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Free and dissipative evolution of squeezed and displaced number states in the third-order nonlinear oscillator

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The evolution of squeezed and displaced number states in the free and dissipative third-order nonlinear oscillator is investigated from the point of view of nonclassical phenomena as the number squeezing in both the strong and the weak sense, the principal squeezing of vacuum fluctuations, and the generation of superposition states. A destructive effect of losses on quantum coherence is demonstrated.

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I. INTRODUCTION

An immense effort has been devoted to the study of the relation of a Kerr medium, modeled as the third-order nonlinear oscillator in quantum optics, to nonclassical states of radiation (see [1] for a review). The quantum dynamics of statistical properties of the dissipative third-order nonlinear oscillator has been investigated for a coherent state, Gaussian pure and mixed states, a displaced number state, and a squeezed and displaced number state as initial states.

Squeezed and displaced number states [2] have been studied from the viewpoint of the standard and principal squeezing of vacuum fluctuations and of the photon statistics; dissipation has been included. These states generalize two-photon coherent states [3], squeezed number states [4–9], and displaced number states [7,10–14]. They exhibit both number squeezing in the strong sense and the quadrature squeezing of vacuum fluctuations. This motivated the use of squeezed and displaced number states as the initial states for the third-order nonlinear oscillator [15]. In this paper we will follow their free and dissipative evolution with regard to the number squeezing in both the strong and the weak sense, the principal squeezing of vacuum fluctuations, and the generation of superposition states. We will take account of phase properties of the resulting states. We would like to complete results obtained in [15].

II. QUANTUM DYNAMICS

Incorporating dissipation to the third-order nonlinear oscillator, we can write the Hamiltonian [16]

$$\hat{H}=\hbar\left\{\omega(\hat{a}^{\dagger}\hat{a}+\tfrac{1}{2})+\kappa\hat{a}^{\dagger 2}\hat{a}^{2}+\sum_{j}\psi_j(\hat{c}_j^{\dagger}\hat{c}_j+\tfrac{1}{2})+\sum_{j}(\eta_j\hat{c}_j\hat{a}^{\dagger}+\text{H.c.})\right\} . \quad (2.1)$$

Here $\hat{a}$ ($\hat{a}^{\dagger}$) is the photon annihilation (creation) operator describing the radiation field of the frequency ω, κ is a real constant for the intensity dependence, $\hat{c}_j$ ($\hat{c}_j^{\dagger}$) are the boson annihilation (creation) operators of the reservoir oscillators with the frequencies ψ_j, and η_j are the coupling constants of the radiation to the reservoir. The dynamics of the compound optical system is described by the reduced density operator $\hat{\rho}_r$ fulfilling the master equation in the interaction representation derived in the standard treatments of the quantum theory of dissipation [17], in which the reservoir is characterized by the damping constant γ and the number of quanta $\bar{n}_d$. Using the classical-quantum correspondence $\tilde{C}\Phi_{\mathcal{A}}=\pi^{-1}\hat{\rho}_r$, related to the quantum correspondence $C^{-1}(\hat{a}^{k}\hat{a}^{\dagger l})=\alpha^{k}\alpha^{*l}$ [18], where the complex amplitude α corresponds to the operator $\exp(i\omega t)\hat{a}$, we obtain the generalized Fokker-Planck equation for the quasidistribution $\Phi_{\mathcal{A}}(\alpha,t)$ related to the antinormal ordering of field operators. This quasidistribution can be expressed in the form [16]

$$\Phi_{\mathcal{A}}(\alpha,t)=\exp(-|\alpha|^2)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\alpha^{m}\alpha^{*n}f_{mn}(t) , \quad (2.2)$$

where

$$f_{mn}(t)=\exp\left\{\left[-2i\kappa(m-n)+\frac{\gamma}{2}\right]t\right\}E_{m-n}^{m+n+1}(t)\sum_{j=0}^{\min(m,n)}\frac{1}{j!}\left[\frac{g_{m-n}(t)}{E_{m-n}^{2}(t)}\right]^{j}\times\sum_{l=0}^{\infty}\frac{1}{l!}\left[\frac{(\bar{n}_d+1)}{\bar{n}_d}g_{m-n}(t)\right]^{l}\frac{(m-j+l)!(n-j+l)!}{(m-j)!(n-j)!}f_{m-j+l,n-j+l}(0) , \quad (2.3)$$

with

$$g_l(t)=\frac{2\bar{n}_d}{\Omega_l+\Delta_l\coth\left[\frac{\gamma}{2}\Delta_l t\right]},$$

$$E_l(t)=\frac{\Delta_l}{\Omega_l\sinh\left[\frac{\gamma}{2}\Delta_l t\right]+\Delta_l\cosh\left[\frac{\gamma}{2}\Delta_l t\right]}, \quad (2.4)$$

$$\Omega_l=1+2\bar{n}_d+i\frac{2}{\gamma}\kappa l,\quad \Delta_l=\sqrt{\Omega_l^2-4\bar{n}_d(\bar{n}_d+1)}\ ; \quad (2.5)$$

$f_{mn}(0)$ are the coefficients of the expansion of $\Phi_{\mathcal{A}}(\alpha,0)$ characterizing an initial state,

$$f_{mn}(0)=\frac{1}{m!n!}\frac{\partial^{m+n}}{\partial\alpha^m\partial\alpha^{*n}}[\exp(|\alpha|^2)\Phi_{\mathcal{A}}(\alpha,t)]|_{\alpha=0,\alpha^*=0}\ . \quad (2.6)$$

The matrix elements of the reduced density operator are related to the coefficients $f_{mn}(t)$,

$$\rho_{nm}(t)=\langle n|\hat{\rho}_r|m\rangle=\pi\sqrt{n!m!}f_{mn}(t)\ . \quad (2.7)$$

The exact formula for the quasidistribution $\Phi_{\mathcal{A}}(\alpha,t)$ is highly instructive for revealing specific quantum features of the evolution. An analysis of $\Phi_{\mathcal{A}}(\alpha,t)$ for the free nonlinear oscillator provides an explanation of the origin of the periodicity of an initial state. All relevant quantum statistics repeat after the time interval π/κ. The quantum coherence is sensitive to dissipation. On incorporating the damping ($\bar{n}_d=0$), a quasiperiodic behavior develops from the original periodic recurrences of the initial state. The initial state undergoes nonlinear oscillations without losses for a time $1/\gamma$. The intermediate state is then attenuated during a time $k\pi/\kappa$ and this state undergoes time-reversed lossless nonlinear oscillations during a time $1/\gamma$. For $\gamma\gg 0$ or $\bar{n}_d\gg 0$, the periodic behavior of the system is destroyed.

Using the quasidistribution $\Phi_{\mathcal{A}}(\alpha,t)$, we can write the moments of the antinormally ordered field operators in the form [19]

$$\langle\hat{a}^k\hat{a}^{\dagger l}\rangle=\langle\alpha^k\alpha^{*l}\rangle_{\mathcal{A}}=\pi\sum_{n=0}^{\infty}(n+l)!f_{n+l-k,n}(t)\ ,\quad k\le l\ , \quad (2.8)$$

where for $k>l$ we consider the complex-conjugate quantity. On substitution of the functions $f_{mn}(t)$ from (2.3) into (2.8), we obtain an explicit formula for the moments $\langle\hat{a}^k\hat{a}^{\dagger l}\rangle$, which can be simplified in the cases $k=0$, $l\neq 0$ and $k=1$, $l=1$ [16].

The squeezed and displaced number states [2] $|\beta,M\rangle_g$, $\beta=\mu\xi(0)+\nu\xi^*(0)$, $|\mu|^2-|\nu|^2=1$ have the coherent-state representation

$$\langle\alpha|\beta,M\rangle_g=\frac{1}{\sqrt{M!}}\zeta^M H_M\left[\frac{\alpha^*-\xi^*(0)}{2\zeta\mu}\right]\langle\alpha|\beta\rangle_g\ , \quad (2.9)$$

where $\zeta=\sqrt{-\nu^*/2\mu}$, $|\beta\rangle_g\equiv|\beta,0\rangle_g$, the Gaussian pure state, and [3]

$$\langle\alpha|\beta\rangle_g=\frac{1}{\sqrt{\mu}}\exp\left[-\frac{1}{2}|\alpha|^2-\frac{1}{2}|\beta|^2-\frac{\nu}{2\mu}\alpha^{*2}+\frac{\nu^*}{2\mu}\beta^2+\frac{1}{\mu}\alpha^*\beta\right]\ . \quad (2.10)$$

$H_M(x)$ is the Hermite polynomial. The appropriate quasidistribution for the antinormal ordering of field operators reads

$$\Phi_{\mathcal{A}}(\alpha,0)=\frac{1}{\pi}|\langle\alpha|\beta,M\rangle_g|^2\ . \quad (2.11)$$

The number state representation of the initial state under study

$$\langle n|\beta,M\rangle_g=\left[\frac{n!M!}{\mu}\right]^{1/2}\exp\left[-\frac{1}{2}|\beta|^2+\frac{\nu^*}{2\mu}\beta^2\right]\times\sum_{j=0}^{\min(n,M)}\frac{1}{j!(n-j)!(M-j)!}\left[\frac{1}{\mu}\right]^j\times\chi^{n-j}\zeta^{M-j}H_{n-j}\left[\frac{\beta}{2\chi\mu}\right]\times H_{M-j}\left[-\frac{\xi^*(0)}{2\zeta\mu}\right]\ , \quad (2.12)$$

where $\chi=\sqrt{\nu/2\mu}$. The initial values of $f_{mn}(t)$ are of the form

$$f_{mn}(0)=\frac{1}{\pi\sqrt{n!m!}}\langle n|\beta,M\rangle_{g\,g}\langle\beta,M|m\rangle\ . \quad (2.13)$$

The statistical properties of the squeezed and displaced number states propagating in a lossy Kerr medium have been studied by Král [15]. For a lossless medium the quasidistribution $\Phi_{\mathcal{A}}(\alpha,L)$, where L is the propagation length, has been determined and the computation of the photon-number distribution $p(n,t)$ and its factorial moments $\langle W^k\rangle_{\mathcal{N}}$ has been outlined. The behavior of the phase quasidistribution

$$\Phi(\varphi,L)=\int\Phi_{\mathcal{A}}[r\exp(i\varphi),L]r\,dr \quad (2.14)$$

has been investigated in the dependence on the displacement $\xi(0)$. The effect of quantum fluctuations has been involved in the quasidistribution $\Phi_{\mathcal{A}}(\alpha,L)$ only approximately (as in the linear oscillator case).

III. NUMBER SQUEEZING

The photon-number distribution reads [19]

$$p(n,t)=n!f_{nn}(t) \quad (3.1)$$

and its factorial moments

$$\left\langle\frac{n(t)!}{[n(t)-k]!}\right\rangle=\langle W^k(t)\rangle_{\mathcal{N}}=\sum_{n=k}^{\infty}\frac{n!}{(n-k)!}p(n,t)\ , \quad (3.2)$$

where $\langle W^k(t)\rangle_{\mathcal{N}}$ is the kth moment of the integrated intensity.

For the third-order nonlinear oscillator as well as for the linear one the formulas (2.3) and (3.1) in the "diagonal" limit provide [19]

$$p(n,t)=\frac{1}{\bar{n}(t)+1}\sum_{m=0}^{\infty}p(m,0)[\bar{n}(t)+1]^{-m}\sum_{j=0}^{\min(m,n)}\frac{(m+n-j)!}{j!(n-j)!(m-j)!}(-1)^j[\bar{n}(t)-\exp(-\gamma t)]^j\left[\frac{\bar{n}(t)}{\bar{n}(t)+1}\right]^{n-j}\times[\bar{n}(t)+1-\exp(-\gamma t)]^{m-j}, \tag{3.3}$$

where

$$\bar{n}(t)=\bar{n}_d[1-\exp(-\gamma t)]. \tag{3.4}$$

The formula (3.2) simplifies then to the form

$$\langle W^k(t)\rangle_{\mathcal{N}}=\pi\sum_{m=0}^{\infty}p(m,0)\sum_{j=0}^{\min(k,m)}\begin{bmatrix}k\\j\end{bmatrix}\frac{(m+k-j)!}{(m-j)!}(-1)^j[\bar{n}(t)-\exp(-\gamma t)]^j[\bar{n}(t)]^{k-j}. \tag{3.5}$$

As for the asymptotic behavior, the formulas (3.3) and (3.5) take on the simple forms

$$p(n,\infty)=\frac{\bar{n}_d^n}{(\bar{n}_d+1)^{n+1}}, \tag{3.6}$$

$$\langle W^k(\infty)\rangle_{\mathcal{N}}=k!\bar{n}_d^k, \tag{3.7}$$

respectively, and characterize the Bose-Einstein statistics of the reservoir.

Phase properties of this single-mode optical field will be studied with the aid of the operators [20,21]

$$\hat{u}=\widehat{\exp}(i\varphi),\quad \hat{u}^\dagger=\widehat{\exp}(-i\varphi) \tag{3.8}$$

defined as

$$\hat{u}=(\hat{n}+\hat{1})^{-1/2}\hat{a},\quad \hat{u}^\dagger=\hat{a}^\dagger(\hat{n}+\hat{1})^{-1/2}, \tag{3.9}$$

with the properties

$$\hat{u}\hat{u}^\dagger=\hat{1},\quad \hat{u}^\dagger\hat{u}=\hat{1}-|n=0\rangle\langle n=0|. \tag{3.10}$$

From the relations (3.8) it is evident that the requirement of the unitarity for the phase operators can be attained algebraically by using the antinormal ordering of the operators $\hat{u},\hat{u}^\dagger$ [22,23]. This is equivalent to the classical quantum correspondence assigning to every phase function $M(\varphi)$ the operator

$$\hat{M}=\frac{1}{2\pi}\int_{\theta}^{\theta+2\pi}M(\varphi)|\varphi\rangle\langle\varphi|d\varphi, \tag{3.11}$$

based on the vectors

$$|\varphi\rangle=\sum_{n=0}^{\infty}\exp(in\varphi)|n\rangle, \tag{3.12}$$

which are approximately orthogonal

$$\langle\varphi|\varphi'\rangle=\pi\delta(\varphi-\varphi')+\{1-\exp[-i(\varphi-\varphi')]\}^{-1}. \tag{3.13}$$

The operator $\hat{M}$ does not depend on θ when $M(\varphi)$ has a 2π-periodic continuation. For the phase distribution it holds that

$$P(\varphi)=\frac{1}{2\pi}\langle\varphi|\hat{\rho}_r|\varphi\rangle, \tag{3.14}$$

or equivalently in terms of the coefficients (2.3)

$$P(\varphi)\equiv P(\varphi,t)=\tfrac{1}{2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sqrt{m!n!}\exp[i(m-n)\varphi]f_{mn}(t). \tag{3.15}$$

Upon the computation of the average value of the operator $\hat{M}$, the phase distribution is used as follows:

$$\langle\hat{M}\rangle=\mathrm{Tr}\{\hat{\rho}_r\hat{M}\}=\int_{\theta}^{\theta+2\pi}P(\varphi)M(\varphi)d\varphi=\langle M(\varphi)\rangle_a, \tag{3.16}$$

where the subscript a indicates the antinormal ordering of the operators $\hat{u},\hat{u}^\dagger$. The phase dispersion is measured by the quantity V [24,22]

$$V=1-|\langle\widehat{\exp}(i\varphi)\rangle|^2 \tag{3.17}$$

and an appropriate uncertainty relation [25,22]

$$[\langle(\Delta\hat{n})^2\rangle+\tfrac{1}{4}][1-|\langle\widehat{\exp}(i\varphi)\rangle|^2]\geq\tfrac{1}{4} \tag{3.18}$$

is respected.

In quantum optics it is recognized that a phase stretching leads to the number squeezing when some number-phase intelligence is conserved. The corresponding state is similar to a crescent one. The term number squeezed state denotes any state whose variance of the photon number is less than the mean photon number. Such a state may be called a crescent state in the strong sense. The deviation from the Poisson photon-number distribution is expressed by the Fano factor

$$d=\frac{\langle(\Delta\hat{n})^2\rangle}{\langle\hat{n}\rangle}. \tag{3.19}$$

The normalized variance d is related to the second reduced factorial moment

$$f=\frac{d-1}{\langle\hat{n}\rangle}, \tag{3.20}$$

or equivalently

$$f\equiv f(t)=\frac{\langle(\Delta W(t))^2\rangle_{\mathcal{N}}}{\langle W(t)\rangle_{\mathcal{N}}^2}. \tag{3.21}$$

For a coherent state it holds that $d=1$, $f=0$. Nonclassical (sub-Poissonian) behavior occurs for $d<1$, $f<0$.

For the squeezed and displaced number states $|\beta,M\rangle_g$

it holds that [2]

$$\langle \hat{n} \rangle = (|\mu|^2+|\nu|^2)M+|\nu|^2+|\xi(0)|^2 ,$$
$$\langle (\Delta\hat{n})^2 \rangle = |\mu\xi(0)-\nu\xi^*(0)|^2(2M+1) + 2|\mu\nu|^2(M^2+M+1) . \quad (3.22)$$

These states exhibit number squeezing in the strong sense dependent on the parameters $\xi(0)$, ν, and M. This nonclassical property is periodically revealed by the free nonlinear evolution and is smoothed out in the lossy case. In Fig. 1 we can see the dissipative evolution of the second reduced factorial moment in the dependence on $|\nu|\in[0,1]$ for fixed values of other parameters. On increasing $|\nu|$, the sub-Poissonian behavior is changing to the super-Poissonian one. The asymptotic behavior of f is in accordance with the formula (3.7).

The nonlinear oscillator cannot produce crescent states in the strong sense due to the conservation of the photon number, but the crescent topography of the quasidistribution $\Phi_{\mathcal{A}}(\alpha,t)$ suffices to an interferometric generation of number squeezed states [26]. An investigation of crescent states in the weak sense may be based upon the set of displaced number operators [23]

$$\hat{n}_c=\hat{a}_c^\dagger\hat{a}_c , \quad \hat{a}_c=\hat{a}-c\hat{1} , \quad (3.23)$$

where

$$c=\frac{(\langle \Delta\hat{a}^\dagger\Delta\hat{a} \rangle+\frac{1}{2})E-\langle(\Delta\hat{a})^2\rangle E^*}{(\langle \Delta\hat{a}^\dagger\Delta\hat{a} \rangle+\frac{1}{2})^2-|\langle(\Delta\hat{a})^2\rangle|^2} , \quad (3.24)$$

with

$$E=\tfrac{1}{2}(\langle \Delta\hat{n}\Delta\hat{a} \rangle+\langle \Delta\hat{a}\Delta\hat{n} \rangle) , \quad (3.25)$$

can be interpreted as the center of curvature of the corresponding contour diagram. By comparing the photon-number variance $\langle(\Delta\hat{n}_c)^2\rangle$ with the mean photon number $\langle \hat{n}_c \rangle$,

$$d_c=\frac{\langle(\Delta\hat{n}_c)^2\rangle}{\langle \hat{n}_c \rangle} , \quad (3.26)$$

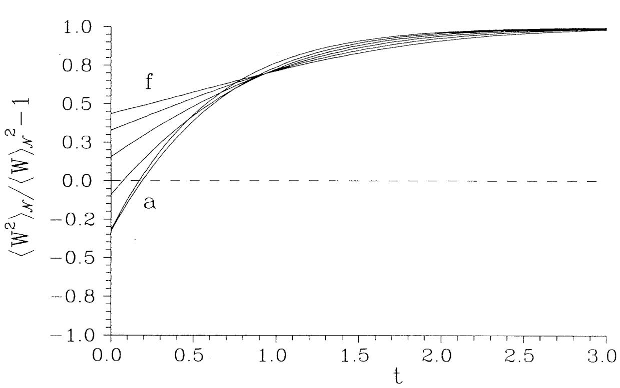

FIG. 1. The evolution of the second reduced factorial moment $\langle W^2\rangle_{\mathcal{N}}/\langle W\rangle_{\mathcal{N}}^2-1$ for $\kappa=100$, $\gamma=1$, $\bar{n}_d=1$, $|\xi(0)|=\frac{1}{2}$, $\psi=0$, $M=1$, $\theta=\pi/2$, and $|\nu|=0$, 0.2, 0.4, 0.6, 0.8, 1 (curves a,b,c,d,e,f, respectively).

we obtain a characteristics of the crescent shape. If the sub-Poissonian behavior is obtained, $d_c<1$, we can characterize the states as crescent in the weak sense.

Another measure of the qualitative properties of crescent states represents an uncertainty product. The operators $\hat{n}_c$ and

$$\hat{P}_c(\tau)=-i[\exp(-i\tau)\hat{a}_c-\exp(i\tau)\hat{a}_c^\dagger] \quad (3.27)$$

are not correlated, which holds for $\hat{n}_c-\hat{P}_c(\tau)$ intelligent states [27]. Because the parameter τ is arbitrary, we choose $\tau=\bar{\varphi}_c$, where $\bar{\varphi}_c$ is the preferred phase,

$$\bar{\varphi}_c=\arg[\langle \widehat{\exp}(i\varphi_c)\rangle] , \quad (3.28)$$

and we linearize the phase operator as

$$\delta\hat{\varphi}_c=\frac{\Delta\hat{P}_c(\bar{\varphi}_c)}{\langle \hat{Q}_c(\bar{\varphi}_c)\rangle} , \quad (3.29)$$

where

$$\hat{Q}_c(\tau)=\exp(-i\tau)\hat{a}_c+\exp(i\tau)\hat{a}_c^\dagger . \quad (3.30)$$

From (3.29) we obtain the measure of the phase dispersion

$$\langle(\delta\hat{\varphi}_c)^2\rangle=\frac{\langle[\Delta\hat{P}_c(\bar{\varphi}_c)]^2\rangle}{\langle \hat{Q}_c(\bar{\varphi}_c)\rangle^2} , \quad (3.31)$$

which can be expressed explicitly in terms of the operators $\hat{a},\hat{a}^\dagger,\hat{a}_c,\hat{a}_c^\dagger$ as follows:

$$\langle(\delta\hat{\varphi}_c)^2\rangle = \frac{(\langle \Delta\hat{a}^\dagger\Delta\hat{a} \rangle+\frac{1}{2})|\langle \hat{a}_c \rangle|^2-\mathrm{Re}[\langle(\Delta\hat{a})^2\rangle\langle \hat{a}_c^\dagger\rangle^2]}{2|\langle \hat{a}_c \rangle|^4} . \quad (3.32)$$

The corresponding uncertainty product u_c reads

$$u_c=\langle(\Delta\hat{n}_c)^2\rangle\langle(\delta\hat{\varphi}_c)^2\rangle\geq\tfrac{1}{4} . \quad (3.33)$$

In analogy to (3.18) we derive the uncertainty relation

$$\bar{u}_c=[\langle(\Delta\hat{n}_c)^2\rangle+\tfrac{1}{4}]V(c)\geq\tfrac{1}{4} , \quad (3.34)$$

where

$$V(c)=\frac{\langle(\delta\hat{\varphi}_c)^2\rangle}{1+\langle(\delta\hat{\varphi}_c)^2\rangle} . \quad (3.35)$$

The minimum value of the uncertainty products u_c and $\bar{u}_c$ equal to $\frac{1}{4}$ is attained by the $\hat{n}_c-\hat{P}_c(\bar{\varphi}_c)$ intelligent states. The uncertainty products u_c and $\bar{u}_c$ along with the Fano factor d_c contribute to an assessment of the crescent shape of a state studied, because not only the sub-Poissonian behavior but also the $\hat{n}_c-\hat{P}_c(\bar{\varphi}_c)$ intelligence characterizes the crescent property.

In the following we will apply the formulas (3.24), (3.26), (3.33), and (3.34) to pictorialize the investigation of crescent states in the weak sense. The limit values of d_c, u_c, and $\bar{u}_c$ for t tending to zero are

$$\lim_{t\to 0} d_c=2 , \quad \lim_{t\to 0} u_c=+\infty , \quad \lim_{t\to 0} \bar{u}_c=\tfrac{1}{4} . \quad (3.36)$$

The center of curvature $c(t)|\langle\hat{a}\rangle(0)|^{-1}$ $[\langle\hat{a}\rangle(0)=\xi(0)]$, $\langle\hat{a}\rangle(t)|\langle\hat{a}\rangle(0)|^{-1}$, and the rotating part of the complex field amplitude

$$\langle\hat{a}\rangle_r(t)=\langle\hat{a}\rangle(t)|\langle\hat{a}\rangle(t)|^{-1} \tag{3.37}$$

as functions of t are demonstrated for chosen parameters $\xi(0)$, ν, and M in Fig. 2. The points corresponding to $t_1=1.644\times10^{-4}$, $t_2=1.496\times10^{-3}$, the end points of the interval $[t_1,t_2]$ in which the field is sub-Poissonian are indicated by the circles on the appropriate curves. The optimum situation occurs for $t=t_0$, $t_0=7.534\times10^{-4}$, in this case $c(t_0)=-0.764+i0.451$. The value $2\kappa t_0$ corresponds to the preferred phase on the output of the Kerr nonlinear interferometer. In this picture the relationship of the center of curvature c and the "center of gravity" $\langle\hat{a}\rangle$ is also demonstrated. It holds that $|c|\lesssim|\langle\hat{a}\rangle|$. It is possible to discern two cases. The case of inequality corresponds to a crescent shape. The case of the approximate equality seems to be related to a shape very similar to that of the two-photon coherent state.

In Fig. 3 appropriate through the choice of the parameters $\xi(0)$, ν, and M to Fig. 2 we can observe the distance from the center of curvature to the origin $|c|\,|\xi(0)|^{-1}$ and the output Fano factor d_c. The curve of d_c indicates the sub-Poissonian behavior and attains a minimum at t_0, $d_c(t_0)=2.351\times10^{-1}$. Further curve represents the output uncertainty product u_c, which arrives at its minimum for a value of t close to t_0. With respect to the length of the period, the effect of the optimum lasts a very short time; it is transient. The quasidistribution $\Phi_{\mathcal{A}}(\alpha,t_0)$ is pictorialized in Fig. 4.

IV. QUADRATURE SQUEEZING

Investigating the squeezing of vacuum fluctuations, we restrict ourselves to the principal squeezing [28,29], which is advantageous in the nonlinear oscillator case because the free-field frequency is modified by self-interaction here and depends on the intensity of the field. The principal quadrature variance, which is phase independent, is not affected by the frequency changes of the free field. The quadrature variance of principal squeezing reads

$$\langle(\Delta\hat{Q}^{(p)})^2\rangle=-1+2[\langle\Delta\hat{a}\Delta\hat{a}^{\dagger}\rangle-|\langle(\Delta\hat{a})^2\rangle|]\,. \tag{4.1}$$

For the squeezed and displaced Fock state $|\beta,M\rangle_g$ the principal squeezing variance is of a simple form

$$\langle(\Delta\hat{Q}^{(p)})^2\rangle=-1+2[(1+2|\nu|^2)M+|\nu|^2+1-|\mu^*\nu|(2M+1)]\,. \tag{4.2}$$

The linear dependence of $\langle(\Delta\hat{Q}^{(p)})^2\rangle$ on M says that for larger M squeezing is achieved for larger $|\nu|$. On increasing $|\nu|$, we can arrive at squeezing until larger values of M. The free evolution of the principal squeezing variance for different values of the angle θ ($\nu=-|\nu|\exp(i2\theta)$, $\theta\in[0,\pi]$) and for the rest parameters fixed is pictorialized in Figs. 5(a) and 5(b) in a neighborhood of the time points $t=0$ and $\pi/2\kappa$, respectively. From Fig. 5(b) it is obvious that in addition to the appearance at $t=0$ and π/κ, the squeezing phenomenon also occurs at $t=\pi/2\kappa$ and in its neighborhood. Similar is the dependence on the angle of displacement $\xi(0)=|\xi(0)|\exp(i\psi)$ under the

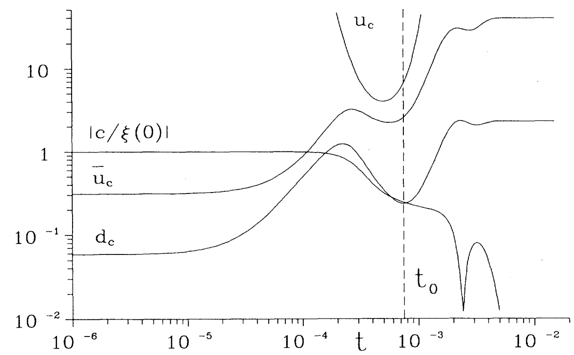

FIG. 3. The distance from the center of curvature to the origin $|c|\,|\xi(0)|^{-1}$, the normalized photon-number variance d_c, and the minimum-uncertainty products $u_c,\bar{u}_c$ in the dependence on t for $\kappa=100$, $\gamma=0$, $\bar{n}_d=0$, $|\xi(0)|=4$, $\psi=0$, $M=1$, $|\nu|=0.1$, and $\theta=\pi/2$.

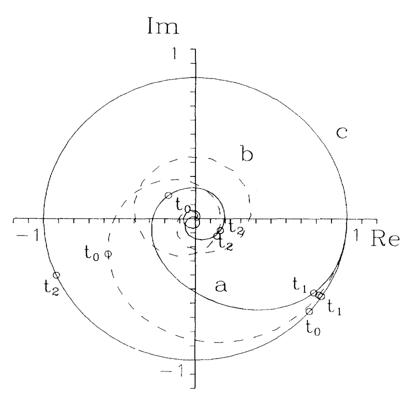

FIG. 2. The plot of curves for $c(t)|\xi(0)|^{-1}$ (curve a), $\langle\hat{a}\rangle(t)|\xi(0)|^{-1}$ (curve b), and $\langle\hat{a}\rangle_r(t)$ (curve c) for $\kappa=100$, $\gamma=0$, $\bar{n}_d=0$, $|\xi(0)|=4$, $\psi=0$, $M=1$, $|\nu|=0.1$, and $\theta=\pi/2$.

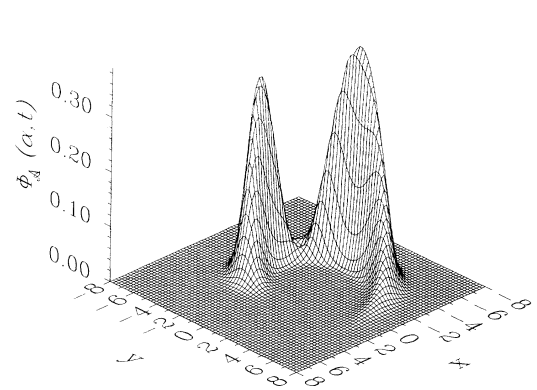

FIG. 4. The quasidistribution $\Phi_{\mathcal{A}}(\alpha,t_0)$ for $\kappa=100$, $\gamma=0$, $\bar{n}_d=0$, $|\xi(0)|=4$, $\psi=0$, $M=1$, $|\nu|=0.1$, $\theta=\pi/2$, and $t_0=7.534\times10^{-4}$.

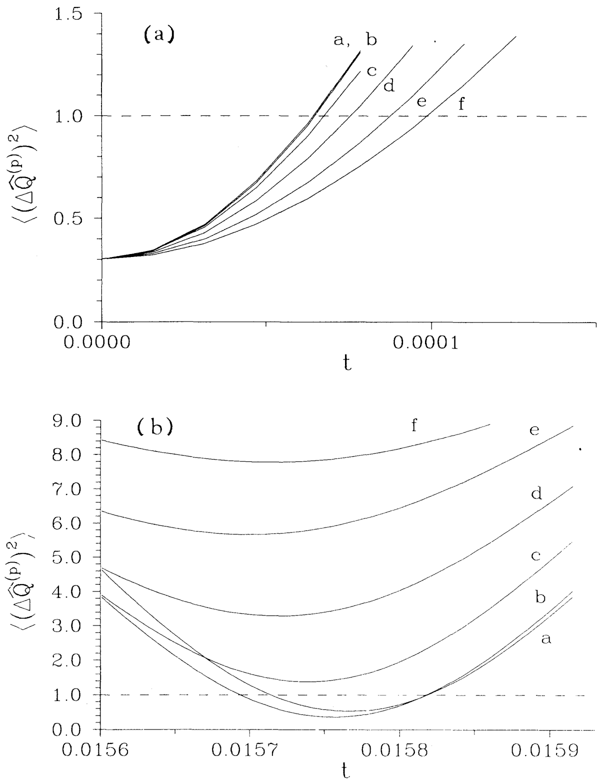

FIG. 5. (a) and (b) The free evolution of the principal squeezing variance in the dependence on $\theta\in[0,\pi/2]$ [$\Delta\theta=\pi/12,\theta=0$ (curve a), $\ldots,\theta=\pi/2$ (curve f)] for $\kappa=100$, $\gamma=0$, $\bar{n}_d=0$, $|\xi(0)|=\sqrt{2}$, $\psi=\pi/16$, $M=1$, and $|\nu|=\sqrt{2}$ in a neighborhood of the time points: (a) $t=0$, (b) $t=\pi/2\kappa$.

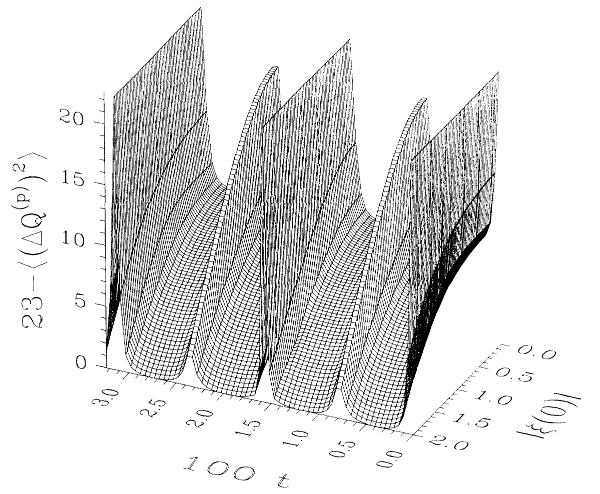

FIG. 6. The effect of $|\xi(0)|\in[0,2]$ on the free evolution of the principal squeezing variance; $\kappa=100$, $\gamma=0$, $\bar{n}_d=0$, $\psi=0$, $M=1$, $|\nu|=\sqrt{2}$, and $\theta=0$.

fixed values of other parameters. For $\psi\in[0,\epsilon)$ squeezing can be observed in a neighborhood of $t=\pi/2\kappa$.

The dependence of the principal squeezing on the magnitude of displacement $\xi(0)$ and that of parameter ν have also been studied. The increase of $|\xi(0)|$ leads to the disappearance of squeezing. This tendency is obvious from Fig. 6. For $\psi=0,\pi,2\pi$ the periodicity of the squeezing is $\pi/2\kappa$, and the curves at $t=\pi/4\kappa$ and $3\pi/4\kappa$ indicate the tendency of $\langle(\Delta\hat{Q}^{(p)})^2\rangle$ to rise. An analogous time dependence could be observed with the effect of $|\nu|$ included. For $\theta=0$ the periodicity of the principal squeezing variance is $\pi/2\kappa$. The case $\nu=0$ corresponds to the initial displaced Fock state $|\xi(0),M\rangle$, whose Kerr evolution has been considered in [30]. No quadrature squeezing in the standard sense has been predicted there, which is in accordance with the observation of no principal squeezing. The amount of squeezing increases with increasing $|\nu|$. The influence of quantum fluctuations on squeezing is illustrated in Fig. 7. The quantum noise attenuates squeezing continuously. In order to evidence all these tendencies, the complementary dependence to the maximum value of the principal squeezing variance is chosen.

Taking into account the formula (2.8) and the coefficients $f_{mn}(t)$ in the lossless case

$$f_{mn}(t)=\exp[i\kappa(m-n)(m+n-1)t]f_{mn}(0)\ , \qquad (4.3)$$

we obtain for the complex field amplitude

$$\langle\hat{a}\rangle(t)=\pi\sum_{n=0}^{\infty}(n+1)!\exp(-i\kappa 2nt)f^{*}_{n+1,n}(0)\ . \qquad (4.4)$$

From (4.4) it is obvious that $\langle\hat{a}\rangle(t)$ is a (π/κ)-periodical function of t for arbitrary initial state. Hence the magnitude of complex field amplitude collapses and revives during the interaction with the nonlinear Kerr medium. The revivals occur at the time points $t=k\pi/\kappa$, with k an integer. The effect of dissipation on the evolution of $|\langle\hat{a}\rangle(t)|$ can be traced in Fig. 8. The quantum noise at-

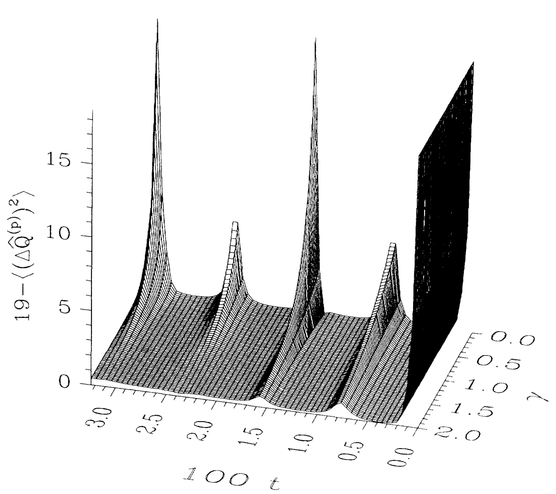

FIG. 7. The attenuation of squeezing under the conditions $\kappa=100$, $|\xi(0)|=\sqrt{2}$, $\psi=0$, $M=1$, $|\nu|=\sqrt{2}$, $\theta=0$, $\gamma\in[0,2]$, and $\bar{n}_d=5$.

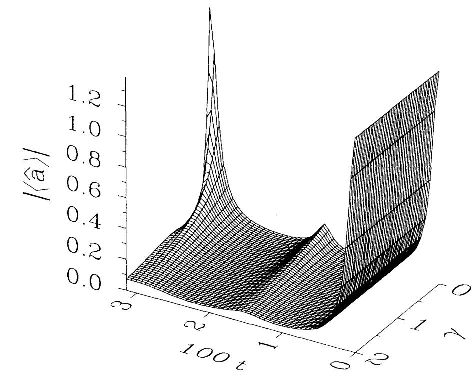

FIG. 8. The effect of dissipation $\gamma\in[0,2]$, $\bar{n}_d=5$ on the evolution of $|\langle\hat{a}\rangle(t)|$ for $\kappa=100$, $|\xi(0)|=\sqrt{2}$, $\psi=0$, $M=1$, $|v|=\sqrt{2}$, and $\theta=0$.

tenuates the revivals analogously as squeezing (for comparison see Fig. 7).

V. GENERATION OF SUPERPOSITION STATES

The possibility of generating quantum-mechanical superpositions of macroscopically distinguishable states in the course of evolution of the anharmonic oscillator has been discussed for initial coherent light [31–34], a two-photon coherent state [35,36], an initial displaced number state [30], and for a phase state [37].

From the Schrödinger equation for the free third-order nonlinear oscillator

$$i\hbar\frac{\partial}{\partial t}|\psi_{\mathcal{N}}(t)\rangle=\hat{H}_{\mathcal{N}}|\psi_{\mathcal{N}}(t)\rangle\ , \tag{5.1}$$

describing the evolution of the system in the interaction picture, we obtain for an initial state $|\psi_{\mathcal{N}}(0)\rangle$,

$$|\psi_{\mathcal{N}}(t)\rangle=\exp[-i\kappa t\hat{n}(\hat{n}-\hat{1})]|\psi_{\mathcal{N}}(0)\rangle\ . \tag{5.2}$$

It is possible to prove that for $t=(L/N)(\pi/\kappa)$, L and N are prime integers, the operators

$$\hat{U}=\exp\left[-i\frac{\pi}{N}\hat{n}(\hat{n}-\hat{1})\right]\ ,\quad \hat{U}_{\mathrm{har}}=\exp\left[-i\frac{\pi}{N}\hat{n}\right] \tag{5.3}$$

form $2N$-cyclic groups, and it holds that

$$\hat{U}^L=\sum_{k=0}^{2N-1}c_k^*\hat{U}_{\mathrm{har}}^k\ , \tag{5.4}$$

where

$$c_k^*=\frac{1}{2N}[1+(-1)^{k+L(N-1)}]\times\sum_{l=0}^{N-1}\exp\left\{-i\frac{\pi}{N}[Ll(l-1)-kl]\right\}\ . \tag{5.5}$$

This explicit expression (5.5) implies that for $L(N-1)$, an even number, it holds that $c_k^*\neq 0$ for k even and $c_k^*=0$ for k odd. For $L(N-1)$, an odd number, it is valid that $c_k^*=0$ for k even and $c_k^*\neq 0$ for k odd. This means that for $t=(L/N)(\pi/\kappa)$ the sum (5.4) reduces to N terms.

Expanding the initial state in the number state basis and applying (5.4), we obtain from (5.2)

$$\left|\psi_{\mathcal{N}}\left(t=\frac{L}{N}\frac{\pi}{\kappa}\right)\right\rangle=\sum_{k=0}^{2N-1}c_k^*|\psi_{\mathcal{N}}(0),\varphi_n^{(k)}\rangle\ , \tag{5.6}$$

where the generalized $\psi_{\mathcal{N}}(0)$ state is defined as follows:

$$|\psi_{\mathcal{N}}(0),\varphi_n^{(k)}\rangle=\sum_{n=0}^{\infty}\exp(i\varphi_n^{(k)})\langle n|\psi_{\mathcal{N}}(0)\rangle|n\rangle\ , \tag{5.7}$$

with

$$\varphi_n^{(k)}=-\frac{\pi}{N}kn\ . \tag{5.8}$$

Hence, every initial state evolves in the lossless Kerr medium at a superposition of its generalizations.

For the squeezed and displaced number state $|\beta,M\rangle_g$ at the input of the nonlinear oscillator we have

$$|\beta,M\rangle_g\left(t=\frac{L}{N}\frac{\pi}{\kappa}\right)=\sum_{k=0}^{2N-1}c_k^*|\beta,M,\varphi_n^{(k)}\rangle_g\ , \tag{5.9}$$

where the generalized squeezed and displaced number states read

$$|\beta,M,\varphi_n^{(k)}\rangle_g=\sum_{n=0}^{\infty}\exp(i\varphi_n^{(k)})\langle n|\beta,M\rangle_g|n\rangle\ . \tag{5.10}$$

The quasidistribution

$$\Phi_{\mathcal{A}}\left(\alpha,t=\frac{L}{N}\frac{\pi}{\kappa}\right)=\left|\langle\alpha|\beta,M\rangle_g\left(t=\frac{L}{N}\frac{\pi}{\kappa}\right)\right|^2=\exp(-|\alpha|^2)\left|\sum_{k=0}^{2N-1}c_k^*\sum_{n=0}^{\infty}\frac{1}{\sqrt{n}}\left[\alpha^*\exp\left(-\frac{\pi}{N}k\right)\right]^n\langle n|\beta,M\rangle_g\right|^2 \tag{5.11}$$

visualizes well the generation of superposition states. It exhibits regular structures when the component states are entangled, as is obvious from Fig. 9, where the evolution time is chosen as the fraction L/N of the period. In Fig. 9(b) an increasing effect of interference terms can be observed when the number of component states is larger than $N_{\max}$.

The phase distribution

$$P(\varphi)\equiv P(\varphi,t)=\tfrac{1}{2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sqrt{m!n!}\exp[i(m-n)\varphi]f_{mn}(t) \tag{5.12}$$

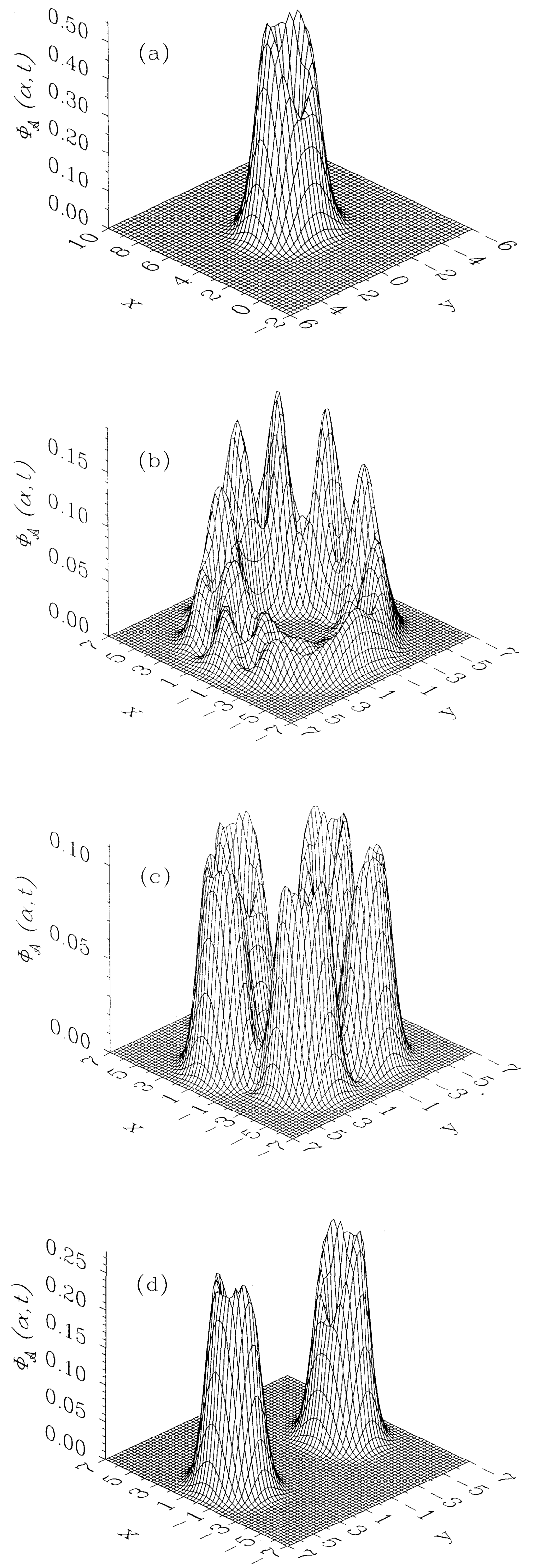

FIG. 9. The quasidistribution $\Phi_{\mathcal{A}}(\alpha,t)$ for $\kappa=100$, $\gamma=0$, $\bar{n}_d=0$, $|\xi(0)|=4$, $\psi=0$, $M=1$, $|\nu|=0.1$, and $\theta=\pi/2$: (a) $t=0$, (b) $t=\frac{1}{10}\pi/\kappa$, (c) $t=\frac{1}{5}\pi/\kappa$, and (d) $t=\frac{1}{2}\pi/\kappa$.

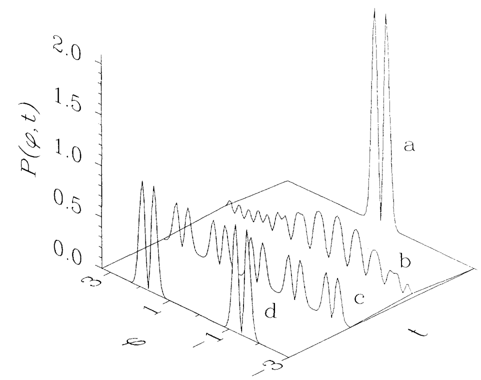

FIG. 10. The phase distribution $P(\varphi,t)$ for $\kappa=100$, $\gamma=0$, $\bar{n}_d=0$, $|\xi(0)|=4$, $\psi=0$, $M=1$, $|\nu|=0.1$, and $\theta=\pi/2$: $t=0$ (curve *a*), $t=\frac{1}{10}\pi/\kappa$ (curve *b*), $t=\frac{1}{5}\pi/\kappa$ (curve *c*), and $t=\frac{1}{2}\pi/\kappa$ (curve *d*).

indicates distinctly superpositions of generalized squeezed and displaced number states, because it exhibits k—two-peak rotational symmetry in the case of superpositions of k states.

We will illustrate graphically in Fig. 10 the evolution of the phase distribution for the same values of parameters as in Fig. 9. This distribution as well as the quasidistribution $\Phi_{\mathcal{A}}(\alpha,t)$ split into the sums of their counterparts for the individual components of the superposition. In the case of phase distribution, these counterparts have $(M+1)$ peaks, the property typical of the displaced number state $|\xi(0),M\rangle$ [11,13]. This is convincingly seen when we plot the contours of the phase distribution versus φ in the polar coordinate system as is obvious from Fig. 11. $(M+1)$ lobes characterize the displaced number state $|\xi(0),M\rangle$. Dissipation prevents the generation of superposition states, as is demonstrated on the phase distribution in Fig. 12.

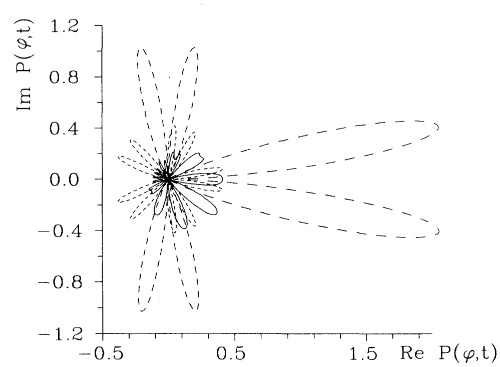

FIG. 11. The plot of the phase distributions $P(\varphi,t)$ in the polar coordinates for $t=0$ (the line with the longest dash), $t=\frac{1}{10}\pi/\kappa$ (the full line), $t=\frac{1}{5}\pi/\kappa$ (the line with the shortest dash), $t=\frac{1}{2}\pi/\kappa$ (the line with a longer dash) and $\kappa=100$, $\gamma=0$, $\bar{n}_d=0$, $|\xi(0)|=4$, $\psi=0$, $M=1$, $|\nu|=0.1$, and $\theta=\pi/2$.

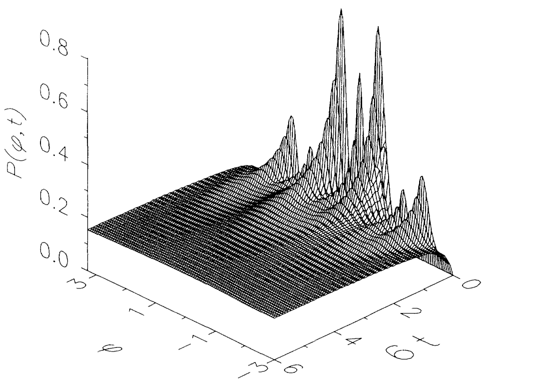

FIG. 12. The dissipative evolution of the phase distribution $P(\varphi,t)$ for $\kappa=100$, $\gamma=1$, $\bar{n}_d=1$, $|\xi(0)|=2$, $\psi=0$, $M=2$, $|\nu|=0.1$, and $\theta=\pi/2$.

VI. CONCLUSION

The evolution of squeezed and displaced number states in the free and dissipative third-order nonlinear oscillator has been investigated. Statistical properties of resulting states have been expressed in terms of the coefficients of the quasidistribution of the complex field amplitude related to the antinormal ordering of field operators. A thorough study of crescent states in both the strong and the weak sense generated in this optical system has been based not only on the Fano factor as a measure of the sub-Poissonian behavior but also on a number-phase intelligence. The occurrence of the principal squeezing of vacuum fluctuations and its dependence on the parameters of the initial state have been discussed. The generation of superpositions of generalized squeezed and displaced number states during the free evolution has been studied and evidenced with the aid of the quasidistribution of the complex field amplitude related to the antinormal ordering of field operators and of the phase distribution. A destructive effect of losses on quantum coherence has been demonstrated. The application of squeezed and displaced number states at the input of the Kerr medium reveals many nonclassical phenomena.

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