# Elementary approximate derivations of some retarded Casimir interactions involving one or two dielectric walls

Larry Spruch and Yoel Tikochinsky*
*Physics Department, New York University, New York, New York 10003*
(Received 6 April 1993)

The original derivation by Lifshitz [Sov. Phys. **2**, 73 (1956)] of $P_{DD}$, the force per unit area between two plane parallel dielectric walls, is *extremely* complicated; the later derivations are simpler but still difficult. The standard derivation of the interaction $V_{\mathrm{At}D}$ of an atom and a dielectric wall uses the expression for $P_{DD}$ as its starting point. The results are valid for all values of the separation $\ell$. For $\ell \sim \infty$, where the interactions are retarded, we obtain reasonably accurate approximate expressions for $P_{DD}$ and for $V_{\mathrm{At}D}$—and also for $V_{\mathrm{El}D}$, the retarded interaction of an electron and a dielectric wall—by the elementary procedure of assuming simple forms with one or two open parameters, adjusted to give the known results for retarded interactions which do *not* include dielectric walls. These include $P_{MM}$ (the force per unit area between two parallel plate metallic walls), $V_{\mathrm{At}M}$ (the atom-metal) interaction, $V_{\mathrm{AtAt}}$ (the atom-atom interaction), and $V_{\mathrm{El}M}$ (the electron-metal interaction). We also consider the possibility of obtaining an improved estimate of $P_{DD}$ by using known properties of $V_{\mathrm{At}D}$. The explicit results obtained by Lifshitz for the various interactions are also very complicated. The simple approximate forms of the interactions can be particularly useful for the wall-wall interaction, since $P_{DD}$ is a double integral with a complicated integrand which depends upon two parameters, the zero-frequency dielectric constants of each of the walls.

PACS number(s): 31.30.Jv, 12.20.Ds, 77.90.+k, 41.20.−q

## I. INTRODUCTION

The force per unit area between two plane parallel metallic walls at a separation $\ell$ was found by Casimir [1], as long ago as 1948, to be

$$P_{MM} = -\frac{\pi^2}{240}\frac{\hbar c}{\ell^4} . \tag{1.1}$$

[We use the subscripts $D$, $M$, At, and El to denote dielectric walls, metallic (perfectly conducting) walls, atoms, and electrons, respectively. A metallic wall is of course a special case of a dielectric wall. Subscripts 1 and 2 will refer to the two interacting systems. Primes on pressures and potentials indicate approximations. We use $P$ for the force per unit area, or pressure, rather than the $F$, which is often used [2,3]. We consider a static potential in Sec. III. All other potentials, and all pressures, are retarded.] Equation (1.1) is valid for all $\ell$. Casimir also obtained the interaction [4]

$$V_{\mathrm{At}M}(\ell) = -\frac{3}{8\pi}\frac{\hbar c \alpha_{10}}{\ell^4} \tag{1.2}$$

between an atom and a metallic wall at an asymptotically large separation $\ell$; $\alpha_{10}$ is the static electric dipole polarizability of the atom, here taken to be system 1. Casimir and Polder found the interaction of two atoms at an asymptotically large separation $r$ to be [5]

$$V_{\mathrm{AtAt}}(r) \sim -\frac{23}{4\pi}\frac{\alpha_{10}\alpha_{20}\hbar c}{r^7} . \tag{1.3}$$

The three derivations are *relatively* simple. (See, for example, the book by Power [6].) Interactions involving dielectric walls, on the other hand, are, as might be expected, extremely complicated. In particular, the original derivation by Lifshitz [2] of $P_{DD}$ is so complicated that Landau and Lifshitz [2] give the result without giving the complete derivation; they simply cite the Lifshitz paper. As pointed out by Ginzburg [7], it is one of the very few results given in the entire series of books by Landau and Lifshitz in which they do that. The derivation of the interaction $V_{\mathrm{At}D}(\ell)$ between an atom and a dielectric wall, at a separation $\ell$, is usually obtained by considering one of the two walls to be a dilute gas of atoms. $V_{DD}(\ell)$ is then a superposition of atom-dielectric wall interactions $V_{\mathrm{At}D}(r)$ and one demands that $V_{\mathrm{At}D}(r)$ be such that the sum of $V_{\mathrm{At}D}$'s reproduces $V_{DD}(\ell)$. Subsequent derivations [3,7,8] of $V_{DD}$ are rather less difficult than the original Lifshitz derivation, but still difficult.

A wall is characterized by its "dielectric constant" or electric permittivity $\epsilon$. One of the complicating features in the determination of $V_{DD}(\ell)$ is that one is normally interested in its value for all $\ell$, and that demands that one know both the real and imaginary parts of $\epsilon$, for all frequencies. If one restricts one's attention to asymptotically large values of $\ell$, one need know only the real zero-frequency component $\epsilon(w = 0) \equiv \epsilon_0$. Rather than proceeding by finding $V_{DD}(\ell)$ for all $\ell$ and taking the limit as $\ell \sim \infty$, one recognizes at the outset that for asymptotically large $\ell$ only $\epsilon_0$ enters; the evaluation of $V_{DD}(\ell)$ for $\ell \sim \infty$ could thereby be greatly simplified [9].

Even this latter calculation is sufficiently tortuous that one can easily lose sight of the "physics." We therefore believe it to be useful to have derivations of $V_{DD}$ and of $V_{\mathrm{At}D}$ which, if only approximate, if only partially "physical," and if valid only for $\ell \sim \infty$, are at least elementary. We will also obtain an approximation for the interaction $V_{\mathrm{El}D}(\ell)$ of an electron and a dielectric wall, at an asymptotically large separation $\ell$. (The exact result for this case was derived only recently [9]. This derivation too is somewhat tortuous.) The derivation of $V_{\mathrm{El}D}(\ell)$ for $\ell \sim \infty$ will rely on a knowledge of the retarded interaction of an electron and a metallic wall [10–12]

$$V_{\mathrm{El}M}(\ell) \sim \frac{e^2\hbar}{4\pi mc\ell^2} \tag{1.4}$$

and the retarded interaction of an electron and an atom [13]

$$V_{\mathrm{ElAt}}(r) \sim \frac{11}{4\pi}\,\frac{\alpha_{20}e^2\hbar}{mcr^5}\ . \tag{1.5}$$

This last result is also applicable to an electron and an ion [14].

There is only one instance, in Sec. VII, in which an interaction involving a dielectric wall is used as input data; we there include information on $V_{\mathrm{At}D}$ as input data in obtaining an improved estimate of $P_{DD}$. Section VIII is the only section in which we consider, though only very briefly, values of the separation $\ell$ of the interacting systems which are not in the asymptotic domain. Asymptotic values of $\ell$ are values which are larger than any wavelength for which either of the two systems undergoes significant absorption.

## II. THE PROCEDURE TO BE FOLLOWED

For each interaction we will begin by obtaining the dependence of the interaction upon the (not all independent) dimensional quantities $e$, $m$, $c$, $\hbar$, $\alpha_0$, and $\ell$. The procedure for doing so is analogous to that used in the past [15] for obtaining the interaction for systems involving atoms, electrons, and metallic walls; the argument is much the same if one replaces a metallic wall by a dielectric wall since $\epsilon_0$ is dimensionless. One extracts some of the physics and one *then* uses dimensional analysis; dimensional analysis alone is not always sufficient. To extract the physics we note that for $\ell \sim \infty$ a wall is completely characterized by $\epsilon_0$, an atom by $\alpha_0$, and an electron by $\alpha(w) = -e^2/m\omega^2$, its dynamic electric dipole polarizability. (The validity of the last statement is not restricted to asymptotically large values of $\ell$.) Furthermore, we recognize that the asymptotic interaction for the cases under consideration is completely dominated by contributions associated with two photon exchanges, and that the normalization of each of the associated electric fields contains a factor $\hbar^{1/2}$; each of the Casimir interactions is therefore proportional to $\hbar$. The $\hbar$ factor also follows for most Casimir effects by noting that they can normally be written as the difference between sums over all frequencies of perturbed and unperturbed energies, namely as

$$\hbar\sum_n(\omega_n' - \omega_n)\ .$$

After the physics has been extracted and dimensional analysis has been performed, the problem reduces to the determination of one real function for each of the three interactions, a function of $\epsilon_{20}$ for $V_{\mathrm{At}D}$ and for $V_{\mathrm{El}D}$, and of $\epsilon_{10}$ and $\epsilon_{20}$ for $P_{DD}$. We impose some obvious restrictions on these functions. Thus, a wall with $\epsilon_0 = 1$ is really a vacuum and does not interact. This *suggests* that we introduce a factor $\epsilon_0 - 1$ for each wall. The fact that the various interactions remain finite as $\epsilon_0 \to \infty$ again *suggests*—we are not here attempting to prove anything—that each factor of $\epsilon_0 - 1$ appear with a denominator of the form $\epsilon_0 + B$; thus, for each wall we insert a factor

$$(\epsilon_0 - 1)\,/\,(\epsilon_0 + B)\ ,$$

with $B$ an as yet unspecified dimensionless constant whose value will depend upon whether the wall is interacting with another wall, an atom, or an electron. We will then use the known expression for $P$ or for $V$ (whichever is appropriate) as $\epsilon_{10}$ and/or $\epsilon_{20} \to \infty$, the fact that for $\epsilon_0 \sim 1$ the wall can be thought of as consisting of independent atoms, and the known metallic-wall–metallic-wall, atom–metallic-wall, atom-atom, and electron-atom interactions to determine the various $B$'s. (The condition imposed for $\epsilon_0 \approx 1$ encompasses the condition used above that the interaction vanishes for $\epsilon_0 = 1$.)

## III. A TRIVIAL EXAMPLE: THE CLASSICAL STATIC INTERACTION OF AN ELECTRON AND A DIELECTRIC WALL

As an illustrative example of the procedure to be used in obtaining approximations of (quantum relativistic) Casimir interactions involving dielectric walls, we use the procedure to estimate the *classical static* interaction $v(\ell)$ of an electron at a distance $\ell$ from a dielectric wall. [$v(\ell)$ is of course well known.] To do so, we use static interactions which do *not* involve dielectric walls. In particular, the static interaction of an electron at a distance $\ell$ from a *metallic* wall (dielectric constant infinite) is assumed to be known to be $(-e^2/4\ell)$, and the static potential energy of an electron at a distance $r$ from an atom with a static electric dipole polarizability $\alpha_{20}$ is assumed to be known to be

$$-(1/2)\alpha_{20}e^2/r^4\ .$$

The wall is completely characterized by its real zero-frequency dielectric constant $\epsilon_2(\omega = 0) \equiv \epsilon_{20}$ and the (stationary) electron by its charge $e$. Dimensional analysis gives

$$v = v(e, \ell, \epsilon_{20}) = \frac{e^2}{\ell}\,f(\epsilon_{20})\ ,$$

where $f(\epsilon_{20})$ is an as yet arbitrary real function. Since $f(1) = 0$ and $f(\infty) = -1/4$, a reasonable choice for our approximation to $v$ is

$$v' = -\frac{e^2}{4\ell}\,\frac{\epsilon_{20} - 1}{\epsilon_{20} + b'}\ . \tag{3.1}$$

To fix $b'$, we formally consider the case for which the wall consists of a dilute gas of atoms. For this rarified medium situation we have

$$\epsilon_{20} - 1 \approx 4\pi N_{2\mathrm{At}}\alpha_{20} , \tag{3.2}$$

where $N_{2\mathrm{At}}$ is the number of atoms per cubic centimeter in the wall and $\alpha_{20}$ is the static electric dipole polarizability of an atom in the wall, and Eq. (3.1) becomes

$$v'(\epsilon_{20} \approx 1) \approx -\frac{e^2}{4\ell}\frac{4\pi N_{2\mathrm{At}}\alpha_{20}}{1+b'} . \tag{3.3}$$

Now $v(\epsilon_{20} \approx 1)$ can be obtained by summing the (independent) interactions of the electron and the atoms of which the wall is composed. This approach gives

$$v(\epsilon_{20} \approx 1) \approx N_{2\mathrm{At}} \int_0^\infty dz_2 \int_0^\infty 2\pi\,\rho_2\,d\rho_2 \left(-\frac{\alpha_{20}e^2}{2r^4}\right) ;$$

with the origin chosen to be the point on the surface of the wall closest to the electron (at a distance $\ell$ from the wall), $z_2$ and $\rho_2$ (both non-negative) are cylindrical coordinates of a point within the wall, and

$$r = \left[(z_2+\ell)^2 + \rho_2^2\right]^{1/2} . \tag{3.4}$$

The integrations are trivial and lead to

$$v(\epsilon_{20} \approx 1) \approx -\frac{e^2}{4\ell}\, 2\pi\, N_{2\mathrm{At}}\alpha_{20} . \tag{3.5}$$

A comparison of Eqs. (3.3) and (3.5) gives $b' = 1$ and therefore

$$v'(\ell) = -\frac{e^2}{4\ell}\frac{\epsilon_{20}-1}{\epsilon_{20}+1} \quad [= v(\ell)] . \tag{3.6}$$

This result is unusual in that it is exact, that is, $v' = v$, a possibility which arose because the approximate form, Eq. (3.1), encompasses the exact result. It should not be thought, however, that the procedure is useful only if one can choose a form which encompasses the exact result. Thus, consider an alternative approximate form

$$v''(\ell) = -\frac{e^2}{4\ell}\frac{\epsilon_{20}^{1/2}-1}{\epsilon_{20}^{1/2}+b''} = -\frac{e^2}{4\ell}\frac{\epsilon_{20}-1}{(\epsilon_{20}^{1/2}+b'')(\epsilon_{20}^{1/2}+1)}$$

with $b''$ determined by the requirement that $v''(\ell)$ give the exact result for the dilute case. The $\epsilon_{20}$-dependent factors in the denominator in the expression on the far right reduce for the dilute case to $2(1+b'')$, as opposed to the $1+b'$ factor which arose when studying $v'$ for the dilute case. Since both $v''$ and $v'$ must reduce to $v$ for the dilute case, $b''$ must satisfy $2(1+b'') = 1+b' = 2$, that is, $b'' = 0$, so that

$$v'' = -\frac{e^2}{4\ell}\frac{\epsilon_{20}^{1/2}-1}{\epsilon_{20}^{1/2}} , \tag{3.7}$$

which differs from $v$ by at most about 15% over the entire range $1 \le \epsilon_{20} \le \infty$. This good agreement is hardly surprising. Not only is $v''$ exact *at* both end points, $\epsilon_{20} = 1$ and $\epsilon_{20} = \infty$, but $v''$ was also adjusted to be exact for $\epsilon_{20}$ in the immediate neighborhood of unity—more precisely, the first derivative of $v''(\epsilon_{20})$ is exact at $\epsilon_{20} = 1$—*and* there is nothing in the physics to suggest any rapid variation of $v$ with $\epsilon_{20}$.

For later reference—see Sec. VII—we consider the behavior of $v$ and $v''$ as $\epsilon_{20} \sim \infty$. While $v''$ is exact *at* $\epsilon_{20} = \infty$, it does not have the correct form for $\epsilon_{20} \sim \infty$, since

$$v(\epsilon_{20}) \sim -\frac{e^2}{4\ell}\left(1-\frac{2}{\epsilon_{20}}\right) , \quad \epsilon_{20} \sim \infty , \tag{3.8}$$

while

$$v''(\epsilon_{20}) \sim -\frac{e^2}{4\ell}\left(1-\frac{1}{\epsilon_{20}^{1/2}}\right) , \quad \epsilon_{20} \sim \infty .$$

Suppose that in some fashion one determines the correct asymptotic form. [In the present context, one might be able to determine it by starting with metallic walls and using perturbation theory with $1/\epsilon_{20}$ as the perturbation parameter. In our analysis of $P_{DD}(\epsilon_{10},\epsilon_{20})$, in Sec. VII, we will obtain some information on $P_{DD}(\epsilon_{10},\epsilon_{20} \sim \infty)$ for $\epsilon_{10} \approx 1$ by assuming a knowledge of $V_{\mathrm{At}D}(\epsilon_{20})$.] The question is the extent to which a knowledge of the asymptotic behavior can help to improve our estimate of $v$. Let us then choose an approximation $v'''$ which cannot reduce to $v$ and which gives corrections of order $1/\epsilon_{20}$ for $\epsilon_{20} \gg 1$, for example,

$$v''' = -\frac{e^2}{4\ell}\frac{\epsilon_{20}^3-1}{\epsilon_{20}^3+\alpha\epsilon_{20}^2+\beta} .$$

$v'''(\epsilon_{20} = \infty)$ is exact. The demand that $v'''(\epsilon_{20} \approx 1) = v(\epsilon_{20} \approx 1)$, with the latter given by Eq. (3.5), leads to $1+\alpha+\beta = 6$. The new requirement, that $v'''$ have the asymptotic form given by Eq. (3.8), leads to $\alpha = 2$, which then gives $\beta = 3$. $v'''$, with $\alpha = 2$ and $\beta = 3$, is a considerable improvement over $v''$, especially, as is to be expected, for $\epsilon_{20} \gg 1$.

## IV. AN ATOM AND A DIELECTRIC WALL

We begin our concrete considerations of Casimir interactions involving one or two dielectric walls with the case of an atom and a dielectric wall. In line with the discussion above on the extraction of the physics, we recognize that

$$V_{\mathrm{At}D}(\ell) = V_{\mathrm{At}D}(e,m,c,\hbar,\alpha_{10},\epsilon_{20},\ell)$$

must have the particular form

$$V_{\mathrm{At}D}(\ell) = \alpha_{10}\hbar f_{\mathrm{At}D}(\ell,c,\epsilon_{20}) ,$$

with $f_{\mathrm{At}D}$ an arbitrary real function; the only dependence upon the charge $e$ and mass $m$ of an electron is that contained in $\alpha_{10}$ and $\epsilon_{20}$. Dimensional analysis then leads to

$$V_{\mathrm{At}D}(\ell)=\frac{\hbar c\alpha_{10}}{\ell^4}\,g_{\mathrm{At}D}(\epsilon_{20})\ ,$$

with $g_{\mathrm{At}D}$ an arbitrary real function. We choose to satisfy two of the restrictions that must be satisfied by $g_{\mathrm{At}D}$ by writing

$$V'_{\mathrm{At}D}(\ell)=-\frac{3}{8\pi}\frac{\hbar c\alpha_{10}}{\ell^4}\frac{\epsilon_{20}-1}{\epsilon_{20}+B'_{\mathrm{At}D}}\ , \tag{4.1}$$

where the numerical coefficient was chosen so that $V_{\mathrm{At}D}$ reduces to $V_{\mathrm{At}M}$ of Eq. (1.2) for $\epsilon_{20}=\infty$. To fix $B'_{\mathrm{At}D}$, we assume that $\epsilon_{20}\approx 1$, so that $\epsilon_{20}-1$ is then given by Eq. (3.2), and

$$V'_{\mathrm{At}D}(\ell;\epsilon_{20}\approx 1)\approx-\frac{3}{8\pi}\frac{\hbar c\alpha_{10}}{\ell^4}\frac{4\pi N_{2\mathrm{At}}\alpha_{20}}{1+B'_{\mathrm{At}D}}\ . \tag{4.2}$$

But we can also write

$$V_{\mathrm{At}D}(\ell;\ \epsilon_{20}\approx 1)$$

$$\approx N_{2\mathrm{At}}\int_0^\infty dz_2\int_0^\infty 2\pi\,\rho_2\,d\rho_2\,V_{\mathrm{AtAt}}(r)\ , \tag{4.3}$$

where $r$ is given by Eq. (3.4) and $V_{\mathrm{AtAt}}(r)$ is given by Eq. (1.3). The integrations are trivial and Eq. (4.3) becomes

$$V_{\mathrm{At}D}(\ell;\epsilon_{20}\approx 1)\approx-\frac{23}{40}\frac{N_{2\mathrm{At}}\alpha_{10}\alpha_{20}\hbar c}{\ell^4}\ . \tag{4.4}$$

Comparison with Eq. (4.2) gives

$$B'_{\mathrm{At}D}=37/23\ ,$$

and our approximation is

$$V'_{\mathrm{At}D}(\ell)=-\frac{3}{8\pi}\frac{\hbar c\alpha_{10}}{\ell^4}\frac{\epsilon_{20}-1}{\epsilon_{20}+(37/23)}\ . \tag{4.5}$$

A special case of interest is that of an atom interacting with a liquid-helium wall [16].

$V_{\mathrm{At}D}(\ell)$ can be obtained analytically and has been evaluated numerically. Dzyaloshinskii, Lifshitz, and Pitaevskii [3] write it in the form

$$V_{\mathrm{At}D}(\ell)=-\frac{3}{8\pi}\frac{\hbar c\alpha_{10}}{\ell^4}\frac{\epsilon_{20}-1}{\epsilon_{20}+1}\phi_{\mathrm{At}D}(\epsilon_{20}) \tag{4.6}$$

and plot $\phi_{\mathrm{At}D}(\epsilon_{20})$. We tabulate the value of the ratio

$$\rho'_{\mathrm{At}D}(\epsilon_{20})\equiv\frac{V'_{\mathrm{At}D}(\epsilon_{20})}{V_{\mathrm{At}D}(\epsilon_{20})}=\frac{\epsilon_{20}+1}{\epsilon_{20}+(37/23)}\frac{1}{\phi_{\mathrm{At}D}(\epsilon_{20})} \tag{4.7}$$

for a few values of $\epsilon_{20}$ in Table I. One sees that the agreement is rough but meaningful.

To check that the agreement is not merely accidental, we consider a second form of the approximation,

TABLE I. Ratios of estimates to exact values for the atom-wall and wall-wall interactions. Each ratio is equal to unity for $\epsilon_0=1$ and $\epsilon_0=\infty$, and the derivative of each ratio is equal to unity at $\epsilon_0=1$.

| Ratio \ $\epsilon_0$ | 4 | 9 | 16 |
|---|---|---|---|
| $\rho'_{\mathrm{At}D}(\epsilon_0)$, Eq. (4.7) | 1.14 | 1.16 | 1.10 |
| $\rho''_{\mathrm{At}D}(\epsilon_0)$, Eq. (4.10) | 0.93 | 0.93 | 0.88 |
| $\rho_{D=D}(\epsilon_0)$, Eq. (6.7) | 0.80 | 0.94 | 1.0 |
| $\rho_{DM}(\epsilon_0)$, Eq. (6.14) | 1.4 | 1.4 | 1.1 |
| $\bar{\rho}_{D=D}(\epsilon_0)$, Eq. (7.15) | 0.70 | 0.86 | 0.94 |

$$\begin{aligned}V''_{\mathrm{At}D}(\ell)&=-\frac{3}{8\pi}\frac{\hbar c\alpha_{10}}{\ell^4}\frac{\epsilon_{20}^{1/2}-1}{\epsilon_{20}^{1/2}+B''_{\mathrm{At}D}}\\&=-\frac{3}{8\pi}\frac{\hbar c\alpha_{10}}{\ell^4}\frac{\epsilon_{20}-1}{(\epsilon_{20}^{1/2}+B''_{\mathrm{At}D})(\epsilon_{20}^{1/2}+1)}\ .\end{aligned} \tag{4.8}$$

(The motivation for introducing $\epsilon_{20}^{1/2}$ is its appearance in a recent derivation [9] of $V_{\mathrm{At}D}$ based on quantized Fresnel modes [17]. One considers quantized electromagnetic waves incident on the dielectric-vacuum interface, from both directions, and for the quantized modes, as for the classical modes, the reflection and transmission coefficients are functions of the zero-frequency index of refraction $n$, that is, $\epsilon_{20}^{1/2}$.)

To agree with $V'_{\mathrm{At}D}$ (and thereby with $V_{\mathrm{At}D}$) for $\epsilon_{20}\approx 1$, we must have

$$2(1+B''_{\mathrm{At}D})=1+B'_{\mathrm{At}D}=60/23\ ,$$

that is,

$$B''_{\mathrm{At}D}=7/23$$

and therefore

$$V''_{\mathrm{At}D}(\ell)=-\frac{3}{8\pi}\frac{\hbar c\alpha_{10}}{\ell^4}\frac{\epsilon_{20}-1}{\epsilon_{20}+(30/23)\epsilon_{20}^{1/2}+(7/23)}\ . \tag{4.9}$$

Values of the ratio

$$\begin{aligned}\rho''_{\mathrm{At}D}(\epsilon_{20})&\equiv\frac{V''_{\mathrm{At}D}}{V_{\mathrm{At}D}}\\&=\frac{\epsilon_{20}+1}{\epsilon_{20}+(30/23)\epsilon_{20}^{1/2}+(7/23)}\frac{1}{\phi_{\mathrm{At}D}(\epsilon_{20})}\end{aligned} \tag{4.10}$$

are given in Table I for a few values of $\epsilon_{20}$.

Since we have been concerned with $\ell\sim\infty$ and therefore with $\epsilon_{20}$, there is no reason to expect any rapid variation of $V_{\mathrm{At}D}$ with $\epsilon_{20}$; the reasonable agreement obtained with both $V'_{\mathrm{At}D}$ and $V''_{\mathrm{At}D}$ is then to be expected, for the same reasons it was expected for the classical static electron-wall interaction studied in Sec. III.

The form for $V_{\mathrm{At}D}$ chosen by Dzyalashinskii, Lifshitz, and Pitaevskii [3] is arbitrary, though of course the numerical value of $V_{\mathrm{At}D}$ as a function of $\epsilon_{20}$ is not. If one wanted an accurate if still approximate analytic expression for $V_{\mathrm{At}D}$, there would be some slight advantage in using the form

$$V_{\mathrm{At}D} = -\frac{3}{8\pi}\frac{\hbar c\alpha_{10}}{\ell^4}\frac{\epsilon_{20}-1}{\epsilon_{20}+(37/23)}\tilde{\phi}_{\mathrm{At}D},$$

suggested by Eq. (4.5) [or the form suggested by Eq. (4.8)]. $\tilde{\phi}_{\mathrm{At}D}$, related to $\phi_{\mathrm{At}D}$ by

$$\tilde{\phi}_{\mathrm{At}D} = \frac{\epsilon_{20}+(37/23)}{\epsilon_{20}+1}\phi_{\mathrm{At}D}\ ,$$

varies less than $\phi_{\mathrm{At}D}$ and could therefore be parametrized more easily.

An alternative procedure [18] is to use the Clausius-Mosotti approximation (which should more properly be called the Mosotti-Clausius approximation) in the expression for $V_{\mathrm{At}D}(\epsilon_{20} \approx 1)$ obtained by starting with Eq. (4.3), namely Eq. (4.4). Thus, replacing Eq. (3.2) by

$$4\pi N_{2\mathrm{At}}\alpha_{20} \approx 3\frac{\epsilon_{20}-1}{\epsilon_{20}+2}\ , \tag{4.11}$$

and using this in Eq. (4.4), we have the approximation

$$V'''_{\mathrm{At}D} = -\frac{69}{160\pi}\frac{\alpha_{10}\hbar c}{\ell^4}\frac{\epsilon_{20}-1}{\epsilon_{20}+2}\ . \tag{4.12}$$

The advantages of this approach are that it is simple and that one knows precisely what approximation has been made. It has the further slight advantage of having built in a rather good approximation for $\epsilon_{20}$ close to unity, but the approximation in Eq. (4.11) is really good only for $\epsilon_{20}$ quite close to unity. The disadvantage of $V'''_{\mathrm{At}D}$ relative to $V'_{\mathrm{At}D}$ of Eq. (4.5) is that no provision was made to account for the behavior of $V_{\mathrm{At}D}$ as $\epsilon_{20} \sim \infty$ and $V'''_{\mathrm{At}D}(\epsilon_{20}=\infty) = (23/20)V_{\mathrm{At}D}$. The ratio of $V'''_{\mathrm{At}D}$ to $V'_{\mathrm{At}D}$ ranges from 1 at $\epsilon_{20}=1$ to 23/20 at $\epsilon_{20}=\infty$. In summary, $V'''_{\mathrm{At}D}$ is slightly better for $\epsilon_{20}$ quite close to unity, but both approximations are quite accurate in that domain. (They each have the correct value and the correct first derivative at $\epsilon_{20}=1$.) Further, $V'''_{\mathrm{At}D} \geq V'_{\mathrm{At}D}$ for all $\epsilon_{20}$, and since $V'_{\mathrm{At}D} > V_{\mathrm{At}D}$ for larger values of $\epsilon_{20}$, as seen from Table I, $V'_{\mathrm{At}D}$ is the better approximation at those larger values.

## V. AN ELECTRON AND A DIELECTRIC WALL

An electron at a distance $\ell$ from a dielectric wall has an interaction $v$ with its image, where $v$ is given by Eq. (3.6). This classical static interaction persists at arbitrarily large $\ell$. The retarded interaction $V_{\mathrm{E1}D}$ to be considered now is in addition to $v(\ell)$.

We here have

$$V_{\mathrm{E1}D}(\ell) = V_{\mathrm{E1}D}(e,m,c,\hbar,\epsilon_{20},\ell) = \left(\frac{e^2}{m}\right)\hbar\, f_{\mathrm{E1}D}(\ell,c,\epsilon_{20})\ .$$

Dimensional analysis leads to

$$V_{\mathrm{E1}D}(\ell) = \frac{e^2\hbar}{mc\ell^2}\, g_{\mathrm{E1}D}(\epsilon_{20})\ .$$

We elect to satisfy two of the restrictions by choosing as our approximation to $V_{\mathrm{E1}D}$

$$V'_{\mathrm{E1}D}(\ell) = \frac{1}{4\pi}\frac{e^2\hbar}{mc\ell^2}\frac{\epsilon_{20}-1}{\epsilon_{20}+B'_{\mathrm{E1}D}}\ ,$$

with the numerical coefficient chosen so that the result for the dielectric wall reduces to that of the metallic wall, given by Eq. (1.4), for $\epsilon_{20}=\infty$. We then have

$$V'_{\mathrm{E1}D}(\ell;\ \epsilon_{20}\approx 1) \approx \frac{1}{4\pi}\frac{e^2\hbar}{mc\ell^2}\frac{4\pi N_{2\mathrm{At}}\alpha_{20}}{1+B'_{\mathrm{E1}D}}$$

and also

$$V_{\mathrm{E1}D}(\ell;\epsilon_{20}\approx 1) \approx N_{2\mathrm{At}}\int_0^\infty dz_2\int_0^\infty 2\pi\,\rho_2\, d\rho_2\, V_{\mathrm{E1At}}(r) = \frac{11}{12}\frac{N_{2\mathrm{At}}\alpha_{20}e^2\hbar}{mc\ell^2}\ , \tag{5.1}$$

where we have used Eq. (1.5) for $V_{\mathrm{E1At}}$. Comparison of the two forms gives

$$B'_{\mathrm{E1}D} = 1/11\ ,$$

so that our approximation is

$$V'_{\mathrm{E1}D}(\ell) = \frac{1}{4\pi}\frac{e^2\hbar}{mc\ell^2}\frac{\epsilon_{20}-1}{\epsilon_{20}+(1/11)}\ . \tag{5.2}$$

(A special case of interest is that of an electron and a liquid-helium wall. Some slight analysis of this case has been given [12]. Very interesting experimental results were obtained for this problem [19], but the data are not good enough to go beyond verifying the classical static interaction.)

For the same reasons as for $V'_{\mathrm{At}D}$ versus $V_{\mathrm{At}D}$, we expect $V'_{\mathrm{E1}D}$ to be a reasonable approximation to $V_{\mathrm{E1}D}$. An explicit expression for $V_{\mathrm{E1}D}$, in the form of an integral, was recently obtained, [9], but no numerical results were given. We have now evaluated $V_{\mathrm{E1}D}(\epsilon_{20})$ for a number of values of $\epsilon_{20}$. We write

$$V_{\mathrm{E1}D}(\epsilon_{20}) = \frac{1}{4\pi}\frac{e^2\hbar}{mc\ell^2}\frac{\epsilon_{20}-1}{\epsilon_{20}+1}\phi_{\mathrm{E1}D}(\epsilon_{20})\ , \tag{5.3}$$

and give some values of $\phi_{\mathrm{E1}D}(\epsilon_{20})$ in Table II. [Some additional values of $\phi_{\mathrm{E1}D}(\epsilon_{20})$ are the following: 1.930, 1.677, and 1.3844 for $\epsilon_0 = 4.0$, 49, and 400, respectively.] We introduce

$$\rho_{\mathrm{E1}D}(\epsilon_{20}) \equiv \frac{V'_{\mathrm{E1}D}(\epsilon_{20})}{V_{\mathrm{E1}D}(\epsilon_{20})} = \frac{\epsilon_{20}+1}{\epsilon_{20}+(1/11)}\frac{1}{\phi_{\mathrm{E1}D}(\epsilon_{20})}\ , \tag{5.4}$$

TABLE II. Numerical values of $\phi_{\mathrm{E1}D}(\epsilon_{20})$; values of $V_{\mathrm{E1}D}(\epsilon_{20})$ follow from Eq. (5.3). $\rho_{\mathrm{E1}D}(\epsilon_{20})$ is the ratio of the approximate and exact values of the interaction energy.

| $\epsilon_{20}$ | $\phi_{\mathrm{E1}D}(\epsilon_{20})$ | Eq. (5.4) |
|---|---|---|
| 1 | 11/6 | 1 |
| 1.1 | 1.848 | 0.954 |
| 1.44 | 1.878 | 0.849 |
| 9 | 1.891 | 0.582 |
| 16 | 1.832 | 0.577 |
| 100 | 1.569 | 0.643 |
| 10 000 | 1.129 | 0.886 |
| $\infty$ | 1 | 1 |

and give some values of $\rho_{\mathrm{E1}D}(\epsilon_{20})$ in Table II. By construction, $\rho_{\mathrm{E1}D}(1) = 1$ and $\rho_{\mathrm{E1}D}(\infty) = 1$, and the derivative of $\rho_{\mathrm{E1}D}$ at $\epsilon_{20} = 1$ is also equal to 1, but the convergence at large values of $\epsilon_{20}$ is very slow. To obtain better results for $\epsilon_{20}$ large, one would have to build into an approximation to $V_{\mathrm{E1}D}(\epsilon_{20})$ a form which built in the asymptotic dependence on $\epsilon_{20}$. Since expressions for $V_{\mathrm{E1}D}(\epsilon_{20})$ are available, in integral form [9] and in integrated form [15], this can be done easily, but since that would be somewhat contrary to our approach of studying complicated cases by using results for simpler cases—we would here be using exact results for $V_{\mathrm{E1}D}(\epsilon_{20})$ to help parametrize $V_{\mathrm{E1}D}(\epsilon_{20})$—we have not bothered to do so.

The use of the approximation Eq. (4.11) in Eq. (5.1) gives

$$V''_{\mathrm{E1}D} = \frac{11}{16\pi}\,\frac{e^2\hbar}{mc\ell^2}\,\frac{\epsilon_{20}-1}{\epsilon_{20}+2}\ . \tag{5.5}$$

This is very much in error, by a factor 11/4, at $\epsilon_{20} = \infty$.

The shift in energy $\Delta E_{n\mathrm{E1}D}$ of the $n$th bound state of an electron and a dielectric wall, generated by $V_{\mathrm{E1}D}$, is given in first-order perturbation theory by

$$\Delta E_{n\mathrm{E1}D} = \langle \psi_n \mid V_{\mathrm{E1}D}(\ell,\epsilon_{20}) \mid \psi_n\rangle\ ;$$

$\psi_n$ is the normalized bound state for the nonrelativistic electron-wall interaction $V_{\mathrm{NR}}(\ell,\epsilon_{20})$. $\Delta E_{nElM}$ has been estimated for the case of a metallic wall, with $V_{\mathrm{NR}}(\ell,\infty)$ taken to be $-e^2/4\ell$ for the electron outside the wall and the wall assumed to be impenetrable [12]; $\psi_n(\epsilon_{20}=\infty)$ is trivially related to the three-dimensional Coulomb $s$-state wave function. For $\epsilon_{20}$ finite and the nonrelativistic electron impenetrable-wall interaction given by $v(\ell)$ of Eq. (3.6), one can again readily obtain the wave function, now denoted by $\psi_n(\epsilon_{20})$. The ratio of $\psi_n(\epsilon_{20})$ and $\psi_n(\epsilon_{20}=\infty)$ is thus known. Furthermore, for $n$ large enough for both $\psi_n(\epsilon_{20})$ and $\psi_n(\epsilon_{20}=\infty)$ to be concentrated asymptotically far from the dielectric wall, $V'_{\mathrm{E1}D}$ and $V_{\mathrm{E1}M}$ differ only by the factor

$$\frac{\epsilon_{20}-1}{\epsilon_{20}+(1/11)}\ .$$

(The use of the exact form [9] for $V_{\mathrm{E1}D}$ provides the exact ratio of the two potentials.) It follows that one can easily obtain the ratio of $\Delta E_{n\mathrm{E1}D}$ and $\Delta E_{n\mathrm{E1}M}$.

## VI. TWO WALLS: SOME SPECIAL CASES

The form of the force per unit area between two dielectric walls follows from dimensional analysis alone; we need not "extract" the physics other than to recognize that the only dependence of $P_{DD}$ on $e$ and $m$ is that contained in $\epsilon_{10}$ and $\epsilon_{20}$. It is given by

$$\begin{aligned} P_{DD} &= P_{DD}(e,m,c,\hbar,\ell,\epsilon_{10},\epsilon_{20}) \\ &= \frac{\hbar c}{\ell^4}\, g_{DD}(\epsilon_{10},\epsilon_{20})\ . \end{aligned}$$

We will consider two cases, $\epsilon_{10} = \epsilon_{20} \equiv \epsilon_0$, where $\epsilon_0$ is arbitrary, and $\epsilon_{20} = \infty$, cases for which $P_{DD}$ has been evaluated numerically.

### A. Identical zero-frequency dielectric constants

For $\epsilon_{10} = \epsilon_{20} \equiv \epsilon_0$, we choose the approximate form

$$\begin{aligned} P'_{DD}(\epsilon_{10}=\epsilon_{20}\equiv\epsilon_0) &\equiv P'_{D=D}(\epsilon_0) \\ &= -\frac{\pi^2}{240}\,\frac{\hbar c}{\ell^4}\left(\frac{\epsilon_0-1}{\epsilon_0+B'_{D=D}}\right)^2\ , \end{aligned} \tag{6.1}$$

with the subscript $D = D$ indicating that the zero-frequency dielectric constants are the same. The numerical coefficient was chosen so that $P'_{D=D}$ reduces to $P_{MM}$ of Eq. (1.1) for $\epsilon_{10} = \epsilon_{20} = \infty$. $B'_{D=D}$ can be determined by considering $\epsilon_0 \sim 1$. With $\alpha_{10} = \alpha_{20} \equiv \alpha_0$ and $N_{1\mathrm{At}} = N_{2\mathrm{At}} \equiv N_{\mathrm{At}}$, Eq. (6.1) becomes

$$P'_{D=D}(\epsilon_0\sim 1) \sim -\frac{\pi^2}{240}\,\frac{\hbar c}{\ell^4}\left(\frac{4\pi\, N_{\mathrm{At}}\alpha_0}{1+B'_{D=D}}\right)^2\ . \tag{6.2}$$

In cylindrical coordinates, with $z_1$ and $z_2$ (both nonnegative) defined by a line perpendicular to each of the two walls, each measured by the distance from the surface of the wall in which it lies, and with $\rho_1$ and $\rho_2$ the distances from the line, the force per unit area exerted by wall 2 on a semi-infinite cylinder in wall 1 centered on the $z_1$ axis and extending from $z_1 = 0$ to $z_1 = \infty$, for $\epsilon_0 \sim 1$, is

$$\begin{aligned} P_{D=D}(\epsilon_0\sim 1) &\sim -\frac{\partial}{\partial\ell}(N_{\mathrm{At}})^2\int_0^\infty dz_1\int_0^\infty dz_2\int_0^\infty 2\pi\,\rho_2\,d\rho_2\,V_{\mathrm{AtAt}}(r) \\ &= -\frac{23}{40}\,\frac{\hbar c(N_{\mathrm{At}}\alpha_0)^2}{\ell^4}\ , \end{aligned} \tag{6.3}$$

where we used Eq. (1.3) for $V_{\rm AtAt}$. Comparison of the two expressions gives

$$P'_{D=D} = \frac{4\pi^2}{(138)^{1/2}} - 1 \approx 2.36 \;, \tag{6.4}$$

and our approximate expression becomes

$$P'_{D=D}(\epsilon_0) = -\frac{\pi^2}{240}\frac{\hbar c}{\ell^4}\left(\frac{\epsilon_0 - 1}{\epsilon_0 + 2.36}\right)^2 , \quad \epsilon_{10} = \epsilon_{20} \equiv \epsilon_0 \;. \tag{6.5}$$

The exact value $P_{D=D}(\epsilon_0)$ was evaluated numerically [3], with the results expressed in the form

$$P_{D=D}(\epsilon_0) = -\frac{\pi^2}{240}\frac{\hbar c}{\ell^4}\left(\frac{\epsilon_0 - 1}{\epsilon_0 + 1}\right)^2 \phi_{DD}(\epsilon_0) \;. \tag{6.6}$$

$\phi_{DD}(\epsilon_0)$ was plotted; its value cannot be read very accurately from the plot given. [The choice of the factor $(\epsilon_0+1)^2$ for the denominator is arbitrary.] Values of the ratio

$$\rho_{D=D}(\epsilon_0) \equiv \frac{P'_{D=D}(\epsilon_0)}{P_{D=D}(\epsilon_0)} = \left(\frac{\epsilon_0 + 1}{\epsilon_0 + 2.36}\right)^2 \frac{1}{\phi_{DD}(\epsilon_0)} \tag{6.7}$$

are given in Table I for a few values of $\epsilon_0$. We once again expect at least rough agreement since, by construction, $P'_{D=D}(\epsilon_0)$ and $P_{D=D}(\epsilon_0)$ agree exactly at the end points ($\epsilon_0 = 1$ and $\epsilon_0 = \infty$), have the same first derivative at $\epsilon_0 = 1$—it vanishes there—and have the same second derivative at $\epsilon_0 = 1$, and since there is nothing in the problem to suggest any rapid variation of $P_{DD}(\epsilon_0)$ with $\epsilon_0$. In fact, $P_{D=D}(\epsilon_0)$ is monotonic in the range 1 to $\infty$ and the two expressions are in meaningful if very rough agreement.

The use of Eq. (4.11) in Eq. (6.3) gives [18]

$$P''_{D=D} = -\frac{207}{640\pi^2}\frac{\hbar c}{\ell^4}\left(\frac{\epsilon_0 - 1}{\epsilon_0 + 2}\right)^2 ,$$

which is off by about 20% at $\epsilon_0 = \infty$.

### B. A metallic wall and a dielectric wall

We turn now to the case of a metallic wall and a dielectric wall. We choose as our approximate form

$$P'_{DD}(\epsilon_{10} \equiv \epsilon_0,\, \epsilon_{20} = \infty) \equiv P'_{DM}(\epsilon_0) = -\frac{\pi^2}{240}\frac{\hbar c}{\ell^4}\frac{\epsilon_{10} - 1}{\epsilon_{10} + B'_{DM}} \;. \tag{6.8}$$

We then have

$$P'_{DM}(\epsilon_0 \sim 1) \sim -\frac{\pi^2}{240}\frac{\hbar c}{\ell^4}\frac{4\pi\, N_{1\rm At}\alpha_{10}}{1 + B'_{DM}} \;. \tag{6.9}$$

$P_{DD}$ for this case is given by $P_{DD}(\epsilon_{10} \equiv \epsilon_0, \epsilon_{20} = \infty) \equiv P_{DM}(\epsilon_0)$. For $\epsilon_0 \sim 1$, we have

$$P_{DM}(\epsilon_0 \sim 1) \sim -\frac{\partial}{\partial \ell} N_{1\rm At}\int_0^\infty dz_1\, V_{{\rm At}M}(r) = -\frac{3}{8\pi}\frac{N_{1\rm At}\alpha_{10}\hbar c}{\ell^4} \;, \tag{6.10}$$

where we used Eq. (1.2) for $V_{{\rm At}M}(r)$, with $r$ given by $r = \ell + z_1$. Equating Eqs. (6.9) and (6.10) gives

$$B'_{DM} = \frac{2\pi^4}{45} - 1 \approx 3.33 \;, \tag{6.11}$$

and our approximate expression becomes

$$P'_{DM}(\epsilon_0) = -\frac{\pi^2}{240}\frac{\hbar c}{\ell^4}\frac{\epsilon_0 - 1}{\epsilon_0 + 3.33} \;. \tag{6.12}$$

$P_{DM}(\epsilon_0)$ has been evaluated numerically [3] and expressed in the form

$$P_{DM}(\epsilon_0) = -\frac{\pi^2}{240}\frac{\hbar c}{\ell^4}\frac{\epsilon_0 - 1}{\epsilon_0 + 1}\phi_{MD}(\epsilon_0) \;. \tag{6.13}$$

(The choice of the form $\epsilon_0+1$ in the denominator is again arbitrary.) We define the ratio $\rho_{DM}(\epsilon_0)$ as

$$\rho_{DM}(\epsilon_0) = \frac{P'_{DM}(\epsilon_0)}{P_{DM}(\epsilon_0)} = \frac{\epsilon_0 + 1}{\epsilon_0 + 3.33}\frac{1}{\phi_{MD}(\epsilon_0)} \;. \tag{6.14}$$

Values of $\rho_{DM}(\epsilon_0)$ for various values of $\epsilon_0$ are listed in Table I. Once again the agreement is meaningful but rough, somewhat rougher than for the previous case of $\epsilon_{10} = \epsilon_{20}$. The use of Eq. (4.11) in Eq. (6.10) gives

$$P''_{DM} = -\frac{9}{32\pi^2}\frac{\hbar c}{\ell^4}\frac{\epsilon_0 - 1}{\epsilon_0 + 2} ,$$

which is off by about 30% at $\epsilon_0 = \infty$.

## VII. TWO WALLS: COMMENTS ON THE GENERAL CASE

A thoroughgoing analysis of the force per unit area $P_{DD}(\epsilon_{10}, \epsilon_{20})$ between walls characterized by $\epsilon_{10}$ and $\epsilon_{20}$ is very much hampered by the paucity of numerical values of $P_{DD}$ available, values being known only for $\epsilon_{10} = \epsilon_{20}$ and for $\epsilon_{20} = \infty$. We intend to determine additional numerical values and then to undertake such an analysis, but we have here largely limited our considerations to some general comments on the forms that might be used in constructing an estimate $P'_{DD}$ of $P_{DD}$ for arbitrary values of $\epsilon_{10}$ and $\epsilon_{20}$; we do give one relatively simple if crude approximation to $P_{DD}$.

### A. Restrictions on the form of the approximation $P'_{DD}(\epsilon_{10}, \epsilon_{20})$

In this subsection we take as our objective the determination of some possible forms of $P'_{DD}$ which encompass *all* of the relevant information which does *not* involve dielectric walls. That information is the following:

$$P_{DD}(\epsilon_{10},\epsilon_{20}) = P_{DD}(\epsilon_{20},\epsilon_{10}) \ , \qquad (7.1)$$

$$P_{DD}(1,\epsilon_{20}) = 0 \ , \qquad (7.2)$$

$$P_{DD}(\infty,\infty) = -\frac{\pi^2}{240}\frac{\hbar c}{\ell^4} \equiv P_{MM} \ , \qquad (7.3)$$

$$P_{DD}(\epsilon_{10}\sim 1,\epsilon_{20}\sim 1) \sim -\frac{23}{40}\frac{N_{1\mathrm{At}}N_{2\mathrm{At}}\alpha_{10}\alpha_{20}\hbar c}{\ell^4} \ , \qquad (7.4)$$

$$P_{DD}(\epsilon_{10}\sim 1,\epsilon_{20}=\infty) \sim -\frac{3}{8\pi}\frac{N_{1\mathrm{At}}\alpha_{10}\hbar c}{\ell^4} \ . \qquad (7.5)$$

Equation (7.4) is a slight extension of Eq. (6.3), and Eq. (7.5) is just Eq. (6.10).

We implement the conditions imposed by Eqs. (7.1) and (7.2) by choosing the form

$$P_{DD}(\epsilon_{10},\epsilon_{20}) = P_{MM}(\epsilon_{10}-1)(\epsilon_{20}-1)\,R(\epsilon_{10},\epsilon_{20}) \ , \qquad (7.6)$$

where

$$R(\epsilon_{10},\epsilon_{20}) = R(\epsilon_{20},\epsilon_{10}) \ . \qquad (7.7)$$

We will now show that, as opposed to the two special cases considered in Sec. VI, we *cannot*, if Eqs. (7.1) through (7.5) are to be satisfied, choose $R(\epsilon_{10},\epsilon_{20})$ to have the product form

$$R(\epsilon_{10},\epsilon_{20}) = F(\epsilon_{10})\,F(\epsilon_{20}) \ . \qquad (7.8)$$

[The difference between the general case and the two special cases is that not all of the equations (7.1)–(7.5) are applicable for the special cases. Equation (7.5) is not applicable if $\epsilon_{10}=\epsilon_{20}$, and Eq. (7.4) is not applicable if $\epsilon_{20}=\infty$.] That Eq. (7.8) is not allowable follows from the conditions on $R(\epsilon_{10},\epsilon_{20})$ imposed by Eqs. (7.3)–(7.5), namely

$$\lim_{\epsilon_{10}\to\infty}\lim_{\epsilon_{20}\to\infty}\epsilon_{10}\epsilon_{20}\,R(\epsilon_{10},\epsilon_{20}) = 1 \ , \qquad (7.9)$$

independently of the order in which the limits are taken,

$$R(1,1) = \frac{69}{8\pi^4} \qquad (7.10)$$

and

$$\lim_{\epsilon_{20}\to\infty}\epsilon_{20}\,R(1,\epsilon_{20}) = \frac{45}{2\pi^4} \ . \qquad (7.11)$$

The assumption made in Eq. (7.8) then gives

$$\left(\lim_{\epsilon_{20}\to\infty}[\epsilon_{20}F(\epsilon_{20})]\right)^2 = 1 \ ,$$

$$F^2(1) = \frac{69}{8\pi^4} \ ,$$

$$F(1)\lim_{\epsilon_{20}\to\infty}[\epsilon_{20}F(\epsilon_{20})] = \frac{45}{2\pi^4} \ .$$

Squaring the last equation and using the first two leads to a contradiction.

Many simple approximate forms for $R(\epsilon_{10},\epsilon_{20})$ are possible. One possibility is

$$R'(\epsilon_{10},\epsilon_{20}) = \frac{1}{(\epsilon_{10}+\beta)(\epsilon_{20}+\beta)}\left(1+\frac{\gamma}{\epsilon_{10}+\epsilon_{20}}\right) \ . \qquad (7.12)$$

Note that, as is necessary, $\gamma$, whose presence makes it impossible to write $R(\epsilon_{10},\epsilon_{20})$ as $F(\epsilon_{10})\,F(\epsilon_{20})$, plays no role as $\epsilon_{10}\sim\infty$ and $\epsilon_{20}\sim\infty$. A second possible form is

$$R''(\epsilon_{10},\epsilon_{20}) = F(\epsilon_{10})\,F(\epsilon_{20}) + G(\epsilon_{10})\,G(\epsilon_{20}) \ . \qquad (7.13)$$

Let us consider the force per unit area, defined by Eqs. (7.6) and (7.12), to be denoted by $\overline{P}'_{DD}(\epsilon_{10},\epsilon_{20})$. On setting $\epsilon_{20}=\infty$ and letting $\epsilon_{10}\sim 1$, $\overline{P}'_{DD}$ assumes the same form as does $P'_{DM}$ of Eq. (6.8) for $\epsilon_{20}=\infty$ and $\epsilon_{10}\sim 1$, namely Eq. (6.9), with $\beta$ replacing $B'_{DM}$. We must therefore have

$$\beta = B'_{DM} \approx 3.33 \ .$$

We then also have

$$\overline{P}'_{DD}(\epsilon_{10}=\epsilon_{20}\equiv\epsilon_0\sim 1) \sim P_{MM}\frac{(4\pi\,N_{\mathrm{At}}\alpha_0)^2}{(1+B'_{DM})^2}\left(1+\frac{\gamma}{2}\right) \ .$$

A comparison with Eq. (6.2) for $P'_{D=D}(\epsilon_0\sim 1)$ gives

$$\frac{1}{(1+B'_{DM})^2}\left(1+\frac{\gamma}{2}\right) = \frac{1}{(1+B'_{D=D})^2} \ ,$$

where $B'_{D=D}\approx 2.36$, so that $\gamma=1.32$ and

$$\overline{P}'_{DD}(\epsilon_{10},\epsilon_{20}) = P_{MM}\frac{(\epsilon_{10}-1)(\epsilon_{20}-1)}{(\epsilon_{10}+3.33)(\epsilon_{20}+3.33)} \times\left(1+\frac{1.32}{\epsilon_{10}+\epsilon_{20}}\right) \ . \qquad (7.14)$$

$\overline{P}'_{DD}(\epsilon_{10},\epsilon_{20})$ reduces to $P'_{DM}$ of Eq. (6.12) for $\epsilon_{20}=\infty$. We introduce the ratio

$$\bar{\rho}_{D=D}(\epsilon_0) \equiv \frac{\overline{P}'_{DD}(\epsilon_{10}=\epsilon_{20}\equiv\epsilon_0)}{P_{DD}(\epsilon_{10}=\epsilon_{20}\equiv\epsilon_0)} = \left(\frac{\epsilon_0+1}{\epsilon_0+3.33}\right)^2\left(1+\frac{0.66}{\epsilon_0}\right)\frac{1}{\phi_{DD}(\epsilon_0)} \ , \qquad (7.15)$$

where we used Eqs. (6.6), (7.3), and (7.14). Some values of $\bar{\rho}_{D=D}$ are given in Table I; we see that $\overline{P}'_{DD}(\epsilon_{10},\epsilon_{20})$ is not as good an estimate, for $\epsilon_{10}=\epsilon_{20}$, as $P'_{D=D}$. We will seek a better estimate of $P_{DD}$ in the near future.

### B. Use of the atom-dielectric wall interaction

The words "elementary derivation" in the title were meant to imply that the analysis *in this paper* of Casimir effects involving dielectric walls is elementary; our analysis does rely, however, on not particularly elementary previous determinations of a number of Casimir effects which are much simpler than those involving dielectric walls, but namely those which do *not* involve dielectric walls. From a logical point of view we can, in the same spirit, use $V_{\text{At}D}(\epsilon_{20})$ in a determination of $P_{DD}(\epsilon_{10},\epsilon_{20})$ since the former is certainly simpler than the latter. Thus, as noted above, the direct determination [9] of $V_{\text{At}D}(\epsilon_{20})$ is much simpler than its derivation from $P_{DD}(\epsilon_{10},\epsilon_{20})$ by letting $\epsilon_{10}\sim 1$, and the derivation of $V_{\text{At}D}$ might have—and logically should have—preceded that of $P_{DD}$.

With $V_{\text{At}D}$ assumed to be known—given by Eq. (4.6), with analytic approximations and with numerical values of $\phi\text{At}D$ both available — we seek an improved estimate of $P_{DD}(\epsilon_{10},\epsilon_{20})$. To do so we replace Eq. (7.5), which considers $\epsilon_{10}\sim 1$ and $\epsilon_{20}=\infty$, by the more restrictive equation which considers $\epsilon_{10}\sim 1$ and *all* $\epsilon_{20}$, namely

$$P_{DD}(\epsilon_{10}\sim 1,\epsilon_{20}) \sim -N_{1\text{At}}\frac{3}{8\pi}\hbar c\alpha_{10}\,\frac{\epsilon_{20}-1}{\epsilon_{20}+1}\,\phi_{\text{At}D}(\epsilon_{20})\,\frac{1}{\ell^4}\ . \tag{7.16}$$

Equation (7.16), which differs from Eq. (7.5) by the presence of the factor

$$\frac{\epsilon_{20}-1}{\epsilon_{20}+1}\,\phi_{\text{At}D}(\epsilon_{20})\ ,$$

is obtained by using Eq. (4.6) for $V_{\text{At}D}(\epsilon_{20})$ rather than Eq. (1.2) for $V_{\text{At}M}=V_{\text{At}D}(\infty)$.

Of the many possible uses of Eq. (7.16), we will concentrate on its value in helping to choose a $P'_{DD}$ for $\epsilon_{10}\sim 1$ and $\epsilon_{20}\sim\infty$; we have used our knowledge of the value of $P_{DD}$ and therefore of $P'_{DD}$ for $\epsilon_{10}\sim 1$ and $\epsilon_{20}$ *equal to* $\infty$, but we have not built in any information on the asymptotic form of $P_{DD}$ for $\epsilon_{10}\sim 1$ and $\epsilon_{20}\sim\infty$, and that is the region where $P_{DD}$ can be expected to (and does) change most rapidly. (The behavior for $\epsilon_{20}\approx 1$ can be expected to be, and is, simply linear in $\epsilon_{20}-1$.) To utilize Eq. (7.16) for $\epsilon_{20}\sim\infty$ we note that

$$V_{\text{At}D}(\epsilon_{20})\sim V_{\text{At}M}\left(1-\frac{5}{4\epsilon_{20}^{1/2}}\right)\ ,\quad \epsilon_{20}\sim\infty\ . \tag{7.17}$$

(See the Appendix.) From Eq. (7.16) we conclude that, ignoring terms of $O(1/\epsilon_{20})$,

$$P_{DD}(\epsilon_{10},\epsilon_{20})\sim-\frac{3}{8\pi}\,\frac{\hbar c N_{1\text{At}}\alpha_{10}}{\ell^4}\left(1-\frac{5}{4\epsilon_{20}^{1/2}}\right)\ ,\qquad \epsilon_{10}\sim 1,\ \ \epsilon_{20}\sim\infty\ . \tag{7.18}$$

## VIII. DISCUSSION

The approximations obtained above for $V_{\text{At}D}$, $V_{\text{El}D}$, and $P_{DD}$ are a bit crude, but their determinations are some orders of magnitude simpler than the exact determinations and, as opposed to the exact determinations, they make at least some of the "physics" more or less understandable. Furthermore, the general procedure should make it possible to obtain a simple approximation $P_{DD}$, for arbitrary values of $\epsilon_{10}$ and $\epsilon_{20}$. This would be quite useful since, as opposed to $V_{\text{At}D}$ and $V_{\text{El}D}$, $P_{DD}$ is a double (not single) integral which contains not one parameter but two ($\epsilon_{10}$ and $\epsilon_{20}$); the integrand is quite complicated and the double integral must be evaluated numerically. It would also be useful to have simple approximations for $V_{\text{At}D}$ and $V_{\text{El}D}$, even though they can be evaluated analytically, because the analytic expressions are very complicated.

Our primary interest has been in the exploration of relations between different retardation effects, one particular aspect being the examination of the accuracy attainable in going from the simple to the complex—from $V_{\text{At}D}$ to $P_{DD}$, for example—rather than, as often is done, in going from the complex to the simple [2,3,20].

It would be straightforward to extend the present analysis to other situations. These include the interactions of atoms or of electrons with walls with magnetic as well as electric properties; exact results for these cases, in the form of complicated integrals, have been obtained recently [21]. They also include the case of the force per unit area $P(\epsilon_{10},\epsilon_{20},\epsilon_{30})$ between two walls when the region between them is not a vacuum but a medium with a dielectric constant $\epsilon_{30}$. Given a good approximation $P'_{DD}$ to $P_{DD}$, the extension in this case would be immediate, since it has been shown [3] that

$$P(\epsilon_{10},\epsilon_{20},\epsilon_{30})=\frac{1}{\epsilon_{30}^{1/2}}P_{DD}\left(\frac{\epsilon_{10}}{\epsilon_{30}}\ ,\ \frac{\epsilon_{20}}{\epsilon_{30}}\right)\ ,$$

where the $P_{DD}$ on the right refers to the two-wall case.

A weakness of our approach is that it does not have a solid theoretical basis—the forms chosen are somewhat arbitrary even if reasonable. That weakness can be a strength in that it allows one to use experimental data. Assume, for example, that one did not know how to calculate $V_{\text{At}D}$ and wanted to go beyond $V'_{\text{At}D}$ of Eq. (4.5) and that one had measured $V_{\text{At}D}$ for an atom of known $\alpha_{10}$ interacting with a wall of known $\epsilon_{20}$. One could then replace $\epsilon_{20}+37/23$ in the denominator of Eq. (4.5) by $\epsilon_{20}+\gamma\,\epsilon_{20}^{1/2}+\delta$, with the two parameters $\gamma$ and $\delta$ determined by one piece of theoretical information, the behavior of $V_{\text{At}D}$ for $\epsilon_{20}\sim 1$, and one piece of experimental information.

The most interesting possible extension would be to the very difficult case of arbitrary $\ell$. To simplify the discussion we consider the specific case of two walls. The $\ell$ dependence cannot then be extracted and $P_{DD}$ is a very comlicated double integral containing one parameter $\ell$ and two complex *functions* $\epsilon_1(\omega)$ and $\epsilon_2(\omega)$. The $\ell$ dependence can be extracted for $\ell$ small [2,3], and one should be able to obtain approximations for that case by methods similar to those used above for $\ell$ large, and one would then try to use interpolation to obtain estimates for intermediate values of $\ell$. Since the fluctuations in $\epsilon_1(\omega)$ and $\epsilon_2(\omega)$ will induce fluctuations in $P_{DD}$, the estimates would at best be averages over domains of intermediate values of $\ell$. The flucutations of $P_{DD}(\ell)$ with $\ell$ may well be very much less than the fluctuations of $\epsilon_1(\omega)$ and $\epsilon_2(\omega)$ with $\omega$ since $P_{DD}$ contains an integral over $\omega$.

We note in passing that the transition of the atom-surface interaction from $1/\ell^3$ to $1/\ell^4$ was very recently measured for the first time [22].

## ACKNOWLEDGMENT

This research was partially supported by the NSF under Grant No. PHY 90–19745.

## APPENDIX: THE ATOM-WALL INTERACTION FOR $\epsilon_{20} \sim \infty$

The interaction of an atom and a dielectric wall is given by Eq. (4.6), where $\phi_{\mathrm{At}D}(\epsilon_{20})$ is given in integrated form by Eq. (4.38) of Ref. [3]. For determining $V_{\mathrm{At}D}(\epsilon_{20})$ for $\epsilon_{20} \sim \infty$, it is much simpler to start with Eq. (4.3) of Ref. [9],

$$V_{\mathrm{At}D}(\epsilon_{20}) = -\frac{\hbar\alpha_{10}}{2\pi c^3}\int_0^\infty d\xi\,\xi^3\int_1^\infty dp\,e^{-2\xi\ell p/c}\,H(p,\epsilon_{20})\ ,$$

where $H(p,\epsilon_{20})$ is defined by Eq. (3.11) of Ref. [9]. One readily finds that

$$H(p,\epsilon_{20}) \sim 2p^2 + \frac{2-6p^2}{p\epsilon_{20}^{1/2}}\ , \quad \epsilon_{20} \sim \infty\ .$$

Equation (7.15) follows immediately.

---

* Permanent address: Racah Institute of Physics, The Hebrew University of Jerusalem, Jerusalem 91904, Israel.

[1] H. B. G. Casimir, Proc. K. Ned. Akad. Wet. **60**, 793 (1948).

[2] E. M. Lifshitz, J. Exp. Theor. Phys. USSR **29**, 94 (1955) [Sov. Phys. **2**, 73 (1956)]. See also L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media,*, Vol. 8 of *Course of Theoretical Physics* (Pergamon, New York, London, 1960), Chap. 13.

[3] I. E. Dzyaloshinskii, E. M. Lifshitz, and L. Pitaevskii, Adv. Phys. **10**, 165 (1961).

[4] H. B. G. Casimir, J. Chim. Phys. **46**, 407 (1949).

[5] H. B. G. Casimir and D. Polder, Phys. Rev. **73**, 360 (1948).

[6] E. A. Power, *Introductory Quantum Electrodynamics* (Elsevier, New York, 1965).

[7] V. L. Ginzburg, *Applications of Electrodynamics in Theoretical Physics and Astrophysics* (Gordon and Breach, New York, 1989), Chap. 14. Ginzburg also discusses a method involving "surface modes"—see Ref. [8]—which is much simpler for the wall-wall problem than the Lifshitz approach and which may be applicable to the electron-wall problem.

[8] D. Langbein, Solid State Commun. **12**, 853 (1973); K. Schram, Phys. Lett. **43A**, 282 (1973).

[9] Y. Tikochinsky and L. Spruch, following paper, Phys. Rev. A **48**, 4223 (1993).

[10] G. Barton, J. Phys. A **10**, 601 (1977).

[11] L. Spruch and E. J. Kelsey, Phys. Rev. A **18**, 845 (1978).

[12] R. Shakeshaft and L. Spruch, Phys. Rev. A **22**, 811 (1980).

[13] J. Bernabou and R. Tarrach, Ann. Phys. (N.Y.) **102**, 323 (1976).

[14] E. J. Kelsey and L. Spruch, Phys. Rev. A **18**, 1055 (1978).

[15] L. Spruch, Phys. Today **39** (11), 37 (1986), and Chap. I in *Long-Range Casimir Forces: Theory and Recent Experiment in Multiparticle Dynamics*, edited by Frank S. Levin and David A. Micha (Plenum, New York, 1993).

[16] K. A. Milton, L. L. DeRaad, and J. Schwinger, Ann. Phys. (N.Y.) **10**, 165 (1961).

[17] R. J. Glauber and M. Lewenstein, Phys. Rev. A **43**, 467 (1991).

[18] P. W. Milonni and M.-L. Shih, *Contemporary Physics*, edited by J. S. Dugdale (Taylor and Francis, London, 1993).

[19] C. C. Grimes and T. R. Brown, Phys. Rev. Lett. **32**, 280 (1974).

[20] We note parenthetically that $V_{\mathrm{AtAt}}$ was obtained originally [5] via a *microscopic* approach, and was later obtained [2] by using a *macroscopic* approach to obtain $V_{DD}$ and then allowing both $\epsilon_{10}$ and $\epsilon_{20}$ to approach unity. We also note that a knowledge of $V_{DD}$ can, with some ambiguity, lead to $V_{\mathrm{El}D}$ (and $V_{\mathrm{ElAt}}$) [9] and, while the converse is not true, that a knowledge of $V_{\mathrm{ElAt}}$ can, as shown above, be used to provide at least a rough approximation of $V_{\mathrm{El}D}$.

[21] Y. Tikochinsky and L. Spruch, this issue, Phys. Rev. A **48**, 4236 (1993).

[22] C. I. Sukenik, M. G. Boshier, D. Cho, V. Sandoghdar, and E. A. Hinds, Phys. Rev. Lett. **70**, 560 (1993).