Two theorems for the group velocity in dispersive media

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Two theorems on the group velocity are presented in this paper. First a simple proof is given that for any dispersive dielectric, there must be a frequency at which the group velocity of an electromagnetic pulse becomes *abnormal,* i.e., greater than the vacuum speed of light, infinite, or negative. Second, at the frequency at which the attenuation (or gain) is a maximum, the group velocity must be abnormal (or normal). This second theorem is more widely applicable, e.g., to propagation in waveguides or through multilayer dielectrics. To illustrate these theorems we discuss dispersion in a medium with two resonance lines, one absorption and the other gain. We find that the group velocity is abnormal within the absorption line and in a transparent region outside the gain line.

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The dispersive characteristics of absorbing dielectric media are well understood in terms of general theorems such as the Kramers-Kronig relation [1,2] and sum rules [3] which relate the absorption of a dielectric to its dispersion. In the transparent region, it has been shown [2] that pulses with a narrow spectral width propagate at a group velocity which is *normal,* i.e., positive and slower than the speed of light in vacuum. On the other hand, it has been observed for specific cases that the group velocity within an absorption band becomes superluminal, infinite or negative, which we term *abnormal.* In the past the abnormal cases were regarded as unphysical [2,4], but recent theoretical [5,6] and experimental work [7] has determined that abnormal group velocities are indeed physically significant, since they describe important features of pulse propagation inside the absorption band. In view of the newly established significance of abnormal group velocities, it is important to prove general results similar to those known for the normal case. We have made a beginning along these lines by proving (1) that regions of abnormal group velocity are required by the causality arguments that lead to the Kramers-Kronig relations and (2) that information about the location of abnormal regions can be obtained from knowledge of the absorption or gain curves of the medium. In connection with (1) it is remarkable that the association of transparent dielectrics with normal (subluminal) group velocities is well established [2], but it has apparently never been proven that abnormal group velocities always exist at some frequencies. Result (2) is particularly useful, since it provides information about dispersion in terms of the more easily measured amplification or loss. The dispersive characteristics of amplifying media are not as widely known [6,8,9], but the general theorems will still be valid. The results presented below are based on linear response of the medium, but they will clearly be applicable to the dispersion of a weak probe beam propagating in a medium with an effective index of refraction induced by a stronger pump beam.

Theorem 1. The principle of causality guarantees that for a medium described by a linear refractive index there is a range of frequencies in which the group velocity is superluminal, infinite, or negative (i.e., abnormal).

Consider a pulse made up of plane waves of angular frequencies near ω propagating along the z axis through a medium with complex index of refraction $n(\omega)=n_r(\omega)+in_i(\omega)$. The real wave number is $k(\omega)=\omega n_r(\omega)/c$, and the phase of the wave at position z and time t is given by

$$\varphi(\omega)=k(\omega)z-\omega t\ . \tag{1}$$

By setting the frequency derivative of this phase to zero (method of stationary phase), we find the group velocity for a pulse with carrier frequency ω,

$$v_g(\omega)=\left[\frac{dk}{d\omega}\right]^{-1}=\left[\frac{d}{d\omega}\left[\frac{\omega n_r(\omega)}{c}\right]\right]^{-1}\ . \tag{2}$$

Alternatively, we can say a pulse peak crosses a distance z in the transit time

$$t_{\text{ph}}(\omega)=\frac{dk}{d\omega}z=\frac{z}{v_g(\omega)}\ . \tag{3}$$

This is to be compared with a pulse moving through an equal length of vacuum, which takes a transit time z/c. We will call the difference in transit times through the medium and through the vacuum the group delay; it is positive for normal group velocities. When the group velocity is abnormal ($v_g>c$, $v_g=\infty$, or $v_g<0$), the group delay is negative,

$$\Delta t(\omega)=\left[\frac{dk}{d\omega}-\frac{1}{c}\right]z<0\ . \tag{4}$$

Thus we want to examine the difference between the two

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phases as a function of frequency,

$$\Delta\varphi(\omega)=\frac{z}{c}\omega[n_r(\omega)-1] , \tag{5}$$

since its frequency derivative is the group delay.

We impose the principle of causality by means of the dispersion relation for the index of refraction

$$n_r(\omega)=1+\frac{2}{\pi}\mathrm{P}\int_0^\infty d\omega'\frac{\omega' n_i(\omega')}{\omega'^2-\omega^2} , \tag{6}$$

where P stands for Cauchy principal value. The smooth response of the medium at short times implies that the $\omega\rightarrow\infty$ limit can be taken inside the integral to yield the high-frequency dependence of $n_r(\omega)$ in the form

$$n_r(\omega)\rightarrow 1-\frac{\omega_p^2}{2\omega^2} , \tag{7}$$

and the corresponding asymptotic expression for the phase difference

$$\Delta\varphi(\omega)=-\frac{z}{c}\frac{\omega_p^2}{2\omega} . \tag{8}$$

Note that this quantity is always negative, since the electrons are essentially free at sufficiently high frequencies. On the other hand, for frequencies near zero, the index of refraction is determined by the zero-frequency sum rule [2] derived from (6),

$$n_r(0)=1+\frac{2}{\pi}\mathrm{P}\int_0^\infty \frac{n_i(\omega)}{\omega}d\omega . \tag{9}$$

The group delay at zero frequency depends only on the value of the index and not its derivative:

$$\Delta t(0)=\frac{z}{c}\frac{d}{d\omega}[\omega n_r(\omega)-\omega]\Big|_{\omega=0}=\frac{z}{c}[n_r(0)-1] . \tag{10}$$

After substituting Eq. (9) for the refractive index, this becomes

$$\Delta t(0)=\frac{2z}{\pi c}\mathrm{P}\int_0^\infty \frac{n_i(\omega)}{\omega}d\omega . \tag{11}$$

Here there are three possibilities, depending on whether the medium has gain bands or is always absorptive: (a) for sufficiently large gain in some bandwidth, the integral in Eq. (11) is negative (strong gain medium); (b) the gain is weaker and the integral is positive (weak gain medium); (c) the medium is a pure absorber, $n_i(\omega)>0$, and the integral is positive. For case (a) the group delay is already negative at zero frequency [9], $\Delta t(0)<0$ [see Fig. 1(a)]. For cases (b) and (c) the group delay at zero frequency is positive, $\Delta t(0)>0$. Next examine the graph of $\Delta\varphi(\omega)$ [Fig. 1(b)] for (b) or (c). There must be a point (A) at very low frequency for which $\Delta\varphi(\omega)$ is positive, while at sufficiently high frequencies (B), its asymptotic value becomes negative. The index of refraction must be continuous and differentiable, because the absorption (or gain) lines must have a finite width, and the index is related to the absorption by the Kramers-Kronig relation (6). Thus the phase $\Delta\varphi(\omega)$ must also be continuous and differentiable. By the mean value theorem, the derivative of $\Delta\varphi(\omega)$ must take on the value of the slope of the line connecting A and B, for at least one point in that interval. Since this slope is negative, the group delay must be negative on some frequency range lying between those two points. This completes the proof for propagation in an unbounded medium. In experiments performed on a dielectric slab of finite thickness, an additional phase term is acquired due to the boundary conditions. This extra phase is discussed in the Appendix; it complicates the argument slightly, but the conclusion of the theorem remains the same.

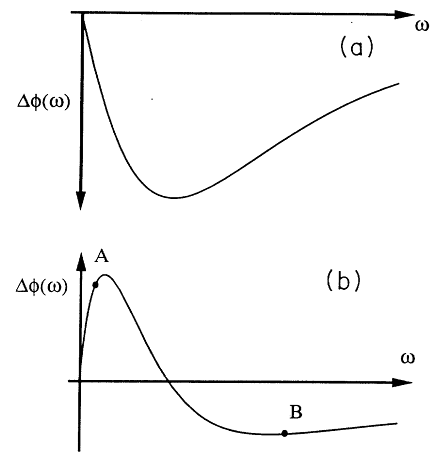

FIG. 1. (a) Phase difference as a function of frequency for a strong gain medium. (b) Phase difference as a function of frequency for either a pure absorber or a weak gain medium. Point A can be established by the zero-frequency limit of the group delay, Eq. (4); point B can be established by the high-frequency limit of the phase difference, Eq. (8). The part of the curve between the points must have a region of negative slope.

To determine the frequencies at which abnormal group velocities occur in an absorbing medium, or at which normal group velocities occur in an amplifying medium, we can use a second theorem which is derived from assumptions similar to the Kramers-Kronig relations.

Theorem 2. In any medium, the group velocity is abnormal for a frequency at which the absorption is an absolute maximum. Also, for a pure absorber, the group velocity is normal for a frequency at which the absorption is an absolute minimum, while for an amplifying medium, the group velocity is normal at the frequency where the gain is maximized.

We begin by defining a function which generalizes the phase difference $\Delta\varphi(\omega)$, and which is analytic in the upper half complex plane,

$$\Delta F(\omega)=\frac{z}{c}\omega[n(\omega)-1] . \tag{12}$$

For real frequencies, the real part is just the difference in phases used to compute the group delay, Eq. (5), and the imaginary part has the interpretation as the product of the absorption (or negative gain) coefficient $\kappa(\omega)$ and the distance propagated,

$$\mathrm{Im}[\Delta F(\omega)]=\frac{z}{c}\omega n_i(\omega)=z\kappa(\omega)\ . \tag{13}$$

Cauchy's integral theorem for the derivative of $\Delta F(\omega)$ is

$$\frac{d\Delta F(\omega)}{d\omega}=\frac{1}{2\pi i}\oint\frac{\Delta F(\omega')}{(\omega'-\omega)^2}d\omega'\ . \tag{14}$$

We choose the (counterclockwise) contour to consist of the large semicircle in the upper half-plane and the real axis, taking the principal part as the double pole. Since the asymptotic form of the index of refraction Eq. (7) is valid along the large semicircle, the contribution to the integral from that part of the contour vanishes. Typically the index of refraction involves taking the square root of the dielectric constant, so we must consider the possibility of power-law branch points along the real axis. Suppose there is a branch point at ω_b; then near that point the function $\Delta F(\omega)$ takes the form

$$\Delta F(\omega)=f(\omega)(\omega-\omega_b)^\nu\ , \tag{15}$$

where $f(\omega)$ is analytic near ω_b. The contribution to the integral from a small semicircle C_ρ above the branch point is

$$\frac{1}{2\pi i}\int_{C_\rho}\frac{\Delta F(\omega')}{(\omega'-\omega)^2}d\omega' =-\frac{1}{2\pi}\lim_{\rho\to 0}\rho^{(1+\nu)}\int_0^\pi\frac{f(\omega_b+\rho e^{i\theta})e^{i(1+\nu)\theta}}{(\omega_b+\rho e^{i\theta}-\omega)^2}d\theta\ , \tag{16}$$

which is zero for $\nu>-1$, allowing square root branch points. We can now take the real part of the remaining integral which is along the real axis, using Eqs. (5) and (13) to interpret the real and imaginary parts of $\Delta F(\omega)$, to obtain the group delay

$$\Delta t(\omega)=\frac{d\Delta\varphi(\omega)}{d\omega}=\frac{z}{2\pi}\mathrm{P}\int_{-\infty}^{\infty}d\omega'\frac{\kappa(\omega')}{(\omega'-\omega)^2}\ . \tag{17}$$

As long as we are only interested in the group delay in a transparent region, it is unnecessary to worry about the principal value in Eq. (17), because the integrand is small around the double pole at ω. However, to obtain exact results, we deal with the double pole as follows. We rewrite the integral as

$$\frac{d\Delta F(\omega)}{d\omega}=\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\mathrm{P}\int_{-\infty}^{\infty}d\omega'\frac{\Delta F(\omega')-\Delta F(\omega)-(\omega'-\omega-i\varepsilon)\dfrac{d\Delta F(\omega)}{d\omega}}{(\omega'-\omega-i\varepsilon)^2} +\Delta F(\omega)\frac{1}{2\pi i}\mathrm{P}\int_{-\infty}^{\infty}\frac{d\omega'}{(\omega'-\omega-i\varepsilon)^2}+\frac{d\Delta F(\omega)}{d\omega}\frac{1}{2\pi i}\mathrm{P}\int_{-\infty}^{\infty}\frac{d\omega'}{(\omega'-\omega-i\varepsilon)}\ . \tag{18}$$

The second and third integrals in Eq. (18) can be done explicitly, yielding the result

$$\frac{d\Delta F(\omega)}{d\omega}=\frac{1}{\pi i}\mathrm{P}\int_{-\infty}^{\infty}d\omega'\frac{\Delta F(\omega')-\Delta F(\omega)-(\omega'-\omega)\dfrac{d\Delta F(\omega)}{d\omega}}{(\omega'-\omega)^2}\ . \tag{19}$$

On taking the real part of Eq. (19), we find a relation between the group delay at a particular frequency and the absorption spectrum,

$$\Delta t(\omega)=\frac{d\Delta\varphi(\omega)}{d\omega} =\frac{z}{\pi}\mathrm{P}\int_{-\infty}^{\infty}d\omega'\frac{\kappa(\omega')-\kappa(\omega)-(\omega'-\omega)\dfrac{d\kappa(\omega)}{d\omega}}{(\omega'-\omega)^2}\ . \tag{20}$$

This gives a method of computing the group delay, and hence the group velocity, at any frequency, but the attenuation spectrum must be completely known. However, the *sign* of the group delay can be determined if we specialize this result to the case in which the attenuation is an absolute maximum or minimum at frequency ω. Then the derivative $d\kappa(\omega)/d\omega$ is zero, and the remaining part of the numerator of the integrand in Eq. (20) is always negative if the attenuation is maximized or positive if it is minimized. Therefore the group velocity is abnormal at maximal attenuation and is normal at minimum attenuation. In a medium with gain (which is just a negative attenuation), the absolute maximum of the gain is a point of normal group velocity. This completes the proof for an unbounded medium; the corrections due to boundary reflections, which are described in the Appendix, do not affect the conclusion.

The second theorem is clearly applicable to any problem in which a wave number is defined and has an asymptotic form similar to Eq. (7). For instance, the group velocity of a pulse traveling through a multilayer dielectric mirror will be abnormal when the light is tuned exactly to the center of the band gap, i.e., transmission is a minimum [10]. Another example is electromagnetic propagation in waveguides. Below the cutoff frequency, the attenuation increases to a maximum at zero frequency, so the group delay is negative there.

As an application of the second theorem and Eq. (17) for the group delay in transparent regions, we consider a medium which has only two resonance lines, one amplifying and one absorbing. We will obtain a qualitative graph of the group delay as a function of frequency by approximating Eq. (17) for the transparent regime and by using the second theorem at the points of maximum gain and maximum absorption. The gain line has resonance frequency ω_g and linewidth γ_g; the other is an absorption line at ω_a with linewidth γ_a. The lines are assumed to be widely spaced compared with either of their widths. We define the maximum gain and maximum absorption, respectively, as

$$g=-\kappa(\omega_g) , \tag{21}$$

$$a=\kappa(\omega_a) . \tag{22}$$

The gain- and absorption-bandwidth products are constrained by the f-sum rule,

$$\int_0^\infty \kappa(\omega')d\omega'=\frac{\pi}{4}c\omega_p^2>0 . \tag{23}$$

Assuming the main contribution to the integral comes from the regions near resonance, this implies

$$a\gamma_a>g\gamma_g . \tag{24}$$

A similar estimate of the integral in (17) gives the group delay,

$$\Delta t(\omega)\approx\frac{1}{2\pi}\left[\frac{a\gamma_a}{(\omega_a-\omega)^2}-\frac{g\gamma_g}{(\omega_g-\omega)^2}\right] . \tag{25}$$

The sign of the group delay in the transparent region is positive or negative depending on the frequency; just outside of the absorption line it will be positive while outside, but near the gain line it will be negative. The frequencies at which it changes sign are found to be

$$\omega=\omega_g\pm\left[\left[\frac{a\gamma_a}{g\gamma_g}\right]^{1/2}\pm1\right]^{-1}(\omega_a-\omega_g) . \tag{26}$$

On the other hand, for frequencies *within* the lines (significant attenuation or amplification), Eq. (25) is no longer valid, but theorem 2 shows the group delay is negative in the absorption line and positive in the gain line.

There are several special cases of the above example that illustrates the possible behavior for the group delay in transparent regions. Those regions are interesting for pulse propagation since little attenuation or amplification will occur. First we consider the weak gain limit $a\gamma_a\gg g\gamma_g$. A typical plot of absorption and group delay versus frequency is shown in Fig. 2(a). Note that negative group delays lie in a symmetric region around (but excluding) the gain line. In the opposite limit, we take a gain line as strong as the f-sum rule permits, so that $a\gamma_a-g\gamma_g\ll a\gamma_a$. If the gain line has the lower frequency, as in Fig. 2(b), then there is a region of negative group delay extending from zero frequency to near the midpoint between the two lines. If the gain line has the higher frequency, Fig. 2(c), the negative group delay regions begin at the midpoint and extend up to a much higher frequency than either line. In both cases the gain line itself is excluded from the region of negative group delay.

Although we have only two lines in these examples, we expect the description of the group delay to be similar for any number of lines. In general, abnormal propagation can occur in two regions: (1) within an absorption line and (2) in a finite transparent region outside a gain line.

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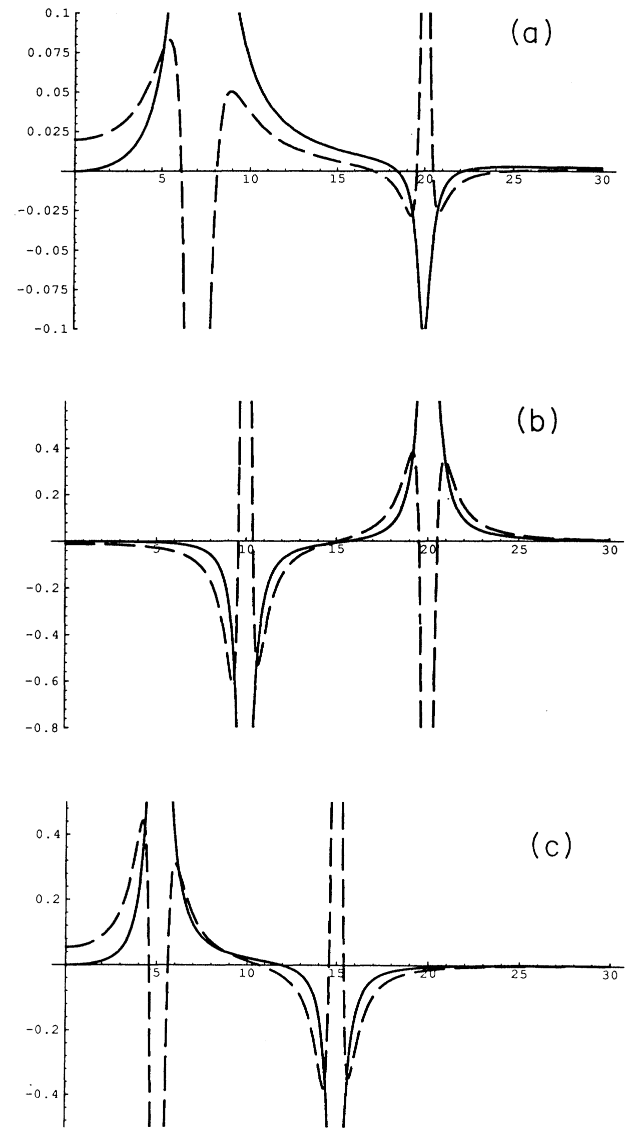

FIG. 2. (a) Group delay (dashed) and absorption (solid curve) as a function of frequency for a medium with a weak gain line and a strong absorption line. (b) Same as (a), but for a strong gain line below an absorption line. (c) Same as (a), but for a strong gain line above an absorption line.

APPENDIX: ADDITIONAL GROUP DELAY FOR A FINITE LENGTH DIELECTRIC

For experiments on a finite sample in vacuum (or air) phase shifts from repeated reflections off the end surfaces must be taken into account. In this appendix we will prove that the first theorem still holds when these boundary effects are included, and indicate how Eq. (17) and the second theorem are modified.

The transmission coefficient for a monochromatic wave passing through a medium of width and index $n(\omega)$ is

$$\ell(\omega)=\left[\frac{t_{12}t_{21}}{1-r_{12}^2e^{i2kz}}\right]e^{ikz}, \qquad (A1)$$

where the reflection and transmission coefficients for a single interface are

$$r_{12}=\frac{1-n}{1+n}, \qquad (A2a)$$

$$t_{12}=\frac{2}{1+n}, \qquad (A2b)$$

$$t_{21}=\frac{2n}{1+n}, \qquad (A2c)$$

and the wave vector is $k(\omega)=\omega/cn(\omega)$. The *extra* phase due to the boundary conditions is the argument of the factor in parentheses in Eq. (A1). We now calculate the extra phase in the two limits of large and zero frequency. When n is real, this phase simplifies to

$$\varphi_b(\omega)=\arctan\left[\frac{(1-n)^2\sin2kz}{(1+n)^2-(1-n)^2\cos2kz}\right]. \qquad (A3)$$

In the high-frequency limit the index of refraction will be nearly real and close to one, so we set

$$n(\omega)=1+\Delta n(\omega),\quad \Delta n(\omega)\ll 1\,. \qquad (A4)$$

Then using Eq. (A3) we find that the extra phase is approximated by

$$\varphi_b(\omega)=\tfrac{1}{4}(\Delta n)^2\sin2kz+O(\Delta n)^3\,. \qquad (A5)$$

The extra group delay associated with this phase is

$$\Delta t_b=\frac{d\varphi_b}{d\omega}\approx\frac{1}{2}\Delta n\frac{d\Delta n}{d\omega}\sin2kz+\frac{z}{2c}(\Delta n)^2\frac{d(\omega n)}{d\omega}\cos2kz\,. \qquad (A6)$$

Substituting the asymptotic expression Eq. (7) for the index of refraction into Eq. (A5), we find

$$\varphi_b(\omega)\to\frac{\omega_p^4}{16\omega^4}\sin2kz\,. \qquad (A7)$$

Since this approaches zero more rapidly than the propagative phase in Eq. (8), it can be neglected and we again find that the phase is negative in this limit. For static fields ($\omega=0$), we find from Eq. (A3) the extra phase is exactly zero. In general the derivative of the phase can have either sign, but the same arguments given to prove the first theorem will still apply in either case.

In Eq. (17) and the second theorem, the only modification required is to include the losses due to reflections by adding a term to the absorption length product,

$$\kappa_b(\omega)z=\kappa(\omega)z-\ln\left|\frac{t_{12}t_{21}}{1-r_{12}^2e^{i2kz}}\right|. \qquad (A8)$$

All of the results can then be restated using $\kappa_b(\omega)z$ in place of $\kappa(\omega)z$.

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