Other studies which involve retardation interactions include the (elastic and inelastic) scattering of atoms by surfaces [21] and the reflection of light by a surface in the presence of a gas [22].

We note in passing that the atom-wall retarded interaction for an atom which is not spherically symmetric but which has axial symmetry and which has equal likelihood of having any given orientation can be obtained [23,24] from the results for spherically symmetric atoms by replacing $\alpha_1(0)$ by $[2\alpha_1^{\perp}(0) + \alpha_1^{\parallel}(0)]/3$, where $\alpha_1^{\perp}(0)$ and $\alpha_1^{\parallel}(0)$ are the static electric dipole polarizabilities for an electric field perpendicular to and parallel to the symmetry axis of the atom.

The quantized Fresnel modes could be used to study the change in half-life of an excited atom near a dielectric

wall and of an electron in an excited bound state near a dielectric wall. The change in half-life of an atom in an excited state embedded in a uniform dielectric has been determined [16].

The various interactions which have been obtained in which walls are involved, with magnetic effects neglected, have been extended to include magnetic effects [25].

*Note added in proof.* Extensions of many of the above results have been obtained. They include interactions with one wall ($V_{\mathrm{At}D}$ and $V_{\mathrm{El}D}$) for *small* $l$, and interactions of an atom or an electron placed between *two* dielectric walls for arbitrary separations of the walls and an arbitrary location of the atom or electron [F. Zhou and L. Spruch (unpublished)].

## ACKNOWLEDGMENTS

One of us (L.S.) was at the Institute of Theoretical Atomic and Molecular Physics of the Harvard-Smithsonian Center for Astrophysics when this work was begun, and would like to express his gratitude for the many courtesies extended to him while there; the work was finished while he was at the Aspen Center for Physics. We would like to thank Dr. D. Pines for a useful remark and Dr. L. Rosenberg for an instructive discussion. This research was partially supported by the NSF under Grant No. PHY90-19745 and by the NSF through a grant for the Institute for Theoretical Atomic and Molecular Physics at Harvard University and the Smithsonian Astrophysical Observatory.

## APPENDIX A: ORTHONORMALITY OF THE FRESNEL MODES

As noted by Glauber and Lewenstein [16], the orthogonality of the modes $\mathbf{f}_q$, with frequency-independent weight factor $\epsilon(\mathbf{r})$, follows from the fact that the $\epsilon^{1/2}(\mathbf{r})\mathbf{f}_q(\mathbf{r})$'s are eigenmodes of a Hermitian operator. However, the expression for the normalization constant $N$, for the mode $q$, given by these authors in their Eqs. (7.26a) and (7.26b), seems to contain typographical errors. Moreover, since no derivation of $N$ is given and since a simpler form of $N$ than that given is in fact obtained, we will sketch the determination of $N$. In the course of that process one checks that the different $\epsilon^{1/2}\mathbf{f}_q$'s are indeed orthogonal.

That the normalization of the modes is determined solely by the region in which the incident wave exists — see Eq. (2.14) — is hardly surprising. Thus it is a standard result in scattering theory that the normalization is independent of the absence or presence of a potential which vanishes sufficiently rapidly; if

$$\psi_{\mathbf{q}}(\mathbf{r}) = (2\pi)^{-3/2}\exp(i\mathbf{q}\cdot\mathbf{r})$$

or if

$$\psi_{\mathbf{q}}(\mathbf{r}) = (2\pi)^{-3/2}[\exp(i\mathbf{q}\cdot\mathbf{r}) + w_{\mathbf{q}}(\mathbf{r})]$$

with $w_{\mathbf{q}}(\mathbf{r})$ the scattered wave, one has

$$\int \psi_{\mathbf{q}}(\mathbf{r})\psi^*_{\mathbf{p}}(\mathbf{r})\,d\mathbf{r} = \delta(\mathbf{q}-\mathbf{p}) \ .$$

Our problem, with $\epsilon_0(z=\infty) \equiv \epsilon_{10}$, $\epsilon_0(z=-\infty) \equiv \epsilon_{20}$, and $\epsilon_{10} \neq \epsilon_{20}$, corresponds to a potential which does not vanish at infinity. That that is not relevant with regard to normalization can be made clear by considering quantum one-dimensional scattering by a potential $V(z)$. We then have

$$v_q(z) \sim \begin{cases} N_q(e^{iqz} + R_q e^{-iqz}) \ , & z \sim -\infty \\ N_q T_q e^{iQz} \ , & z \sim +\infty \ . \end{cases}$$

We first consider the simple case, with $V(-\infty) = V(\infty) = 0$, for which $Q = q$. We multiply the Schrödinger equation for $v_q$ by $v_p^*$, the equation for $v_p^*$ by $v_q$, subtract, and integrate from $-Z$ to $+Z$. With a prime integrate denoting a derivative with respect to $z$, and with

$$I(q,p,Z) \equiv \int_{-Z}^{Z} v_q v_p^*\,dz \ ,$$

we find, for $V(\pm\infty) = 0$,

$$\begin{aligned}(q^2-p^2)I(q,p,Z) &= -\frac{\hbar^2}{2m}(v_p^* v_q' - v_p^{*\prime} v_q)\Big|_{-Z}^{Z} \\ &\equiv (q+p)\,F(q,p)\end{aligned}$$

or

$$I(q,p,Z) = F(q,p)/(q-p) \ .$$

$V(z)$ does not appear explicitly. $F(q,p)$ contains terms $\exp[\pm i(q+p)]Z$, with coefficients $R_q$ and $R_p^*$, and terms $\exp[\pm i(q-p)Z]$, with coefficients unity, $R_qR_p^*$, and $T_qT_p^*$. For $q$ different from $p$, all the terms wash out on integrating over $q$ and letting $Z \sim \pm\infty$, and we have the expected orthogonality of $v_q$ and $v_p^*$. The terms in $R_q$ and $R_p^*$ wash out for $Z \sim \pm\infty$ even for $q \sim p$, but the other terms peak there. In that neighborhood we can replace $R_qR_p^*$ by $|R_q|^2$ and $T_qT_p^*$ by $|T_q|^2$. Using $|R_q|^2 + |T_q|^2 = 1$, one finds that the choice $N_q = (2\pi)^{-1/2}$ leads to

$$\int_{-\infty}^{\infty} v_q(z)\,v_p^*(z)\,dz = \delta(q-p) \ ,$$

independent of $V(z)$. The integral for one wave incident from $z=-\infty$ and one from $z=+\infty$ vanishes identically, so that the result just given is valid for all situations.

Now assume that $V(z=+\infty) = V_0 \neq 0$ and $V(z=-\infty) = 0$, and consider a wave incident from $z=-\infty$ with an energy $E = q^2\hbar^2/2m > V_0$. The argument is much the same; one need merely replace $|R_q|^2 + |T_q|^2 = 1$ by the flux conservation condition now appropriate, namely

$$q|R_q|^2 + Q|T_q|^2 = 1 \ ,$$

where $Q^2\hbar^2/2m = E - V_0$. For $E < V_0$, $v_q \sim 0$ as $z \sim +\infty$ and $T_q$ plays no role, but now $|R_q|^2 = 1$ and the normalization condition again remains the same.

For our problem a factor $\delta(k_x - k_x')\,\delta(k_y - k_y')$ arises

since $k_x$ and $k'_x$ must be the same, as must be $k_y$ and $k'_y$. Since flux is conserved for either polarization, for either class, and for any values of $\epsilon_{10}$ and $\epsilon_{20}$, the factor $\delta(k_z - k'_z)$ also arises, and one finds that the normalizations are indeed given by Eq. (2.14).

## APPENDIX B: PROPERTIES OF $H(P, \epsilon(\omega))$

It will be useful to collect a few properties of $H(p, \epsilon(\omega))$. By Eqs. (3.11) and (3.7), we have

$$H(p,\epsilon) = \frac{s-p}{s+p} + (1-2p^2)\frac{s-\epsilon p}{s+\epsilon p} \quad (\epsilon \text{ real}), \tag{B1}$$

$$s = (\epsilon - 1 + p^2)^{1/2} . \tag{B2}$$

[For simplicity, in this appendix but not elsewhere, we drop the subscript 2. Thus $\epsilon_2(\omega) = \epsilon_2 \to \epsilon(\omega) = \epsilon$, $\epsilon_{20} \to \epsilon_0$, $s_2 \to s$, and $s_{20} \to s_0$.] We begin by noting that $H(p,\epsilon)$ can be rewritten as

$$H(p,\epsilon) = \frac{(s-p)^2}{s^2-p^2} + (1-2p^2)\frac{(s-\epsilon p)^2}{s^2-\epsilon^2 p^2} \tag{B3}$$

$$= \frac{1}{\epsilon - 1}\left((s-p)^2 + (1-2p^2)\frac{(s-\epsilon p)^2}{1-(\epsilon+1)p^2}\right) .$$

Let $p \to iy$, where $y$ is real and positive. Then

$$s(p) \to s(iy) = (\epsilon - 1 - y^2)^{1/2} . \tag{B4}$$

For $y < \Delta$, with $\Delta$ defined by Eq. (3.24), $s(iy)$ is real and Im $H(iy,\epsilon)$ comes only from cross terms in $(s-iy)^2$ and $(s-i\epsilon y)^2$. We therefore have

$$\text{Re}\,[iH(iy,\epsilon)] = -\text{Im}\,H(iy,\epsilon)$$

$$= \frac{2s(iy)y}{\epsilon - 1}\left[1 + (1+2y^2)\frac{\epsilon}{1+(\epsilon+1)y^2}\right] ,$$

$$0 \le y \le \Delta . \tag{B5}$$

For $y > \Delta$,

$$s(iy) = i[y^2 - (\epsilon - 1)]^{1/2} \tag{B6}$$

is imaginary, so that $H(iy,\epsilon)$ is real, and

$$\text{Re}[iH(iy,\epsilon)] = 0 , \qquad \Delta \le y \le \infty . \tag{B7}$$

In particular, we therefore have that

$$H(iy,\epsilon)\begin{cases} < 0 & \text{for} \quad \epsilon > 1, \quad y > \Delta \\ > 0 & \text{for} \quad \epsilon < 1, \quad y > \Delta . \end{cases} \tag{B8}$$

We now record the forms assumed by $H(p,\epsilon)$ for various values of $p$ and $\epsilon$:

$$H(p,1) = 0, \tag{B9}$$

$$H(p,\infty) = 2p^2, \tag{B10}$$

$$H(0,\epsilon) = 2, \tag{B11}$$

$$H(p \ne 0,\ \epsilon \approx 1) \sim \frac{\epsilon - 1}{4p^2}\left[1 + (1-2p^2)^2\right] . \tag{B12}$$

The asymptotic form of $H(p,\epsilon)$ as $p \sim \infty$ is a bit complicated, since $\epsilon(\omega)$ can assume very large positive and negative values for values of $\omega$ near a resonance. Specializing to $\epsilon = \epsilon_0$, we have

$$H(p,\epsilon_0) \sim \frac{\epsilon_0 - 1}{\epsilon_0 + 1}2p^2 , \quad p \sim \infty . \tag{B13}$$

We turn now to the question of poles of $H(p,\epsilon_0)$ in the upper right quadrant. Specializing Eqs. (B1) and (B2) to the case $\epsilon = \epsilon_0$ and noting that $\epsilon_0 > 1$, we readily find that $H(p,\epsilon_0)$ cannot have poles in the upper right quadrant of the $p$ plane. Indeed, the denominator $(s_0+p)$ can vanish only if $\epsilon_0 = 1$, and $s_0 + \epsilon_0 p = 0$ implies

$$1 = (\epsilon_0+1)p^2 = (\epsilon_0+1)(p_R^2 - p_I^2 + 2ip_R p_I) ,$$

which, for real $\epsilon_0$, demands that $p_R p_I = 0$. But $p_R = 0$ implies $1 = -(\epsilon_0+1)p_I^2$ while $p_I = 0$ implies, reverting to the form $s_0 + \epsilon_0 p = 0$, that $(\epsilon_0 - 1 + p_R^2)^{1/2} = -\epsilon_0 p_R$. Neither of these conditions can be met for $p$ in the upper right quadrant.

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