Scattering on two solenoids

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Starting from the previously derived scattering matrix [P. Šťovíček, Phys. Lett. A **161**, 13 (1991)] the differential cross section is calculated for electrons scattering on two infinitely thin parallel solenoids. The magnetic fluxes are arbitrary. The wave vector is assumed to lie in the asymptotic domain $k\rho \gg 1$ with ρ being the distance of the solenoids.

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The Aharonov-Bohm effect has attracted a great deal of interest beginning with the original paper [1] and continuing up to now. Scattering on one solenoid (in the idealized setup) has been investigated theoretically already in [1]. But the situation changes drastically for two and more solenoids as the rotational symmetry is lost. This fact makes any computation much more difficult. The present paper aims to fill in this void and presents a formula for the differential cross section for plane-wave scattering on two solenoids. This is done with the assumption that $k\rho \gg 1$, where k is the length of the wave vector and ρ is the distance between the solenoids. In this way the paper extends the results from [2] where a formula for the S matrix has been obtained. However, as explained below, unfortunately it is only of theoretical importance. Here we are going to derive a simplified formula, more appropriate for numerical evaluation. As one can intuitively expect, scattering on two targets should exhibit some sort of interference. The numerical results confirm this assumption. It is worth noting that an AB effect is presented also for two equal but opposite fluxes. This is the limiting case for the two-cylinder problem which has been suggested and treated, at least on the qualitative level, very early [3]. This situation is physically more consistent since the total flux is zero. It turns out that this is the only case when the differential cross section does not diverge for forward scattering.

The two-solenoid AB effect is considered here in the idealized setup, i.e., the solenoids are assumed to be infinite, infinitely thin, and parallel. As usual, m, e, and E designate the mass, the elelctric charge, and the energy of the scatering electron, respectively, and we set $k = (2mE/\hbar^2)^{1/2}$. Denote by $\alpha, \beta \in [0,1)$ are those numbers for which $\exp(2\pi i\alpha) = \exp(-ie\Phi_A/\hbar c)$ and similarly for β and Φ_B, where Φ_A and Φ_B are the magnitudes of the corresponding two magnetic fluxes. The geometry is arranged in such a way that the solenoids are parallel with the z-coordinate axis and intersect the x-coordinate axis in the points $(a,0,0)$ and $(b,0,0)$, respectively, $\rho = b-a$. But clearly, the problem can be reduced to a two-dimensional one and so we are going to consider the scattering in the plane. The wave vectors for the incoming and outcoming particle are $\mathbf{k}_0 = k(\cos\theta_0, \sin\theta_0)$ and $\mathbf{k} = k(\cos\theta, \sin\theta)$, respectively. Besides, we assume that the incoming plane wave is entering from the upper half plane and so $\theta_0 \in (\pi, 2\pi)$.

In the paper [2] the scattering matrix $S(\theta,\theta_0)$ has been presented in a suitable gauge. The choice of the gauge was discussed in the same paper, but one can consult also [4–6]. Here we recall briefly that usually one works with the vector potential $\mathbf{A} = (\hbar c/e)\,\mathrm{grad}(\alpha\phi_A + \beta\phi_B)$, with ϕ_A (ϕ_B) being the angle for the polar coordinates centered in the first (second) target. The Hamiltonian is given by $\hat{H}' = -(\hbar^2/2m)[\boldsymbol{\nabla} + i(\hbar c/e)\mathbf{A}]^2$. In our gauge, we first cut the plane along the first coordinate axis and then transform off the potential $\mathbf{A}$ with the help of the unitary mapping $U = U_\alpha U_\beta$, $U\psi = \exp(i\alpha\phi_A)\exp(i\beta\phi_B)\psi$. The gauge-transformed Hamiltonian $\hat{H} = U\hat{H}'U^{-1} = -(\hbar^2/2m)\Delta$ does not contain the potential $\mathbf{A}$ directly, but it is now hidden in the boundary conditions on the cut. Using the standard scattering theory [7] one can verify a relation between the scattering operators, namely

$$S' = V_- S V_+ ,\ V_\pm = \lim_{t\to+\infty} \exp(\mp it\hat{H}_0)\, U^{\pm 1} \exp(\pm it\hat{H}_0)\,,$$

where $\hat{H}_0$ is the free Hamiltonian. Since $U = U_\alpha U_\beta$, the limits $V_\pm$ can be obtained from the corresponding one-solenoid cases. An explicit calculation has been done in [2]. An important conclusion is that $V_\pm$ amount to multiplication by some unimodular factors and so $|S'(\theta,\theta_0)|^2 = |S(\theta,\theta_0)|^2$ and the differential cross section remains unaltered.

The derivation of $S(\theta,\theta_0)$ was based on the knowledge of the Green's function obtained in [8,9]. The resulting formula is rather cumbersome. It is expressed as an infinite series, the summands are multiple integrals, and the dimension of the integration domain increases with the summand's order. One can interpret each summand as being related to a diagram which presents a ray coming in from the infinity, hitting one of the solenoids, oscillating between the solenoids several times, and escaping to infinity afterwards. The number n of solenoids being entered by the ray varies from zero to infinity (a solenoid can be entered by the ray several times during the oscillations). The case $n = 0$ corresponds to an unscattered wave, the case $n = 1$ corresponds to a wave scattered by one of the solenoids with an appropriate phase shift, and so on:

$$S(\theta,\theta_0)=\sum_{n=0}^{\infty}S^{(n)}(\theta,\theta_0), \tag{1}$$

$$S^{(0)}(\theta,\theta_0)=\frac{1}{2}\{1+\exp[2\pi i(\alpha+\beta)]\}\ \delta(\theta-\theta_0), \tag{2}$$

$$\begin{aligned}S^{(1)}(\theta,\theta_0)=&\frac{1}{2\pi}[1-\exp(2\pi i\alpha)]\exp[-ika(\cos\theta-\cos\theta_0)]\frac{\exp[i\alpha(\theta-\theta_0)]}{1-\exp[i(\theta-\theta_0)]}\\&+\frac{1}{2\pi}\exp[(\pi\mp\pi)i\alpha][1-\exp(2\pi i\beta)]\exp[-ikb(\cos\theta-\cos\theta_0)]\frac{\exp[i\beta(\theta-\theta_0)]}{1-\exp[i(\theta-\theta_0)]},\end{aligned} \tag{3}$$

$$\begin{aligned}S^{(n)}(\theta,\theta_0)=&\frac{i}{2}\Big(-\frac{1}{\pi}\Big)^n\exp[(\pi\mp\pi)i\alpha]\sum_{(c_n,\ldots,c_2,c_1)}\exp(-ikc_n\cos\theta+ikc_1\cos\theta_0)\\&\times\int_{\mathbb{R}^n}d^n\tau\exp(\varphi\tau_n+\varphi_0\tau_1)\prod_{j=1}^{n}\frac{\sin\pi\sigma_j}{\sin[\pi(\sigma_j+i\tau_j)]}\prod_{j=1}^{n-1}K_{i(\tau_{j+1}-\tau_j)}(-ik\rho),\end{aligned} \tag{4}$$

where $(c_n,\ldots,c_2,c_1)=(\ldots,a,b,a)$, $(\ldots,b,a,b)$ is a finite sequence oscillating between the values a and b $(n\geq 2)$. In these formulas, $\sigma_j=\alpha$ $(\sigma_j=\beta)$ if $c_j=a$ $(c_j=b)$, $\varphi_0=\pi-\theta_0$ $(\varphi_0=2\pi-\theta_0)$ for $c_1=a$ $(c_1=b)$, $\varphi=\theta-\pi\pm\pi$ $(\varphi=\theta-\pi)$ for $c_n=a$ $(c_n=b)$, the upper (lower) sign corresponds to the case $\theta\in(0,\pi)$ $[\theta\in(\pi,2\pi)]$, and $K_\nu(z)$ is the Macdonald function.

The differential cross section $d\sigma(\theta)$ for the scattering in the plane equals $(2\pi/k)\ |S(\theta,\theta_0)|^2d\theta$ $(\theta\neq\theta_0)$. But the complexity of the complete formula for $S(\theta,\theta_0)$ makes the numerical evaluation difficult. Fortunately, the asymptotic behavior of the Macdonald function

$$K_{i\tau}(-iz)=\sqrt{\pi/2z}\exp\left(iz+i\frac{\pi}{4}\right)\ [1+O(z^{-1})]\quad\text{for } z\to+\infty \tag{5}$$

opens the way to the asymptotic domain $k\rho\gg 1$. We are going to calculate $2\pi|S(\theta,\theta_0)|^2$ retaining the terms up to the order $O((k\rho)^{-1})$ and so the error is going to be of the order $O((k\rho)^{-3/2})$. In what follows we assume that θ_0 is not very close to the values π and 2π. Provided θ is also separated enough from the critical values 0, π, and 2π, one can simply replace the Macdonald functions by the leading term in (5) and perform the integration explicitly with the help of the identity

$$\int_{-\infty}^{+\infty}d\tau\ \frac{\exp(\omega\tau)}{\sin[\pi(\sigma+i\tau)]}=2\ \frac{\exp(i\sigma\omega)}{1+\exp(i\omega)}, \tag{6}$$

valid for $\sigma\in(0,1)$ and $|\omega|<\pi$. The situation is more delicate when θ tends to one of the critical values since in that case φ can tend to $\pm\pi$ and the integral in the variable τ_n fails to converge rapidly enough to allow this asymptotic.

To understand this problem better let us consider separately the integral

$$\int_{-\infty}^{+\infty}d\tau\ K_{i(\tau-\nu)}(-iz)\ \frac{\exp(\varphi\tau)}{\sin[\pi(\sigma+i\tau)]} \tag{7}$$

in the asymptotic domain $z\gg 1$ $(|\varphi|<\pi)$. With the help of the Fourier transform we find that (7) equals $\exp(i\sigma\varphi)I$ where

$$I=\int_{-\infty}^{+\infty}ds\ \exp(iz\cosh s\ )\ \frac{\exp[-(\sigma+i\nu)s]}{1+\exp(-s+i\varphi)}. \tag{8}$$

Provided φ is not very close to the border values $\pm\pi$, the stationary phase method yields

$$I\approx(1+e^{i\varphi})^{-1}\sqrt{2\pi/z}\exp\left(iz+i\frac{\pi}{4}\right).$$

To get the correct behavior in the vicinity of the values $\pm\pi$ one can employ the extended Sochocki formula

$$\frac{1}{x+i\epsilon}=\mathrm{P}\frac{1}{x}-i\pi\ (\operatorname{sgn}\epsilon)\ \delta+\epsilon\left((\operatorname{sgn}\epsilon)\ \pi\delta'+i\frac{d}{dx}\mathrm{P}\frac{1}{x}\right)+O(\epsilon^2). \tag{9}$$

Here the symbol P indicates the regularizataion in the sense of Cauchy principal value. Finally one finds that

$$I\approx[\mp i\pi+\sqrt{-2\pi iz}(1+e^{i\varphi})]\ e^{iz}\ \text{for}\ \varphi\to\pm\pi,\ |\varphi|<\pi.$$

Note that the Fresnel integral, written with the help of the error function

$$\Phi(x)=(2/\sqrt{\pi})\int_0^x\exp(-s^2)\ ds,$$

yields a suitable interpolating function involving both asymptotic regions,

$$I\approx\pi\exp\left(-i\frac{1}{2}\varphi-iz\cos\varphi\right)\ \{1-\Phi[e^{-i\pi/4}\sqrt{2z}\cos(\varphi/2)]\}\quad\text{for } z\gg 1. \tag{10}$$

It is so because

$$\Phi(x)\approx\begin{cases}1-(1/\sqrt{\pi}x)\exp(-x^2)&\text{for } x\to+\infty\\(2/\sqrt{\pi})x&\text{for } x\to 0.\end{cases}$$

From this analysis it is clear that starting from the value $n=5$, $S^{(n)}(\theta,\theta_0)$ will not contribute to the retained leading terms in any asymptotic domain. The sum

$$[S^{(2)}(\theta,\theta_0)+S^{(3)}(\theta,\theta_0)+S^{(4)}(\theta,\theta_0)]\ \exp[ikb(\cos\theta-\cos\theta_0)+i\tfrac{1}{2}(\theta-\theta_0)]$$

can be replaced with the error of order $O((k\rho)^{-3/2})$ by the expression

$$\begin{aligned}
&\frac{i}{2\pi}\sin\pi\alpha\sin\pi\beta\left(\exp[ik\rho(\cos\theta-\cos\theta_0)]\ \frac{\exp[i(2\pi\mp\pi-\theta_0)\alpha+i(\theta-\pi)\beta]}{\sin(\theta_0/2)}\right.\\
&\quad\times\{1-\Phi[e^{-i\pi/4}\sqrt{2k\rho}\ \sin(\theta/2)]\}\\
&\quad\left.\pm\ \frac{\exp[i(2\pi-\theta_0)\beta+i\theta\alpha]}{\cos(\theta_0/2)}\ \{1-\Phi[e^{-i\pi/4}\sqrt{2k\rho}\ |\cos(\theta/2)|]\}\right)\left(1+\frac{i}{2\pi k\rho}\sin\pi\alpha\sin\pi\beta\ e^{i2k\rho}\right)\\
&+\frac{i}{2\pi}\frac{\sin\pi\alpha\sin\pi\beta}{\sqrt{2\pi k\rho}}\left[\frac{\sin\pi\beta}{\cos(\theta_0/2)}\exp\left(i(\pi\mp\pi)\alpha+i(\pi+\theta-\theta_0)\beta+ik\rho(1+\cos\theta)-i\frac{\pi}{4}\right)\right.\\
&\quad\times\{1-\Phi[e^{-i\pi/4}\sqrt{2k\rho}\ \sin(\theta/2)]\}\\
&\quad\left.\mp\ \frac{\sin\pi\alpha}{\sin(\theta_0/2)}\ \exp\left(i(\pi+\theta-\theta_0)\alpha+ik\rho(1-\cos\theta_0)-i\frac{\pi}{4}\right)\{1-\Phi[e^{-i\pi/4}\sqrt{2k\rho}\ |\cos(\theta/2)|]\}\right].
\end{aligned}\tag{11}$$

Let us now turn to the numerical analysis. For the sake of convenience, in the presented graphs the function $2\pi|S(\theta,\theta_0)|^2$ depends on the variable $\Theta\in(-\pi,\pi)$, $\Theta\equiv\theta-\theta_0+\pi\ (\mathrm{mod}\ 2\pi)$, rather than on θ. Hence the values $\Theta=0$ and $\Theta=\pm\pi$ correspond to the backward and forward scattering, respectively. The graphs have oscillatory character and thus exhibit an interference between the solenoids. This behavior can be understood already from the first-order approximation (valid outside of the values $\Theta=\pi-\theta_0$ and $\Theta=2\pi-\theta_0$),

$$\begin{aligned}
2\pi|S^{(1)}(\theta,\theta_0)|^2=\frac{1}{2\pi}\{&\sin^2\pi\alpha+\sin^2\pi\beta+2\sin\pi\alpha\ \sin\pi\beta\\
&\times\cos[(\pm\pi+\theta-\theta_0)(\alpha-\beta)-(\pi\mp\pi)\beta\\
&+k\rho(\cos\theta-\cos\theta_0)]\}\sin^{-2}\frac{1}{2}(\theta-\theta_0).
\end{aligned}\tag{12}$$

The oscillations of the differential cross section are caused by the appearance of the term $k\rho\cos\theta$ in the argument of cosinus in (12). One can estimate the number of nodes roughly by the integer part of $2k\rho/\pi$.

Another feature should be mentioned. As in the one-solenoid case, $2\pi|S(\theta,\theta_0)|^2$ diverges for Θ tending to $\pm\pi$ (forward scattering). This happens due to the term $\sin^{-2}[\frac{1}{2}(\theta-\theta_0)]$ in (12). The only exception is the case $\alpha+\beta=1$. Then the expression (12) remains finite for $\theta\to\theta_0$ (note that the lower sign should be accounted). Figure 1 depicts the graph for $\alpha=\beta=0.5$, $\theta_0=3\pi/2$, and $k\rho=8$. It should be emphasized that this case is also rather special and important from the point of view of physical interpretation. Recall that the differential cross section depends on the quantities $\exp(2\pi i\alpha)=\exp(-ie\Phi_A/\hbar c)$ and $\exp(2\pi i\beta)=\exp(-ie\Phi_B/\hbar c)$ rather than directly on Φ_A and Φ_B. To be specific in the performed calculations, we have chosen $\alpha,\beta\in(0,1)$. Thus the case $\beta=1-\alpha$ involves equal and opposite fluxes, i.e., $\Phi_B=-\Phi_A$. This is a limiting case to the two-cylinder problem [3]. Since the total flux passing through the plane is zero, some difficulties with the physical interpretation occurring in the one-cylinder case are removed. The numerical results derived in the present paper confirm the existence of an AB effect also in this special case.

Concerning the dependence of the graph on the parameters, it turns out that the change of θ_0 distorts the graph somewhat while the change of α and β in a rather

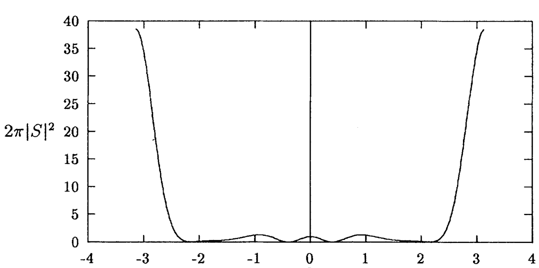

FIG. 1. Dependence of $2\pi|S(\theta,\theta_0)|^2$ on $\Theta\equiv\theta-\theta_0+\pi(\mathrm{mod}\ 2\pi)$ for $\alpha=\beta=0.5$, $\theta_0=3\pi/2$, and $k\rho=8$.

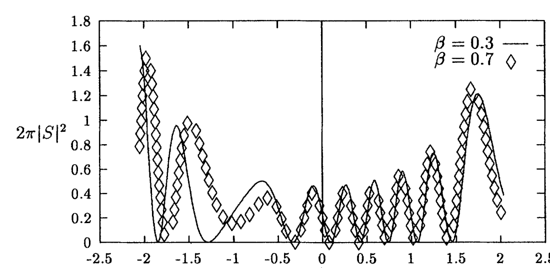

FIG. 2. Dependence of $2\pi|S(\theta,\theta_0)|^2$ on $\Theta\equiv\theta-\theta_0+\pi(\mathrm{mod}\ 2\pi)$ for $\alpha=0.3$, $\beta=0.3$ and 0.7, $\theta_0=1.3\ \pi$, and $k\rho=20$.

wide range [say, in the interval $(0.3, 0.7)$] does not influence the shape so strongly. Figure 2 depicts the case $\alpha = 0.3$, $\beta = 0.3$ and 0.7, $\theta_0 = 1.3\,\pi$, and $k\rho = 20$.

It should be also observed that a tiny discontinuity is still remaining in the points $\Theta = \pi - \theta_0$ and $\Theta = 2\pi - \theta_0$. But it is of the order of the allowed error, namely $O\big((k\rho)^{-3/2}\big)$. On the other hand, this discontinuity provides a numerical test and according to it one can judge that the results are reasonable, starting already from the value $k\rho = 5$. But it is also worth recalling that the scattering matrix $S(\theta, \theta_0)$ describes the scattering of plane waves. In the case of a wave packet the differential cross section obtained is still realistic only provided the width Δx of the packet is large if compared with the distance of solenoids.

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