Energy and β-function solutions to relativistic Hamiltonians with Coulombic and linear potentials

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It is argued that quantum-mechanical β functions can be derived consistently by using renormalization length scales implied by the quasiclassical minimization of Hamiltonian forms. Our renormalization-group results for the Dirac-Coulomb system can then be related analytically to ones obtained in ladder quantum electrodynamics in four space-time dimensions. Involving spinless relativistic two-body Hamiltonians, one can present exact energy- and β-function solutions for the Lorentz scalar linear potential. Inverse-square, other linear, and mixed linear plus Coulomb potentials can also be treated. The exact energy of this last potential has been discussed in conjunction with an alternative β function.

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I. INTRODUCTION

Proofs have been given that the Dirac equation with both Lorentz-vector $[U(x)=-\alpha/x]$ and Lorentz-scalar $[S(x)=-\gamma/x]$ Coulomb potentials, where $x=|\mathbf{x}|$, leads to the nonperturbative β function [1]

$$\beta(\alpha;\gamma)=\alpha\left[1-\frac{\alpha^2}{\alpha_C^2}\right]+\alpha_C\frac{\gamma}{\kappa}\left[1-\frac{\alpha^2}{\alpha_C^2}\right]^{3/2}, \quad (1.1)$$

which relies on the self-similar attributes of the ground-state energy

$$\mathcal{E}_0=\frac{1}{\alpha_C^2}[-\alpha\gamma+\kappa(\alpha_C^2-\alpha^2)^{1/2}], \quad (1.2)$$

in which $l=n_r=0$ and $s=1$ [2]. The length scale of the present renormalization description has been identified with the location $x=x_0$ of the minimum of the effective radial potential [see Eq. (7) in Ref. [1]] characterizing the quantum-mechanical $1/N$ description of bound states [3,4]. Note that the $1/N$ description of Coulomb potentials gives energy results which are exact to first $1/N$ order [5], so that the same remains valid for the above β function. In this respect, x_0 has the meaning of a nonperturbative renormalization scale, which becomes subject to an inherent dependence on the couplings by virtue of the fixing condition of the expansion parameter d_0 of the $1/N$ method. We then have to realize that the coupling-values for which $x_0=0$ and $x_0=\infty$ lead to ultraviolet (UV) and infrared (IR) fixed points, respectively. Additional fixed points are able to be produced by extrapolating the β functions towards intervals in which $x_0<0$. Concerning Eq. (1.1) it has also been assumed that only the α coupling is sensitive to x_0. So the above β function produces the UV fixed point $\alpha_C=(\kappa^2+\gamma^2)^{1/2}$ and the IR fixed point $\alpha=-\gamma$. One has $\kappa=(N-1)/2$, where N stands for the number of space dimensions. Going outside the bound-state region (see the dashed curves) leads to the additional UV fixed point $\alpha=-\alpha_C$, as displayed in Fig. 1. An immediate generalization, incorporating the above $1/N$ approach, can be done just in terms of the quasiclassical minimization of the pertinent Schrödinger Hamiltonian $\mathcal{H}(x,p)=p^2+V(x)=E$. This amounts to minimizing $\mathcal{H}(x,d_0/x)$, where d_0 plays, this time, just the role of a parameter relying on the mapping $p\rightarrow d_0/x$ [6]. Starting from a well-defined analytic expression for the ground-state energy E_0 then gives x_0 via

$$V(x_0)+\frac{x_0}{2}V'(x_0)=E_0\ . \quad (1.3)$$

In general, E_0 may be exact or an approximation, so that the x_0 outcomes are subject to the same statements, respectively.

The above β function can also be viewed as a certain subcritical counterpart of the nonperturbative β function derived before with the help of the ladder approximation to the Schwinger-Dyson equation for the fermion self-energy [7–9]. In this later case, the critical coupling is given by $\alpha_C^*=\pi/3$ in four space-time dimensions. However, the analytic form of this ladder quantum electrodynamics in four space-time dimensions (QED_4) β function can also be reobtained, up to the value of the UV fixed point, by performing the renormalization-group (RG) description of the Dirac-Coulomb system in terms of the collapse of the wave function, now for $\alpha>\alpha_C$ [10]. This indicates that we have to account for nonperturbative field-theoretical phases in which quantum-mechanical RG structures survive vacuum polarization

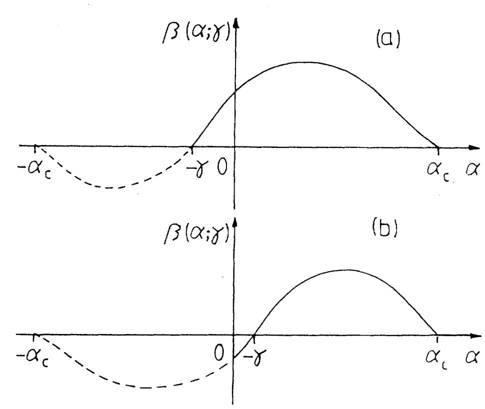

FIG. 1. The α dependence of $\beta(\alpha;\gamma)$ for $\gamma>0$ (a) and $\gamma<0$ (b).

effects. In other words, quantum-mechanical RG structures are able to serve as useful analogies providing a better understanding of nonperturbative renormalizations of related field theories. In this context, other mutual relationships or theoretical intercorrelations, such as the analogies between the φ^4 field theory and the anharmonic oscillator [11], the interpretation of inverse dimensionality as temperature [12], or the practical importance of matrix models [13], can also be mentioned. On the other hand, the exploration of the fixed points, as well as of the phase-transition attributes of quantum-mechanical systems, has its own intrinsic interest. Indeed, having obtained the β functions enables us to derive the corresponding order parameters, as discussed before [14]. Accordingly, we are in a position to extract useful nonperturbative information concerning fixed-point structures, this time from quickly tractable RG formulas for exactly and/or approximately solvable quantum-mechanical systems. The derivation of such results, which are useful for further interrelated investigations, represents the main goal of the preset studies.

This paper is organized as follows. In Sec. II we shall discuss the RG equations for the ground-state $1/N$ expansion parameter

$$d_0=d_0(\alpha)=(\alpha_C^2-\alpha^2)^{1/2} , \tag{1.4}$$

of the Dirac-Coulomb system, as well as for $\mathcal{E}_0$. This results in the exact derivation of corresponding anomalous dimensions. Comparisons with ladder QED_4 results and related parameter-fixing interpretations are presented in Sec. III. Two-body spinless relativistic Hamiltonians such as [15,16]

$$\mathcal{H}_R(x,p)=2\{p^2+[\mu_0+S(x)]^2\}^{1/2}+U(x)=\mathcal{E} , \tag{1.5}$$

where μ_0 is the mass parameter are discussed in Sec. IV, with a special emphasis on exact $\mathcal{E}_0$ and β solutions for the Lorentz scalar linear potential $S(x)=kx/2$. Closed $1/N$ approximations to the potentials $U(x)=k'x$ and $S(x)=-\Lambda_0/x^2$ [17] have also been done. The conclusions are presented in Sec. V.

II. THE DERIVATION OF RG EQUATIONS

It is an instructive exercise to reformulate our knowledge about $\mathcal{E}_0$ and d_0 into RG language. Starting from the definition of the present β function,

$$\beta(\alpha;\gamma)=\mu\frac{\partial}{\partial\mu}\alpha , \tag{2.1}$$

where $\mu=1/x_0$ and using the exact $1/N$ result

$$d_0=\kappa\alpha x_0(\alpha_C^2-\gamma x_0)^{-1} , \tag{2.2}$$

one finds immediately the RG equation

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(\alpha;\gamma)\frac{\partial}{\partial\alpha}-\gamma_{d_0}(\alpha)\right]d_0=0 . \tag{2.3}$$

Accordingly, the exact anomalous dimension of d_0 reads as

$$\gamma_{d_0}(\alpha)=-\frac{\alpha^2}{\alpha_C^2}\left[1+\frac{\gamma\alpha_C}{\kappa\alpha}\left[1-\frac{\alpha^2}{\alpha_C^2}\right]^{1/2}\right] , \tag{2.4}$$

which can also be reobtained identically in terms of Eqs. (1.1) and (1.4). Conversely, the RG solution to (2.3) is

$$d_0(\alpha)=d_0(\alpha_0)\exp\left[\int_{\alpha_0}^{\alpha}\gamma_{d_0}(\alpha')\frac{d\alpha'}{\beta(\alpha';\gamma)}\right] , \tag{2.5}$$

in which α_0 plays the role of an initial coupling. It is then obvious that Eq. (2.5) gives the RG invariant

$$\frac{d_0(\alpha)}{(\alpha_C^2-\alpha^2)^{1/2}}=\frac{d_0(\alpha_0)}{(\alpha_C^2-\alpha_0^2)^{1/2}}=\text{const} , \tag{2.6}$$

which reproduces precisely Eq. (1.4) via const$=1$.

Concerning $\mathcal{E}_0$ one proceeds similarity. Now the RG equation is

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(\alpha;\gamma)\frac{\partial}{\partial\alpha}-\gamma_{\mathcal{E}_0}(\alpha)\right]\mathcal{E}_0=0 , \tag{2.7}$$

in which

$$\gamma_{\mathcal{E}_0}(\alpha)=\frac{d_0}{\kappa\alpha_C^2}\frac{(\kappa\alpha+d_0\gamma)^2}{\gamma\alpha-\kappa d_0} , \tag{2.8}$$

represents the corresponding anomalous dimension. The solution to Eq. (2.7) reads as

$$\mathcal{E}_0(\alpha)=\mathcal{E}_0(\alpha_0)\exp\left[\int_{\alpha_0}^{\alpha}\gamma_{\mathcal{E}_0}(\alpha')\frac{d\alpha'}{\beta(\alpha';\gamma)}\right] , \tag{2.9}$$

so that

$$\frac{\mathcal{E}_0(\alpha)}{\kappa d_0(\alpha)-\gamma\alpha}=\frac{\mathcal{E}_0(\alpha_0)}{\kappa d_0(\alpha_0)-\gamma\alpha_0}=\text{const.} \tag{2.10}$$

Equation (2.10) reproduces precisely Eq. (1.2) as soon as const$=1/\alpha_C^2$. Other states can also be treated in the same manner by starting from Eqs. (8), (9), and (11) in Ref. [1]. Thus, the Dirac-Coulomb system is subject to a well-defined RG description, which opens the way to further analogies with ladder QED_4.

III. ANALOGIES WITH LADDER QED_4

Choosing a physical observable such as $\mu^{D_0}\Omega(\alpha)$, we find the RG equation

$$\left[\beta(\alpha;\gamma)\frac{\partial}{\partial\alpha}+D_0-\gamma_\Omega(\alpha)\right]\Omega(\alpha)=0 , \tag{3.1}$$

where the canonical dimension has been denoted by D_0. Accordingly, the dynamical scaling dimension reads as

$$D_\Omega(\alpha)=D_0-\gamma_\Omega(\alpha) . \tag{3.2}$$

First let us assume that the above observable is RG invariant, as usual. Then, $\gamma_\Omega(\alpha)=0$, so that [14]

$$\Omega(\alpha)=\Omega(\alpha_0)\exp\left[-D_0\int_{\alpha_0}^{\alpha}\frac{d\alpha'}{\beta(\alpha';\gamma)}\right] , \tag{3.3}$$

where α_0 again has the meaning of an initial coupling. Restricting ourselves to α values near α_C yields the leading form

$$\beta(\alpha;\gamma)\cong 2(\alpha_C-\alpha)\left[1+\frac{\gamma}{\kappa}2^{1/2}\left[1-\frac{\alpha}{\alpha_C}\right]^{1/2}\right], \tag{3.4}$$

in accord with Eq. (1.1). This in turn produces the leading behavior

$$\Omega(\alpha)\cong\text{const}\times\left[1-\frac{\alpha}{\alpha_C}\right]^{D_0/2}, \tag{3.5}$$

which shows that $\alpha=\alpha_C$ plays the role of a phase-transition point. Next we would like to generalize Eq. (3.5) by starting from an order parameter such as [18]

$$P=\text{const}\times\mu^{D_0}(\alpha_C-\alpha)^n, \tag{3.6}$$

for $\alpha<\alpha_C$, in which $n>0$ is, this time, an arbitrary power exponent which should be fixed later. Obviously, P represents a "nonconserved" order parameter as long as $n\neq D_0/2$. The anomalous dimension characterizing (3.6) reads as

$$\gamma_P(\alpha)=D_0-\gamma_n(\alpha), \tag{3.7}$$

where

$$\gamma_n(\alpha)=n\frac{\alpha(\alpha+\alpha_C)}{\alpha_C^2}\left[1+\frac{\gamma\alpha_C}{\kappa\alpha}\left[1-\frac{\alpha^2}{\alpha_C^2}\right]^{1/2}\right]. \tag{3.8}$$

Now we are in a position to address the question of whether the above anomalous dimension is able to be related analytically to the ones derived within the ladder approach to the combination between QED_4 and the induced four-Fermi interaction [19]. For this purpose, let us recall that the fermion mass operator $\bar{\psi}\psi$ for which $D_0=3$ exhibits the anomalous dimension

$$\gamma_m(\alpha)=1\pm\left[1-\frac{\alpha}{\alpha_C^*}\right]^{1/2}, \tag{3.9}$$

where the plus and minus signs correspond to the broken and symmetric phases of the theory, i.e., to $\langle\bar{\psi}\psi\rangle\neq 0$ [20] and $\langle\bar{\psi}\psi\rangle=0$ [21,22], respectively. On the other hand (3.7) and (3.8) produce the leading behavior

$$\gamma_P(\alpha)=D_0-2n-2^{3/2}\frac{n\gamma}{\kappa}\left[1-\frac{\alpha}{\alpha_C}\right]^{1/2}, \tag{3.10}$$

for α near α_C. Putting $D_0=3$, we then find that the matching conditions between Eqs. (3.9) and (3.10) read as

$$n=1, \tag{3.11}$$

which indicates that, in this context, our order parameter ceases to be RG invariant and

$$2^{3/2}\gamma=\mp\kappa. \tag{3.12}$$

Combining these results with the equivalence condition $\alpha_C=\alpha_C^*$, i.e., with

$$(\kappa^2+\gamma^2)^{1/2}=\frac{\pi}{3} \tag{3.13}$$

gives

$$|\gamma|=\frac{\pi}{g}\simeq 0.349, \tag{3.14}$$

where $\gamma<0$ ($\gamma>0$) relies on the broken (symmetric) phase. One should also have

$$N=\frac{2^{5/2}}{g}\pi+1\cong 2.975, \tag{3.15}$$

which shows that the mutual relationship between ladder QED_4 and the present RG description of the Dirac-Coulomb problem works by restricting the underlying Dirac equation to $N\cong 2.975$ space dimensions, provided that $|\gamma|\cong 0.349$. This leads to a refinement of the fixing condition for N which was written down earlier [1]. So the mutual relationship between ladder QED_4 and our RG approach is theoretically tractable, provided that the n exponent is subject to the extrapolation $n=\frac{3}{2}\rightarrow n=1$. However, this later point indicates that further clarifications, as well as a more appropriate selection of underlying potentials, remain desirable.

IV. β FUNCTIONS FOR RELATIVISTIC TWO-BODY SYSTEMS

Squaring twice Eq. (1.5) leads to the Schrödinger-equivalent Hamiltonian [15]

$$\mathcal{H}(x,p)=p^2+S^2(x)-\tfrac{1}{4}U^2(x)+2\mu_0 S(x)+\tfrac{1}{2}\mathcal{E}U(x)=E, \tag{4.1}$$

in which

$$E=E(\mathcal{E})=\frac{\mathcal{E}^2}{4}-\mu_0^2, \tag{4.2}$$

as discussed in similar cases before [5]. Our first example is the Lorentz-scalar linear potential $S(x)=kx/2$, which is of interest in the description of quark confinement [23–25]. This potential is exactly solvable for s-wave states, i.e., for $L^2=\frac{1}{4}$, where $L=l+(N-2)/2$. Indeed, putting $l=0$ and $N=3$ and using the energy formula for the "driven" harmonic oscillator discussed recently [26] gives

$$E=(2n_r+1)\frac{k}{2}-\mu_0^2, \tag{4.3}$$

so that $\mathcal{E}$ becomes, surprisingly enough, independent of μ_0,

$$\mathcal{E}=[2k(2n_r+1)]^{1/2}, \tag{4.4}$$

provided that $k>0$. One would then obtain the x_0 parameter as

$$x_0=\frac{1}{2k}[-3\mu_0+(\mu_0^2+4k)^{1/2}], \tag{4.5}$$

by virtue of Eqs. (1.3), (4.1), and (4.3), where $n_r=0$. Applying the RG differentiation,

$$x_0\frac{d}{dx_0}=x_0\frac{\partial}{\partial x_0}-\beta(k)\frac{\partial}{\partial k} \tag{4.6}$$

to Eq. (4.5) yields the β function

$$\beta(k)=k(\mu_0^2+4k)^{1/2}\frac{(\mu_0^2+4k)^{1/2}-3\mu_0}{\mu_0^2+2k-3\mu_0(\mu_0^2+4k)^{1/2}}\ , \tag{4.7}$$

which has been outlined in Fig. 2. We remark that $\beta(k)$ exhibits the IR fixed point $k=0$ as well as the UV fixed point

$$k_C=2\mu_0^2\ , \tag{4.8}$$

which agrees with the $x_0=\infty$ and $x_0=0$ limits mentioned before, respectively. The vertical asymptotes characterizing Eq. (4.7) are located at $k_-=\mu_0^2(4-3\sqrt{2})$ and $k_+=\mu_0^2(4+3\sqrt{2})$. One has also the additional fixed point $k_1=-\mu_0^2/4$, which comes from the extrapolation of $\beta(k)$ beyond the starting $x_0>0$ domain. The superposition between the Lorentz-scalar and Lorentz-vector linear potentials $S(x)=kx/2$ and $U(x)=k'x$ can be treated in a similar manner. Without considering further details, we would like to say that the present s-wave energy-solution reads as

$$\mathcal{E}=\frac{2}{k}[-\mu_0k'+(k^2-k'^2)^{3/4}(n_r+\tfrac{1}{2})^{1/2}] \tag{4.9}$$

if $k^2>k'^2$, which reproduces Eq. (4.4) as soon as $k'=0$.

The next example is the Lorentz-vector linear potential $U(x)=k'x$, which is not exactly solvable, in contradistinction to the above cases. For convenience, we shall then resort again to the quickly tractable $1/N$ method, which has been applied to similar cases before [5,27]. A further simplification arises by performing a tentative extrapolation towards two-body fermion-antifermion systems via $l\to l+s/2$ and $n_r\to n_r-s/2$, where, this time, $s=0,\pm1$ [28]. Indeed, choosing the $^3S_1$ state, for which $l=n_r=0$ and $s=1$, enables us to eliminate the undesirable square-root term [29] from the $1/N$ fixing condition for d_0 via $s=1$. It should also be mentioned that the $1/N$ formulas for Dirac systems with non-Coulombic potentials become improved by inserting an additional spin-orbit term into the corresponding Schrödinger-equivalent Hamiltonian [30]. For the sake of simplicity, such a term will be hereafter ignored. Under such conditions one obtains the equations

$$\mathcal{E}_0^2-3\mathcal{E}_0k'x_0+2k'^2x_0^2-4\mu_0^2=0 \tag{4.10}$$

and

$$\mathcal{E}_0k'x_0^3-k'^2x_0^4=(N-1)^2\ , \tag{4.11}$$

in which $d_0=(N-1)/2$ to first $1/N$ order. Applying the RG differentiation to (4.11) then gives the β function

$$\beta(k')=k'\left[1+\frac{2}{u}(N-1)^2\right]\ , \tag{4.12}$$

where

$$u=\tfrac{1}{2}k'^2x_0^4+\tfrac{1}{2}k'^3x_0^5(k'^2x_0^2+16\mu_0^2)^{-1/2}+(N-1)^2 \tag{4.13}$$

and

$$\mathcal{E}_0=\tfrac{1}{2}[3k'x_0+(k'^2x_0^2+16\mu_0^2)^{1/2}]\ . \tag{4.14}$$

Next, we have to remark that Eq. (4.11) can be approximated as

$$\mathcal{E}_0k'x_0^3\cong(N-1)^2\ , \tag{4.15}$$

if $k'\to0$. This, in turn, means that u should be replaced correspondingly by $u+k'^2x_0^4$. On the other hand, one has $\mathcal{E}_0\to2\mu_0$ for $k'\to0$, so that

$$x_0\to(N-1)^{2/3}(2\mu_0k')^{-1/3}\ , \tag{4.16}$$

for $k'\to0$. Consequently, $\beta(k')$ becomes

$$\beta(k')\cong3k'\left[1-\left[\frac{N-1}{4\mu_0^2}\right]^{2/3}k'^{2/3}\right]\ , \tag{4.17}$$

which yields the IR fixed point $k'=0$ and the UV fixed point

$$k'_C\cong\frac{4}{N-1}\mu_0^2\ , \tag{4.18}$$

which reproduces, somewhat fortunately, Eq. (4.8) as soon as $N=3$. Keeping in mind the fact that Eq. (4.17) represents a quite crude approximation, we would like to emphasize, however, that Eq. (4.18) is able to reflect correctly the analytic dependence of k'_C on N. As in the Coulomb case, the IR fixed point $k'=0$ is the only point which survives the nonrelastivistic limit.

Our third example is the Lorentz-scalar attractive inverse square power-law potential $S(x)=-\Lambda_0/2x^2$. This leads to the Schrödinger-equivalent Hamiltonian

$$\mathcal{H}(x,p)=p^2-\mu_0\frac{\Lambda_0}{x^2}+\frac{\Lambda_0^2}{4x^4}=E=\frac{\mathcal{E}^2}{4}-\mu_0^2\ , \tag{4.19}$$

which can be treated by using Eq. (16) from Ref. [17]. Although this later equation has been established, in combination with the WKB method, to first $1/N$ order, it produces accurate ground-state ($n_r=0$) energy results, as verified previously [31]. Accordingly, one finds

$$E=-\left[\frac{1}{\Lambda_0^2}(\mu_0\Lambda_0-L^2)^{1/2}-\frac{\sqrt{2}}{2}\right]^4\ , \tag{4.20}$$

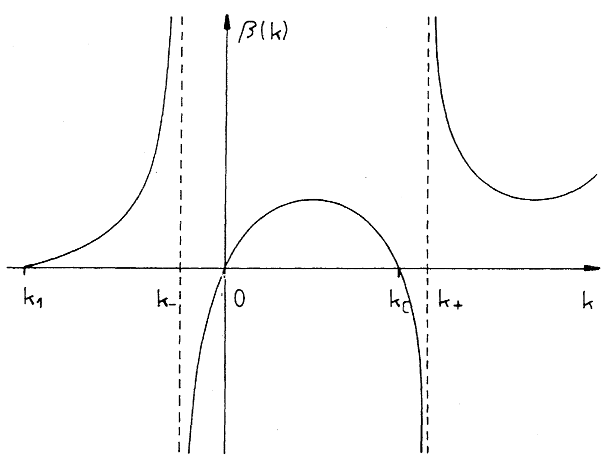

FIG. 2. An outline of the β function for $S(x)=kx/2$.

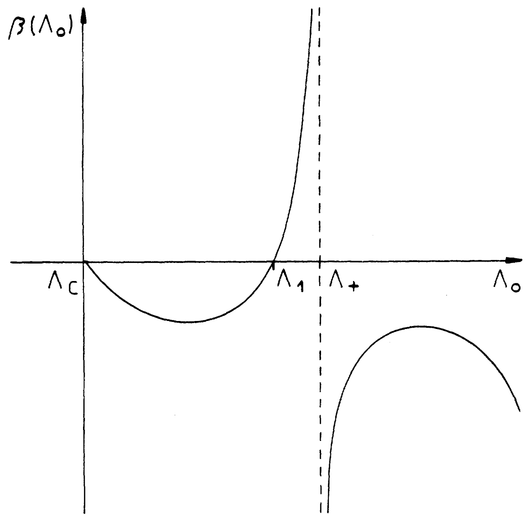

FIG. 3. An outline of the β function for $S(x)=-\Lambda_0/2x^2$.

for $n_r=0$, so that

$$x_0=\frac{\Lambda_0}{\sqrt{2}}\left[(\mu_0\Lambda_0-L^2)^{1/2}-\frac{\sqrt{2}}{2}\right]^{-1}, \qquad (4.21)$$

by virtue of Eq. (1.3). This gives the β function

$$\beta(\Lambda_0)=\Lambda_0\frac{(\mu_0\Lambda_0-L^2)^{1/2}[(\mu_0\Lambda_0-L^2)^{1/2}-\sqrt{2}/2]}{L^2-\mu_0\Lambda_0/2+[(\mu_0\Lambda_0-L^2)/2]^{1/2}}, \qquad (4.22)$$

which exhibits the UV fixed point

$$\Lambda_0=\Lambda_C=\frac{L^2}{\mu_0} \qquad (4.23)$$

and the IR fixed point

$$\Lambda_0=\Lambda_1=\frac{1}{\mu_0}(L^2+\tfrac{1}{2}), \qquad (4.24)$$

as shown in Fig. 3. It should also be specified that Eq. (4.22) has a pole located at

$$\Lambda_0=\Lambda_+=\frac{2}{\mu_0}[L^2+\tfrac{1}{2}+(L^2+\tfrac{1}{2})^{1/2}]. \qquad (4.25)$$

V. CONCLUSIONS

Typical forms of β functions have been established in some detail, as illustrated in Figs. 1–3. All these functions exhibit UV fixed points, which are responsible for the onset of related phase transitions [14]. Now we found it suitable to identify the renormalization scale with the x_0 parameter implied by Eq. (1.3). Indeed, one has a large number of exactly solvable potentials [32–34] which are not able to be treated properly by the $1/N$ method. Accordingly, Eq. (1.3) should be favored to the detriment of the x_0 choice based exclusively on the $1/N$ fixing of the parameter [35]. Exact and/or approximate formulas for the pertinent ground-state energies have also been established. In addition, the x_0 and d_0 parameters can be interpreted as nontrivial generalizations of the Bohr radius and of the principal quantum number, respectively. This also means that studying the β functions leads simultaneously to useful interrelated information about such generalizations. So far, the vertical asymptotes displayed in Figs. 2 and 3 have the role to restrict the number of fixed points by preserving the boundary conditions needed. It has also been proved that our parameters concerning the Dirac-Coulomb problem can be chosen in order to reproduce the UV fixed point, as well as the anomalous dimension of the mass operator, in ladder QED_4. Such agreements are able to support our emphasis on similarities and/or structural relationships between ladder QED_4 and the present RG approach to the Dirac-Coulomb system. Nevertheless, this does not rule out the existence of more complex RG structures in the case of QED. It is also clear that such intercorrelated studies can also be done for other systems.

Finally, it should also be mentioned that other self-similar descriptions, including the derivation of corresponding β functions, have also been proposed [36]. We argue, however, that our x_0 scheme should be favored on general grounds bearing on relevance and simplicity. However, there are systems which exhibit precisely an exact energy in terms of a superimposed mass scale, say $\mu_s(\alpha;k)$, depending on the couplings. Indeed, the two-body relativistic Hamiltonian (1.5) with a superposition between a Lorentz-vector Coulomb potential $U(x)=-\alpha/x$ and a Lorentz-scalar linear potential $S(x)=kx/2$ [23–25] is subject to the exact ground-state ($n_r=0$) energy [37]

$$\mathcal{E}_0=2k^{1/2}\left[\frac{1}{2}+\left[L^2-\frac{\alpha^2}{4}\right]^{1/2}\right]^{1/2}, \qquad (5.1)$$

provided that

$$\mu_0=\mu_s(\alpha;k)=\frac{\alpha}{2}k^{1/2}\left[\frac{1}{2}+\left[L^2-\frac{\alpha^2}{4}\right]^{1/2}\right]^{-1/2}, \qquad (5.2)$$

which plays the role of an actual mass-quantization condition. One proceeds by establishing the superpotential characterizing the Schrödigner-equivalent Hamiltonian. This indicates that one has actual grounds to interpret μ_s as a special renormalization scale. We then find the alternative β function

$$\beta_s(\alpha a)=\mu_s\frac{\partial}{\partial\mu_s}\alpha$$
$$=\alpha\left[L^2-\frac{\alpha^2}{4}\right]^{1/2}\left[\left[L^2-\frac{\alpha^2}{4}\right]^{1/2}+\frac{\alpha^2}{8b_0}\right]^{-1}, \qquad (5.3)$$

where $b_0=\frac{1}{2}+(L^2-\alpha^2/4)^{1/2}$, which produces the UV fixed points $\alpha=\pm 2L$ and the IR fixed point $\alpha=0$. Beta-functions depending on both α and k couplings can also be derived similarly as before [1], now by starting from the complex supercritical form of Eq. (5.2). In other words, one still has a number of open points which are worthy of being explored. We can then hope that such investigations may reveal new patterns towards a better understanding of the phase-transition attributes of many-body systems in terms of related one- or two-body quantum problems and conversely.

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[2] The quantum number of the angular momentum (radial quantum number) is given by $l=0,1,2,\ldots,$ $(n_r=0,1.2,\ldots,)$, as usual. The quantum number of the spin reads as $s=\pm 1$, so that $j=l+s/2$. For the sake of simplicity the rest-mass factor from the r.h.s. of Eq. (1.2) has been disregarded ($\mu_0=1$). As before, one has $x=|\mathbf{x}|$ and $p=|\mathbf{p}|$.

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