Canonical quantization of four- and five-dimensional U(1) gauge theories

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We discuss the canonical quantization of an interacting massless U(1) gauge field using a bosonic gauge-fixing method. We present a way to make the transformation between the Lorentz and the Coulomb gauge of such theories, without using an explicit representation of the fields in terms of creation-annihilation operators. We demonstrate this method in the case of Maxwell photons interacting with Schrödinger electrons and then we treat, with the same methods, a system of higher-dimensional equations appearing in the framework of a manifestly covariant relativistic quantum theory. The nonrelativistic limit of the Coulomb term for such a theory is discussed and compared to the Fokker action appearing in the Wheeler-Feynman action-at-a-distance theory for electromagnetic interactions.

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I. INTRODUCTION

The canonical quantization of a massless gauge vector field contains difficulties due to the fact that not all the components of the vector field correspond to real physical degrees of freedom of the system. Instead, the gauge freedom of the field implies that the Hilbert space of such a system admits an equivalence relation. In the canonical quantization process these problems appear in the fact that there are no canonically conjugate momenta for some of the field components. The usual way to deal with this problem is to add to the Lagrangian "gauge fixing" terms. The method of bosonic gauge fixing (BGF) has been applied by many authors [1–5] to quantum electrodynamics (QED). In addition, Haller and co-workers [1] constructed a unitary transformation which takes the Hamiltonian from Lorentz gauge to the form of Coulomb gauge, and they use this to explain the detailed structure of QED and related theories [1–4]. In his works, Haller uses an explicit representation of the fields A^μ and its conjugate momenta Π^μ in terms of creation and annihilation operators, which is somewhat complicated by the fact that the fields in such a theory satisfy the dipole-ghost equation [6], $\partial^\mu\partial_\mu\partial^\nu\partial_\nu A^\sigma=0$; the A^μ field, therefore, has no natural separation into negative and positive frequency parts. The Hamiltonian is, in this case, not diagonal in the photon-number representation; Haller has taken this into account by defining additional fields [1–4]. In this paper, we shall present a method to carry out the transformation of the Hamiltonian from the Lorentz gauge form to the Coulomb gauge form using only the algebraic properties of the fields, without recourse to their representation in terms of creation and annihilation operators. We then apply, in Sec. III, this method to a five-dimensional pre-Maxwell field theory [7–11] coupled to a spinless particle and discuss the details of the corresponding quantized theory. The relation between the quantized form of the higher-dimensional theory and that of the Maxwell theory, and to the Wheeler-Feynman theory are discussed in Sec. IV. The conclusion and further discussion are given in Sec. V.

II. CANONICAL QUANTIZATION OF THE INTERACTING MAXWELL FIELD

Let us begin with the BGF action for 3+1 Maxwell photons interacting with Schrödinger electrons [2],

$$S=\int_{-\infty}^{\infty} d^4x \left\{ -\tfrac{1}{4}F^{\mu\nu}(x)F_{\mu\nu}(x)-G(x)[\partial_\mu A^\mu(x)]+\tfrac{1}{2}(1-\gamma)G^2(x) + i\psi^\dagger(x)\frac{\partial\psi(x)}{\partial t}+\frac{1}{2m}\psi^\dagger(x)[\partial^i-ieA^i(x)][\partial_i-ieA_i(x)]\psi(x)+e\psi^\dagger(x)A_0(x)\psi(x)\right\} . \tag{2.1}$$

This action contains a gauge-fixing term $-G(x)[\partial_\mu A^\mu(x)]+\frac{1}{2}(1-\gamma)G^2(x)$. In a path-integral method, for example, it provides a Gaussian factor that approaches a "δ function" $\delta(\partial_\mu A^\mu(x))$ in the limit $\gamma\to 1$. For our purposes, we need this term in order to supply a canonically conjugate momentum to the A^0 field, which has no conjugate momentum in the naive Lagrangian. One can see that for this modified Lagrangian [we use the metric $(+---)$],

$$\pi^i=\frac{\delta\mathcal{L}}{\delta\partial_0 A_i}=-F^{0i} ,$$
$$\pi^0=\frac{\delta\mathcal{L}}{\delta\partial_0 A_0}=-G , \tag{2.2}$$
$$\pi_\psi=\frac{\delta\mathcal{L}}{\delta\partial_0\psi}=i\psi^\dagger ;$$

we can impose the canonical equal-time commutation relation (ETCR) for the fields,

$$[\pi^{\mu}(\mathbf{x}), A_{\nu}(\mathbf{y})] = -i\delta^{\mu}_{\nu}\delta(\mathbf{x}-\mathbf{y}) ,$$
$$[i\psi^{\dagger}(\mathbf{x}), \psi(\mathbf{y})] = -i\delta(\mathbf{x}-\mathbf{y}) \tag{2.3}$$

so that we can consider $\pi^{\mu}(\mathbf{x})$ as a translation generator for the wave functional which defined over the A space, i.e., $\pi^{\mu}(\mathbf{x}) = -i\delta/\delta A_{\mu}(\mathbf{x})$. The Hamiltonian, defined by Legendre transformation, is

$$H = \int_{-\infty}^{\infty} d^3\mathbf{x}\{\pi^i(\partial_0 A_i) + \pi^0(\partial_0 A_0) + i\psi^{\dagger}\partial_0\psi - \mathcal{L}\}$$
$$= H_{\gamma} + H_m + H_{\gamma m} , \tag{2.4}$$

where

$$H_{\gamma} = \int_{-\infty}^{\infty} d^3\mathbf{x}\{ -\tfrac{1}{2}\pi^i\pi_i + \tfrac{1}{4}F^{ij}F_{ij} + \pi^i(\partial_i A_0) - \pi^0(\partial_i A^i) - \tfrac{1}{2}(1-\gamma)\pi^0\pi_0\} ,$$

$$H_m = -\frac{1}{2m}\int_{-\infty}^{\infty} d^3\mathbf{x}\psi^{\dagger}\partial^i\partial_i\psi ,$$

and

$$H_{\gamma m} = \int_{-\infty}^{\infty} d^3\mathbf{x}\left\{ -e\psi^{\dagger}A_0\psi + \frac{ie}{2m}\psi^{\dagger}[2A^i\partial_i + (\partial_i A^i)]\psi - \frac{e^2}{2m}\psi^{\dagger}\psi A^i A_i \right\} . \tag{2.5}$$

Of course, the fictitious field π^0 (or G) must have a zero expectation value in the physical Hilbert space, i.e., we shall take at the end of the calculation, $\langle \nu|\pi^0|\nu'\rangle = 0$, where $\{|\nu\rangle, |\nu'\rangle \in \mathcal{H}_{\text{phys}}\}$. This condition is easy to satisfy because (as one can see from the Euler-Lagrange equations of this action) G is a free field which satisfies the massless Klein-Gordon equation. It therefore admits a natural decomposition into negative and positive frequencies, and the physical Hilbert space is defined by the subsidiary condition $G^{(+)}|\nu\rangle = 0$. In order to stabilize this condition, i.e., to ensure that the S matrix does not mix the physical Hilbert space with the nonphysical components, one must satisfy the condition,

$$\dot{G}^{(+)}|\nu\rangle = 0 \tag{2.6}$$

or

$$[H, \pi^0]^{(+)}|\nu\rangle = i(\partial_i\pi^i + J^0)^{(+)}|\nu\rangle = 0 , \tag{2.7}$$

where $J^0(\mathbf{x}) \equiv e\psi^{\dagger}(\mathbf{x})\psi(\mathbf{x})$. It follows from (2.6) that $\langle \nu|\partial_i\pi^i + J^0|\nu'\rangle = 0$. Clearly, this condition is exactly the Gauss law, satisfied in this quantized electromagnetism in terms of matrix elements in physical states and not as an operator identity. The stability of the Gauss law itself is, in fact, self-consistent with the previous assumptions, i.e.,

$$[H, \partial_i\pi^i] = i(\partial^i\partial_i\pi^0 + \partial_i J^i) \tag{2.8}$$

where J^i is the spatial part of the conserved (Noether) current, i.e.,

$$J^i \equiv -\frac{ie}{2m}\{\psi^{\dagger}(\mathbf{x})[(\partial^i - ieA^i)\psi(\mathbf{x})] - [(\partial^i - ieA^i)\psi(\mathbf{x})]^{\dagger}\psi(\mathbf{x})\} . \tag{2.9}$$

Using the continuity equation for the current, one finds that

$$i\partial_i J^i = -i\partial_0 J^0 = -i[H, J^0] \tag{2.10}$$

so that the expectation value of the commutator of the Hamiltonian with the Gauss law operator vanishes in the physical Hilbert space in which $\langle \pi^0 \rangle = 0$.

For what follows, let us separate the spatial part of A into longitudinal and transverse parts,

$$\mathbf{A} = \mathbf{A}_{\perp} + \mathbf{A}_{\parallel} = \mathbf{A}_{\perp} + \nabla\Lambda(\mathbf{x}) \tag{2.11}$$

where $\Lambda(\mathbf{x})$ is a scalar operator valued function. We can now define a unitary operator transformation which makes the wave functional Φ independent of $A_{\parallel}$,

$$\Phi(A_{\perp}, A_0, \psi) = e^{i\chi}\Phi(A_{\parallel}, A_{\perp}, A_0, \psi) , \tag{2.12}$$

that is,

$$e^{i\chi} = \exp\left[i\int d^3\mathbf{x}\, \Lambda(x)[\partial_i\pi^i(\mathbf{x}) + J^0(\mathbf{x})] \right] , \tag{2.13}$$

where π^i, J^0 are operator valued [and here $\Lambda(x)$ is the c-number field which is the spectrum of $\Lambda(\mathbf{x})$, i.e., one should consider the representation of Φ in the basis which diagonalizes the *operator* $\Lambda(\mathbf{x})$]. One finds that this transformation corresponds to

$$\mathbf{A} \to \mathbf{A}_{\perp} , \quad A_0 \to A_0 , \quad \mathbf{J} \to \mathbf{J} + F(M_0)\mathbf{A}_{\parallel} ,$$
$$J^0 \to J^0 , \quad \pi^{\mu} \to \pi^{\mu} , \tag{2.14}$$

and it commutes with the Hamiltonian in the physical subspace. $F(M_0)$ is part of the Schwinger term from the current commutation relations. It arises from the commutator of the time component of the current in the generator χ with the space component of the current in the Hamiltonian and has the form

$$[J_k(\mathbf{x}), J_0(\mathbf{y})] = i\partial_k\delta(\mathbf{x}-\mathbf{y})F(M_0) , \tag{2.15}$$

where

$$F(M_0) = \int_{M_0}^{\infty} dM^2 \frac{p(m^2)}{M^2} \tag{2.16}$$

and M_0 is the infrared cutoff [12]. The part of the transformed current proportional to $\mathbf{A}_{\parallel}$ does not contribute to the Hamiltonian, since it occurs in scalar product with $\mathbf{A}$, which has been truncated to $A_{\perp}$.[1]

The new Hamiltonian then takes the form

$$\tilde{H} = H_{\perp} - H_c = H_{\perp}(A_{\perp}, A_0, J_{\perp}) - \tfrac{1}{2}\int d^3\mathbf{x}\, \pi^i_{\parallel}\pi_{\parallel i} , \tag{2.17}$$

where

$$H_{\perp} = \int_{-\infty}^{\infty} d^3\mathbf{x}\left\{ -\tfrac{1}{2}\pi^i_{\perp}\pi_{\perp i} + \tfrac{1}{4}F^{ij}F_{ij} - A_0(\partial_i\pi^i + J^0) - \tfrac{1}{2}(1-\gamma)\pi^0\pi_0 - \frac{1}{2m}\psi^{\dagger}\partial^i\partial_i\psi - J_{\perp}A_{\perp} - \frac{e^2}{2m}\psi^{\dagger}\psi A^i_{\perp}A_{\perp i} \right\} , \tag{2.18}$$

[1]The proof of the stability of the Gauss law [Eqs. (2.8)–(2.10)] is not effected by the introduction of Schwinger terms, since it is based on the current continuity equation which is necessarily compatible with their existence. In fact, one assumes the current continuity in order to derive the Schwinger terms [12].

where the last term is the seagull term. This form of the Hamiltonian is stable in the physical subspace. Since A_0 commutes with

$$\dot{G}'=e^{i\chi}\dot{G}e^{-i\chi}=\dot{G}=\partial_i\pi^i+J^0$$

the condition

$$\dot{G}'^{(+)}|\tilde{v}\rangle=0 ,$$

where $|\tilde{v}\rangle=e^{i\chi}|v\rangle$ ensures that there is no contribution to the matrix elements of $H_\perp$ from the term $A_0(\partial_i\pi^i+J^0)$ in the physical subspace. One can verify also that after the transformation, A_0 commutes with the Hamiltonian in the physical subspace. Furthermore, all matrix elements of π^0 vanish in physical states, so that π^0 is not an observable; hence A_0 commutes with all observables it is therefore a c number, which we shall take to be zero. The Lorentz gauge condition ($A_0=A_\parallel$) is then satisfied in the physical subspace as well.

We can see also that the expectation value of the part H_c in the physical Hilbert space is exactly the Coulomb interaction term. In order to do that, let us define $G(\mathbf{x}-\mathbf{y})$ as the Green's function of the operator ∇^2. Then (using the Gauss law),

$$\begin{aligned}\langle H_c\rangle&=\left\langle-\tfrac{1}{2}\int d^3\mathbf{x}\,\pi_\parallel(\mathbf{x})\pi^\parallel(\mathbf{x})\right\rangle\\&=\left\langle-\tfrac{1}{2}\int d^3\mathbf{x}\,d^3\mathbf{y}\,\pi_\parallel(\mathbf{x})\nabla^2G(\mathbf{x}-\mathbf{y})\pi^\parallel(\mathbf{y})\right\rangle\\&=\tfrac{1}{2}\int d^3\mathbf{x}\,d^3\mathbf{y}\,\langle J^0(\mathbf{x})G(\mathbf{x}-\mathbf{y})J^0(\mathbf{y})\rangle+\text{const}\ .\end{aligned} \tag{2.19}$$

The correlation terms in H_c give just a constant expectation value in the physical Hilbert space under the assumption that the equal-time commutator of the positive frequency part of $\nabla\cdot\pi$ and its negative frequency part is a c number; this, in turn, is a necessary condition for the definition of the asymptotic states in terms of free fields. Since $\dot{G}^{(+)}|\tilde{v}\rangle=0$, $\langle\tilde{v}|\dot{G}(\mathbf{x})\dot{G}(\mathbf{y})|\tilde{v}'\rangle=0$. It then follows, from this condition, that $\langle\tilde{v}|\nabla\pi(\mathbf{x})\nabla\pi(\mathbf{y})|\tilde{v}'\rangle=\langle\tilde{v}|J^0(\mathbf{x})J^0(\mathbf{y})|\tilde{v}'\rangle+\text{const}$. This Coulomb term comes from the longitudinal part of π; there is no dependence of the Hamiltonian on the momentum conjugate to $\pi_\parallel$ and, hence, it does not occur as a dynamical variable. The transformation which eliminates the longitudinal part of the electromagnetic field effectively replaces $\pi_\parallel$ by the charge-density operator, eliminating reference to those independent-field degrees of freedom. We conclude that by this unitary transformation we have shown that the Lorentz gauge Hamiltonian for such a theory is, in any specific inertial frame, a sum of the Coulomb gauge Hamiltonian and a part of the operator with zero expectation value in the physical Hilbert space. Thus, this physical Hilbert space is stable under the time evolution of the system and the theory is well defined in both gauges.

III. CANONICAL QUANTIZATION OF THE HIGHER-DIMENSIONAL THEORY

Let us consider now the five-dimensional theory of electromagnetism defined by Saad, Horwitz, and Arshansky [11] in order to create a theory which is gauge covariant and provides integrable equation of motion[2] in the framework of the relativistic quantum mechanics discussed, for example, in Refs. [7,15,16].

In this theory the system develops on the four-dimensional manifold $\mathbf{x},t$ according to an evolution parameter ("universal time") τ. Since the Schrödinger type of equation for the evolution of the states contains a derivative with respect to τ (first order) as well as derivatives with respect to $\mathbf{x},t$, the gauge compensation fields are five dimensional. The signature of the five-dimensional metric may be (3,2) or (4,1). We shall adopt the notation where the indices α, β, etc., run over the five components $\mathbf{x},t,\tau$, and where the indices μ,ν, etc. run over the four space-time components only. We take the metric for this theory to be $[\sigma+---]$, where σ, the $g^{\tau\tau}$ term, is $+$ for the O(3,2) theory and $-$ for the O(4,1) theory.

The action for either signature is defined as,

$$S=\int_{-\infty}^{\infty}d^5x\left\{-\frac{\lambda}{4}f^{\alpha\beta}(x)f_{\alpha\beta}(x)-G(x)[\partial_\alpha a^\alpha(x)]+\frac{1}{2\lambda}G^2(x)\right.$$
$$\left.+i\psi^\dagger(x)\frac{\partial\psi(x)}{\partial\tau}-\frac{1}{2M}\psi^\dagger(x)[\partial^\mu-ie'a^\mu(x)][\partial_\mu-ie'a_\mu(x)]\psi(x)+e'\psi^\dagger(x)a_\tau(x)\psi(x)\right\} , \tag{3.1}$$

where λ is a quantity with dimension of length (as we explain later, this quantity corresponds to the τ-correlation length of the wave function in the Maxwell limit). The constant e' is the coupling constant of the theory, which also has dimensions of length. The canonically conjugate momenta are

$$\begin{aligned}\pi^\mu&=\frac{\delta\mathcal{L}}{\delta(\partial_\tau a_\mu)}=-\lambda f^{\tau\mu}\ ,\\\pi^\tau&=\frac{\delta\mathcal{L}}{\delta(\partial_\tau a_\tau)}=-\sigma G\ ,\\\pi_\psi&=\frac{\delta\mathcal{L}}{\delta(\partial_\tau\psi)}=i\psi^\dagger\ ,\end{aligned} \tag{3.2}$$

where a_τ is the fifth component of the five-vector a_α and $f^{\tau\mu}$ is the antisymmetric form $\partial^\tau a^\mu-\partial^\mu a^\tau$. We now impose equal τ commutation relations (EτCR)

[2] F. Rohrlich [13], has pointed out that the N-body problem for $N\geq2$ is intrinsically unstable in the Maxwell theory. This follows from the fact that the conserved currents are defined by integrals over the world lines. A problem with radiation reaction must therefore be treated as a self-consistency problem. The problem of the "runaway electron" [13,14] seems to as to be of the same type, where the lowest-order approximation is not stable.

$$[\pi^\alpha(x), a_\beta(y)] = -i\delta^\alpha_\beta \delta(x-y) ,$$
$$[i\psi^\dagger(x), \psi(y)] = -i\delta(x-y) . \quad (3.3)$$

The Hamiltonian (the τ translation generator K) takes the form

$$K = \sigma \int_{-\infty}^{\infty} d^4x \{ \pi^\mu(\partial_\tau a_\mu) + \pi^\tau(\partial_\tau a_\tau) + i\psi^\dagger \partial_\tau \psi - \mathcal{L} \}$$
$$= K_\gamma + K_m + K_{\gamma m} , \quad (3.4)$$

where

$$K_\gamma = \int_{-\infty}^{\infty} d^4x \left\{ -\frac{1}{2\lambda}\pi^\mu \pi_\mu - \frac{\lambda\sigma}{4} f^{\mu\nu} f_{\mu\nu} + \pi^\mu(\partial_\mu a^\tau) - \pi^\tau(\partial_\mu a^\mu) - \frac{1}{2\lambda}\pi^\tau \pi_\tau \right\} ,$$

$$K_m = \frac{\sigma}{2M} \int_{-\infty}^{\infty} d^4x \, \psi^\dagger \partial^\mu \partial_\mu \psi ,$$

and

$$K_{\gamma m} = \sigma \int_{-\infty}^{\infty} d^4x \left\{ -e'\psi^\dagger a_\tau \psi - \frac{ie'}{2M}\psi^\dagger [2a^\mu \partial_\mu + (\partial_\mu a^\mu)]\psi - \frac{e'^2}{2M}\psi^\dagger \psi a^\mu a_\mu \right\} . \quad (3.5)$$

The subsidiary condition (the stability condition for the restriction $\langle \pi^\tau \rangle = 0$ on the physical Hilbert space) is now

$$\langle \partial_\mu \pi^\mu + j^\tau \rangle = 0 \quad (3.6)$$

which is the new "Gauss law." It is obvious that one can eliminate the longitudinal part of the field a^μ (the part parallel to k^μ) by the same procedure defined above. It is convenient to discuss three cases:

Case 1. The four-vector k^μ is timelike, for which the (4,1) theory is the stable solution. In this case, if one boosts the system into a frame in which k^μ is (1,0,0,0), one sees that he can eliminate, by a unitary transformation (as in the Maxwell case), the *time* component of a^μ. The remaining degrees of freedom contain, except for the Coulomb term, three spacelike polarization components a^i, and the Hilbert space has positive norm.

Case 2. The four-vector k^μ is spacelike, for which the (3,2) theory is the stable solution. By the same method used for the Maxwell field, one can eliminate, in this case, a space component of a^μ (the part parallel to k^μ); in the frame for which $k^0 = 0$. This leaves three degrees of freedom, which now transform under O(2,1). The Fock space for the one-photon polarization states for given k^μ is three dimensional, but must transform under O(2,1); the representation is therefore nonunitary. The indefinite metric preserves the norm under such transformations. We may therefore choose a sector of the polarization space with a non-negative norm; clearly this sector is preserved under the O(2,1) subgroup of O(3,1). The action of O(3,1) is represented by the polarization states as an *induced* representation, for which the O(2,1) little group is the stabilizer of k^μ in O(3,1). Transformations of O(3,1) therefore also preserve the non-negative norm sector. Furthermore, since the dynamical evolution operator K is Lorentz invariant, it is also invariant under O(2,1). It therefore commutes with the O(2,1) Casimir operator $N = -M_{01}^2 - M_{02}^2 + M_{12}^2$, which is then a constant of the motion. The expectation value of the Casimir operator N is states of positive norm is positive and in states of negative norm, it is negative. Since these signs are preserved under the evolution as well as under the action of O(3,1), the space of polarization states with negative norm states removed is a stable invariant subspace. The zero norm states are treated in case 3.

Case 3. The four-vector k^μ is lightlike. The elimination of the longitudinal modes corresponds exactly to the elimination of both a_0 and $a_\parallel$ and there are then only two physical degrees of freedom, as in the usual (on-shell) Maxwell theory.

The "Coulomb" term of these theories is (in close analog to the Maxwell case),

$$\langle K_c \rangle = \left\langle \frac{-1}{2\lambda} \int d^4x \, d^4y \, \partial_x^\mu \pi_\mu(x) \, G(x-y) \, \partial_y^\nu \pi_\nu(y) \right\rangle$$
$$= \frac{e'^2}{2\lambda} \int d^4x \, d^4y \langle \rho^\tau(x) G(x-y) \rho^\tau(y) \rangle + \text{const} , \quad (3.7)$$

where $\rho \equiv \psi_\tau^\dagger(x)\psi_\tau(x)$ [we use the notation $\psi_\tau(x)$ with the implication that ψ has a measure on the four-dimensional manifold $(\mathbf{x}, t)$ only] and $G(x-y)$ is a Green's function of the d'Alembertian operator $\partial^\mu \partial_\mu G(x-y) = \delta(x-y)$. It is natural, for this theory, to take this Green's-function symmetric between past and future so we may take it as half the sum of the advanced and retarded Green's functions.

IV. RELATION BETWEEN THE FIVE-DIMENSIONAL AND THE MAXWELL THEORY: THE MASS-SHELL LIMIT

To see how the term (3.7) can be put into correspondence with some applications of the electrodynamic theory formulated as an "action-at-a-distance" theory, such as the Fokker action appearing in the Wheeler-Feynman treatment [17], we first remark that the interaction with the apparatus, or some *macroscopically* defined object, corresponds to an interaction with a *worldline* [18]. The correlation in τ between the microscopic system and the macroscopic may be destroyed by the structure of the macroscopic object. For example, the conserved current associated with the Steueckelberg action (3.1) satisfies,

$$\partial_\mu j_\tau^\mu(x) + \partial_\tau j^\tau(x) = 0 , \quad (4.1)$$

where

$$j_\tau^\mu(x) = \frac{e'}{2Mi} \{ \psi_\tau^\dagger(x)[(\partial^\mu - ie'a^\mu)\psi_\tau(x)] - [(\partial^\mu - ie'a^\mu)\psi_\tau(x)]^\dagger \psi_\tau(x) \} . \quad (4.2)$$

Integrating over all τ, and assuming that the limit of $j^\tau(x)$ as $\tau \to \pm\infty$, is pointwise zero [8,15], one obtains, from (4.1),

$$\partial_\mu J^\mu(x) = 0 , \quad (4.3)$$

where

$$J^{\mu}(x)=\frac{1}{\lambda}\int_{-\infty}^{\infty}d\tau\, j_{\tau}^{\mu}(x) \tag{4.4}$$

is a four-dimensional conserved current which is identified with the macroscopic Maxwell current, in agreement with standard covariant classical formula [19]. Note that j_{τ}^{μ} is proportional to e'; defining the dimensionless charge $e\equiv e'/\lambda$ we see that J^{μ} is proportional to e, which we then identify as the Maxwell electric charge. This relation follows in the classical theory [11] from the equation of motion

$$\lambda\partial_{\alpha}f^{\alpha\beta}=j^{\beta} . \tag{4.5}$$

Consider the $\beta=\mu$ components; (4.5) then becomes

$$\partial_{\tau}f^{\mu\tau}+\partial_{\nu}f^{\mu\nu}=\frac{1}{\lambda}j^{\mu} . \tag{4.6}$$

Integrating over τ from $-\infty$ to ∞, and assuming that $f^{\mu\nu}(x)\rightarrow 0$ for $\tau\rightarrow\pm\infty$ one obtains

$$\partial_{\nu}F^{\mu\nu}=J^{\mu} , \tag{4.7}$$

where

$$F^{\mu\nu}=\int d\tau\, f^{\mu\nu} , \tag{4.8}$$

and, hence, we infer that the Maxwell potential is given by

$$A^{\mu}=\int d\tau\, a_{\tau}^{\mu}(x) . \tag{4.9}$$

We may understand the construction of a particle with a well-defined mass (on mass shell), as a wave packet $\psi_{\tau}(x)$ for which the Fourier transform $\psi_{\sigma}(x)$ of the wave function $\psi_{\tau}(x)$ is sharply peaked about some definite σ, where $\sigma=m^2/2M$, corresponding to the Klein-Gordon mass squared. On the other hand, the zero-mass photons are recognized as a states of the field $a_{\sigma}^{\mu}(x)$ for which the Fourier transform has support near $\sigma=0$. We shall call this limit the mass-shell limit of the theory.

Let us take, for example, the interaction term in the covariant theory

$$\int d\tau\, d^4x\, j_{\tau}^{\alpha}(x)a_{\alpha,\tau}(x)=\int ds\, d^4x\, j_{s}^{\alpha}(x)a_{\alpha,-s}(x) . \tag{4.10}$$

The condition for the mass-shell limit is

$$\int ds\, d^4x\, j_{s}^{\alpha}(x)a_{\alpha,-s}(x)\sim\frac{1}{\lambda}j_{0}^{\alpha}(x)a_{\alpha,0} , \tag{4.11}$$

where $1/\lambda$ is the width of the mass distribution in the field $a_{\alpha,s}$ around zero. We therefore obtain

$$\int d\tau\, d^4x\, j_{\tau}^{\alpha}(x)a_{\alpha,\tau}(x)\sim\frac{1}{\lambda}\int d\tau\, d\tau' d^4x\, j_{\tau}^{\alpha}(x)a_{\alpha,\tau'}(x) . \tag{4.12}$$

Using (4.4), (4.9), and the definition of the dimensionless charge, we get in the mass-shell limit from (4.10)

$$\int d^4x\, J^{\mu}(x)A_{\mu}(x) . \tag{4.13}$$

To understand this result physically, we remark that if $a_{\alpha,s}$ has support only in a $\Delta s\sim 1/\lambda$ neighborhood of $s=0$, then $j_{s}^{\alpha}(x)$ contributes to the action only in this neighborhood as well. If we define [20]

$$\psi_{\tau}(x)=\int_{-\infty}^{\infty}\frac{d\sigma}{\sqrt{2\pi}}e^{-i\sigma\tau}\psi_{\sigma}(x) ,$$

then the current has the form

$$j_{\tau}^{\mu}(x)=\frac{-ie'}{2M}\int_{-\infty}^{\infty}\frac{d\sigma}{\sqrt{2\pi}}\frac{d\sigma'}{\sqrt{2\pi}}e^{i(\sigma-\sigma')\tau}\times\left\{\psi_{\sigma}^{\dagger}(x)[\partial^{\mu}\psi_{\sigma'}(x)]-[\partial^{\mu}\psi_{\sigma}^{\dagger}(x)]\psi_{\sigma'}(x)\right\} , \tag{4.14}$$

and therefore, if the matter field has support in the $1/\lambda$ neighborhood of the mass-shell, then the photon coupling to it is also in the $1/\lambda$ neighborhood of zero mass. Conversely, the restriction of s to a $1/\lambda$ neighborhood of zero implies that only equal mass components (as for a Lehmann distribution) in the wave function contribute to the interaction.

The Lagrangian of the usual Maxwell theory may be cast into the form of that of the pre-Maxwell theory following this idea. In fact,

$$J^{\mu}A_{\mu}=\frac{1}{\lambda}\int d\tau\, j_{\tau}^{\mu}\int d\tau'\, a_{\mu,\tau'} , \tag{4.15}$$

and if we assume that j_{τ}^{μ} and $a_{\mu,\tau}$ are uncorrelated for $|\tau-\tau'|\geq\lambda$, then (in expectation value)

$$J^{\mu}A_{\mu}\sim\int d\tau\, j_{\tau}^{\mu}a_{\mu,\tau} . \tag{4.16}$$

These arguments provide an interpretation for the scale λ which relates the dimensional charge of the pre-Maxwell theory and the dimensionless Maxwell charge.

If we take, then, the Coulomb term (3.7) in the Hamiltonian back to the action by integration over τ, we find that it contribute in the form

$$\frac{-e'^2}{2\lambda}\int d\tau\, d^4x\, d^4y\,\langle\rho^{\tau}(x)G(x-y)\rho^{\tau}(y)\rangle \tag{4.17}$$

and the mass-shell limit of this term is

$$\frac{-e^2}{2}\int d\tau_1 d\tau_2 d^4x\, d^4y\,\langle\rho^{\tau_1}(x)G(x-y)\rho^{\tau_2}(y)\rangle . \tag{4.18}$$

In the classical limit,

$$\rho^{\tau}(x)\sim\sum_i\delta^4(x-x_i(\tau)) , \tag{4.19}$$

and (4.18) becomes

$$-\frac{e^2}{2}\sum_{i,j}\int d\tau_1 d\tau_2\delta([x_i(\tau_1)-x_j(\tau_2)]^2) . \tag{4.20}$$

The classical limit of the vector current density takes the form

$$j^{\mu\tau}(x)\sim e\sum_i\frac{dx_i^{\mu}(\tau)}{d\tau}\delta^4(x-x_i(\tau)) . \tag{4.21}$$

Let us consider a theory in which instead of the scalar charge densities of (4.18), we use the vector densities of (4.21), which carry the sign of the flow of events in space time, thus taking into account explicitly the distinction between particles and antiparticles. We then obtain

$$\frac{e^2}{2}\sum_{i,j}\int d\tau_1 d\tau_2 d^4x\, d^4y\, \frac{dx_i^{\mu}(\tau_1)}{d\tau_1}\frac{dx_{\mu j}(\tau_2)}{d\tau_2}\delta^4(x-x_i(\tau_1)) \times\delta^4(y-x_j(\tau_2))\delta((x-y)^2)\ , \tag{4.22}$$

where we have chosen the half-advanced, half-retarded Green's function. The terms for which $i=j$ do not affect the dynamical evolution of particles with nonzero mass since $\delta[(x-y)^2]$ has support only on the light cone. The result

$$\frac{e^2}{2}\sum_{i\neq j}\int dx_i^{\mu}dx_{\mu j}\delta((x_i-x_j)^2) \tag{4.23}$$

is the Fokker action term [16]. In the frame in which both particles velocities are *small*, $dx_i^{\mu}dx_{j,\mu}$ approximated (on shell) by $d\tau_i d\tau_j$, and one recovered (4.20) to second order. The Fokker action (4.23) can therefore be understood as on-mass shell covariant form of (4.20) when the relative motion of the particles is small. In fact, the nonrelativistic limit for the particle motion,

$$\frac{dx_i^{\mu}}{d\tau}\simeq\delta_{\mu,0}\ , \tag{4.24}$$

both theories coincide; only the scalar part of the vector field of the Wheeler-Feynman theory contributes [as for the scalar field in (4.18)], and the propagator goes to the Coulomb form $1/R$ where $R=|\mathbf{x}_1-\mathbf{x}_2|$.

Let us discuss now how the distinction between particles and antiparticles manifests itself in the mass-shell limit (4.18) of the pre-Maxwell theory. Antiparticles are events which flow in time from the future to the past, i.e., they propagate in the negative direction of the time as τ increases. We know from quantum field theory that such a particle (e.g., the positron), carry an opposite charge and then, in the static (Coulomb) limit, they attract the forward going (in time) particles such as the electrons. On the other hand, the expression (3.7) seems not to distinguish between these two cases.

To resolve this problem, let us consider the semistatic collision of two charged particles described by Fig. 1. In the mass-shell (decorrelation) limit, the action for this situation is given approximately by the expression (4.20). We can treat, then, each of the particles separately, assuming a knowledge of the other particle world line. For each point on the particle (a) world line, we get, according to (4.20), two contributions from the points on the world line of (b), lying in the light cone of this point. One can see that the contribution to the action, which is proportional, in that limit, to $1/R$, is greater in its absolute value at $x'(\tau')$ than at $x(\tau)$, i.e., the particle tends to move as τ increases from the space-time point $x'(\tau')$ to the point $x(\tau)$. The time, $t(\tau)$, is measured by the laboratory clock when the laboratory records the signal at t emitted by the particle at (universal time) τ; so that, if the particle (a) has positive energy, t increases with τ, and one sees the particle go from x to x'; the process has the form of Fig. 1. On the other hand, if the particle (a) has negative energy, we see it in the laboratory going from x' to x, and the whole process has the form of Fig. 2. We, therefore, see that the direction of evolution along the world line, combine with formula for the energy of the system, provides a dynamical framework in which the charge of the particle becomes evident.

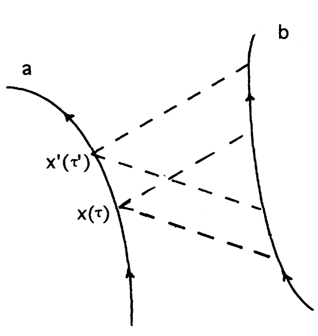

FIG. 1. World lines for particle-particle interaction.

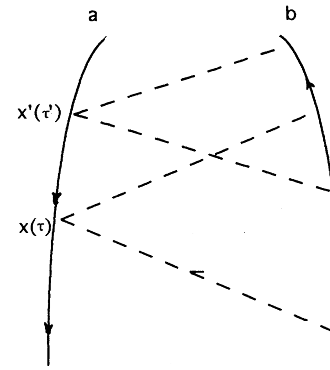

FIG. 2. World lines for particle-antiparticle interaction.

V. CONCLUSIONS AND DISCUSSION

We have shown that canonical quantization of the Maxwell field in interaction with the Schrödinger electrons can be carried out in the algebraic framework of the quantum fields. The results are in precise correspondence with those of Haller and co-workers [2], who introduced an explicit representation of the operators in terms of annihilation creation operators in the momentum representation for the fields. The algebraic method was extended to treat the five-dimensional field necessary for the consistent treatment of the interacting covariant dynamic of quantized Stueckelberg-type fields.

The coupling constant for the higher-dimensional theory has the dimension of length L, and the potentials are of dimension L^{-2}. The kinetic terms in the Lagrangian carry a scale which is consistent with the classical relation between the higher dimensional and the Maxwell field obtained by integration of the linear field equations over τ. In the quantized form of the theory, the scale can be related to a correlation length in the structure of the Maxwell field. We find that, in terms of this interpretation, the Maxwell theory emerges from the higher-dimensional theory when the higher-dimensional fields and currents are incoherent beyond the correlation length λ. The relation between the pre-Maxwell and Maxwell theories can therefore be thought of as follows:

An examination of the action of the higher-dimensional theory implies a relation to the Maxwell theory which is controlled by a correlation length related to the effective mass width of the off-shell photon field. The pre-Maxwell theory contains two-dimensional parameters, a coupling constant for the potential, and a scale parameter for the field strength. The equations of motion for the field strengths (4.5), imply, due to the relations (4.4) and (4.9), that $e'/\lambda=e$, the dimensionless Maxwell charge. The Ward identities, implied by gauge invariance, of the pre-Maxwell as well as for the Maxwell theory, imply a consistent universal renormalization for e', λ, and e.

The Maxwell Lagrangian and, hence, the quantitative predictions of the Maxwell electrodynamics, can coincide with those of the pre-Maxwell theory when, as we have seen, the latter becomes incoherent over a world time scale of the order of λ (in fact, the renormalized value of λ, which should be consistent with the effective decoherence of the theory on a larger scale), coinciding with the measure of the mean fluctuations of the radiation field from its classical (zero) mass-shell value. In the decoherence, or mass-shell limit, the third component of the polarization disappeared, and only the usual transverse polarization of the Maxwell field remain.

One can understand the physical meaning of the coherence length λ as the length for which two relatively timelike, or spacelike, events can exchange an off-shell photon [21]; such an off-shell photon should satisfy $\Delta m \Delta\tau \geq 1$ where $\delta m \sim 1/\lambda$. For such a case, the Coulomb term (3.7) gives no contribution to the Hamiltonian because it contains $\delta((x-y)^2)$ which is nonvanishing only when the two particles are on their relative light cone of each other, and the whole system can be considered as an extended object, free of the self energy divergences of the Coulomb term. On the other hand, when the separation between the particles become larger than λ, the motion may become uncorrelated, and the equal τ lines may then cross the light cone many times, so that the Coulomb term (4.17) contributes. Under these conditions, the contribution to the action

$$e' \int d\tau\, d^4x\, \tilde{j}^{\alpha}_{\tau}(x) a_{\alpha,\tau}(x) \sim e \int d^4x\, \tilde{J}^{\mu}(x) A_{\mu}(x) ,$$

where $\tilde{j}^{\alpha}, \tilde{J}^{\mu}$ defined as the quantum current density, i.e., $e'\tilde{j}^{\alpha}\equiv j^{\alpha}$, $e\tilde{J}^{\mu}\equiv J^{\mu}$, and $J^{\mu}(x), A_{\mu}(x)$ are the usual Maxwell currents and fields [we do not, in this discussion, distinguish the e' parameter occurring in the internal structure of $j^{\mu}_{\tau}(x)$ to ensure gauge invariance, but assume its effective replacement by e in this limit as well; the mechanism, due to the bilinear form of the current, is precisely the same].

With the formation of this decoherence mechanism, the Coulomb term (3.7) of the Hamiltonian goes over to the form (4.20) which is similar to the Fokker action studied by Wheeler and Feynman [17]. The Coulomb term corresponds (in the nonrelativistic approximation) to the dynamics of classical electromagnetic theory; quantum radiative effects are accounted for by the physical polarization degrees of freedom of the quantized field.

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