Bound states in the continuum from supersymmetric quantum mechanics

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Starting from a potential with a continuum of energy eigenstates, we show how the methods of supersymmetric quantum mechanics can be used to generate families of potentials with bound states in the continuum (BIC's). We also find the corresponding wave functions. Our method preserves the spectrum of the original potential except it adds these discrete BIC's at selected energies. Specifically, we compute and graph potentials which have bound states in the continuum starting from a null potential representing a free particle and from both the attractive and the repulsive Coulomb potentials.

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I. INTRODUCTION

In 1929, von Neumann and Wigner [1] realized that it was possible to construct potentials which have quantum-mechanical bound states embedded in the classical energy continuum (BIC's). Further developments by many authors [2–6] have produced more examples and a better understanding of the kind of potential that can have such bound states, although as yet there is no fully systematic approach. These authors have also suggested possible applications to atoms and molecules. Furthermore, these works have shown that such BIC's appear mainly in certain oscillatory potentials whose envelopes fall off fast enough to lead to normalizable wave functions but sufficiently slowly such that the different maxima are able to "conspire" to create a captive state. Friedrich and Wintgen [5] have given the example of two conspiring resonances and also of a hydrogen atom in a uniform magnetic field. Robnik [6] has shown in a similar way that a simple separable Hamiltonian can develop bound states in the continuum. In his examples too, coupled channels are responsible for the creation of the BIC. Such a BIC is a very fragile structure—a small perturbation of the potential transforms it into a decaying resonance. Nevertheless, Capasso *et al.* [7] have recently reported direct evidence for BIC's by constructing suitable potentials using semiconductor heterostructures grown by molecular-beam epitaxy. Finally, it is interesting to note that BIC's have found their way into a text [8], illustrating for students the surprising possibility of the existence of quantum-mechanical bound states in the classical continuum.

Recently, extensive work has been devoted to generating isospectral potentials by the methods of supersymmetric quantum mechanics (SUSYQM) [9–14]. Starting from the Schrödinger equation for a potential, whose ground-state wave function is known, this method permits one to generate families of new potentials, which may look quite different from the original one, but have exactly the same spectrum [12–14]. These methods are based on procedures invented by Darboux [15] and generalized by Crum [16]. In this paper we extend the usual SUSYQM formalism for obtaining isospectral potentials and apply it to potentials with a continuum of scattering states to generate new potentials with bound states in the continuum. We show that, while the wave functions in the continuum of the original potential are non-normalizable, the ones generated by SUSYQM are normalizable thus representing a bound state. In particular, we construct one-parameter and two-parameter families of supersymmetric partner potentials with one and two bound states in the continuum.

In Sec. II, we present the development of the one-parameter family in SUSYQM to generate potentials with a single bound state in the continuum. The procedure is valid for any spherically symmetric potential $V(r)$ which vanishes as $r \to \infty$. We illustrate our procedure with two explicit examples: (a) free particle $V \equiv 0$ on the half-line and (b) attractive and repulsive Coulomb potentials. In Sec. III, we generalize the SUSYQM method to the two-parameter family of isospectral potentials and show that it generates two bound states in the continuum. Section IV contains a summary of our results. The method can readily be extended to an arbitrary number of BIC's.

II. THE ONE-PARAMETER FAMILY OF BIC'S

The radial s-wave Schrödinger equation for the reduced wave function $u(r)$ (in units where $\hbar = 2m = 1$) is

$$-u'' + V(r)\, u(r) = E\, u(r), \tag{1}$$

where we have scaled the energy and radial variables such that all quantities are dimensionless. A prime denotes differentiation with respect to r. For any potential that vanishes at infinity, Eq. (1) has a classical continuum of positive energy solutions which are clearly not normalizable.

Using the formalism of SUSYQM and the Darboux [15] procedure for deleting and then reinstating the ground state $u_0(r)$ of a potential $V(r)$, one can generate a family of potentials $\hat{V}(r; \lambda)$ which have the same eigenvalues as $V(r)$. These isospectral potentials are labeled by a real parameter λ. Since u_0 here is normalizable, the pa-

rameter λ can vary in the ranges $\lambda > 0$ or $\lambda < -1$. The isospectral potential $\hat{V}(r; \lambda)$ is given in terms of the original potential $V(r)$ and the original ground-state wave function $u_0(r)$ by [12–14]

$$\hat{V}(r; \lambda) = V(r) - 2[\ln(I_0 + \lambda)]''$$
$$= V(r) - \frac{4 u_0 u_0'}{I_0 + \lambda} + \frac{2 u_0^4}{(I_0 + \lambda)^2}, \quad (2)$$

where

$$I_0(r) \equiv \int_0^r u_0^2(r') dr'. \quad (3)$$

Let us recall the main steps in obtaining Eq. (2). First, one writes the Hamiltonian $H = -\frac{d^2}{dr^2} + V(r)$ in factorized form $H = A^\dagger A$, with the operators

$$A = \frac{d}{dr} + W(r), \quad A^\dagger = -\frac{d}{dr} + W(r).$$

The superpotential is given by $W(r) = -u_0'/u_0$. The supersymmetric partner Hamiltonian is

$$H_+ = AA^\dagger = -\frac{d^2}{dr^2} + V_+(r),$$

where

$$V_+(r) = W^2 + W' = V(r) - 2\left(\frac{u_0'}{u_0}\right)'.$$

If the potential $V(r)$ has eigenfunctions $u_n(r)$ at energies E_n, then the SUSY partner potential $V_+(r)$ has the same energy eigenvalues as $V(r)$ with eigenfunctions $Au_n(r)$, except that there is no ground state at $E = 0$ since $Au_0(r) = 0$. This is the standard procedure for deleting the ground state and obtaining $V_+(r)$. To reinsert the ground state, one asks for the most general superpotential $\hat{W}(r)$ such that

$$V_+(r) = \hat{W}^2 + \hat{W}'$$

and this can be shown to be [17]

$$\hat{W}(r; \lambda) = W(r) + \frac{d}{dr} \ln [I_0(r) + \lambda],$$

with I_0 given in Eq. (3). Thus the entire family of potentials $\hat{V}(r; \lambda) = \hat{W}^2(r; \lambda) - \hat{W}'(r; \lambda)$ has the same supersymmetric partner potential $V_+(r)$ obtained by deleting the ground state.

In all previous work, u_0 was taken to be the nodeless, normalizable ground-state wave function of the starting potential $V(r)$. However, for the purposes of this paper, we can generalize the above equations to the case where $u_0(r)$ is any solution of Eq. (1) with arbitrary energy E_0. If $u_0(r)$ has nodes, this leads to singular superpotentials and to singularities in the partner potential $V_+(r)$. However, when the original state at E_0 is reinserted, the resulting family of potentials $\hat{V}(r; \lambda)$ is free of singularities [18]. Our results are best summarized in the following statement.

Theorem. Let $u_0(r)$ and $u_1(r)$ be any two nonsingular solutions of the Schrödinger equation for the potential $V(r)$ corresponding to arbitrarily selected energies E_0 and E_1, respectively. Construct a new potential $\hat{V}(r; \lambda)$ as prescribed by Eq. (2). Then, the two functions

$$\hat{u}_0(r; \lambda) = \frac{u_0(r)}{I_0 + \lambda} \quad (4)$$

and

$$\hat{u}_1(r; \lambda) = (E_1 - E_0)u_1 + \hat{u}_0 \, W(u_0, u_1) \quad (5)$$

[where W denotes the Wronskian function $W(u_0, u_1) \equiv u_0 u_1' - u_1 u_0'$] are solutions of the Schrödinger equation for the new potential $\hat{V}(r; \lambda)$ corresponding to the same energies E_0 and E_1.

While the new potential in Eq. (2) and the new wave functions in Eq. (4) were originally inspired by SUSYQM, the easiest proof of the above theorem is by direct substitution. One simply computes $-\hat{u}_i'' + \hat{V}(r; \lambda)\hat{u}_i$ (i=0,1), with the wave functions $\hat{u}_i$ given in the theorem. After straightforward but tedious algebraic manipulations, one gets $E_i \hat{u}_i$, thus establishing the theorem. The algebra is considerably simplified by using the following identity for the Wronskian function of two solutions of the Schrödinger equation:

$$\frac{d}{dr} W(u_0, u_1) = (E_0 - E_1)u_0 u_1. \quad (6)$$

In the present work, we take u_0 to be a scattering solution at a positive energy $E_0 = k^2$ of a potential $V(r)$ which vanishes at $r = \infty$. Taking $u_0(r = 0) = 0$ satisfies one of the required boundary conditions, but clearly u_0 oscillates as $r \to \infty$ and has an amplitude which does not decrease. Consequently, the integral $I_0(r)$ in Eq. (3) now grows like r at large r and $\hat{u}_0$ is now square integrable for $\lambda > 0$, while the original wave function u_0 was not. Negative values of λ are no longer allowed. Therefore, we see that all the potentials $\hat{V}(r; \lambda)$ have a BIC with energy E_0. Note from Eq. (4) that $\hat{u}_0$ has the same zeros as the original u_0. At zeros of u_0, $\hat{V}(r; \lambda)$ and $V(r)$ are equal. All the other oscillatory solutions of the Schrödinger equation with $V(r)$ get transformed into oscillatory solutions to the new Schrödinger equation with $\hat{V}(r; \lambda)$ with the same energy. In particular, note that $\hat{u}_1(r; \lambda)$ remains a non-normalizable scattering solution of the corresponding Schrödinger equation.

We note that the new potential $\hat{V}(r; \lambda)$ in Eq. (2) and the BIC at energy E_0 are formed using the corresponding wave function $u_0(r)$. Any other state, say $u_1(r)$, is transformed into a solution of the new Schrödinger equation by the operation given in Eq. (4) which involves both u_0 and u_1. Column 3 of Table I gives a convenient overview of the relationship of the potentials V and $\hat{V}$ and the solutions of the corresponding Schrödinger equations.

We now give two examples to explicitly illustrate how one applies the above procedure to obtain potentials possessing one BIC.

A. Free particle on the half-line

Here we consider the case $V \equiv 0$, the free particle on the half-line $0 \le r < \infty$. We choose $u_0 = \sin kr$, the

TABLE I. The one-parameter family of potentials $\hat{V}(r;\lambda)$ (central column) depends on the parameter λ and has one bound state in the continuum at energy E_0 with wave function $\hat{u}_0$. Note that all other new states represented by $\hat{u}_1$ at E_1 are not normalizable. The right column shows the two-parameter family of potentials $\hat{\hat{V}}(r;\lambda,\lambda_1)$, depending on the parameters λ and λ_1, which now has two normalizable states $\hat{\hat{u}}_0$ and $\hat{\hat{u}}_1$ in the continuum. Both families of potentials are generated from the non-normalizable scattering states u_0 and u_1 of the original potential $V(r)$. Using the theorem described in the text, in the first step one produces a BIC at energy E_0 and in the second step a BIC at energy E_1. While it is customary to denote the lower-energy state by E_0, this is not necessary for our approach; E_0 can also be above E_1.

		Potentials	
$V(r)$		$\hat{V} = V - 2[\ln(I_0 + \lambda)]''$	$\hat{\hat{V}} = \hat{V} - 2[\ln(\hat{I}_1 + \lambda_1)]''$
		Wave functions	
E_1	u_1	$\hat{u}_1 = (E_1 - E_0)u_1 + \hat{u}_0 W(u_0, u_1)$	$\hat{\hat{u}}_1 = \dfrac{1}{\hat{I}_1 + \lambda_1}\hat{u}_1$
E_0	u_0	$\hat{u}_0 = \dfrac{1}{I_0 + \lambda}u_0$	$\hat{\hat{u}}_0 = (E_0 - E_1)\hat{u}_0 + \hat{\hat{u}}_1 W(\hat{u}_1, \hat{u}_0)$

spherical wave solution, corresponding to energy $E_0 = k^2 > 0$, which vanishes at $r = 0$. The integral I_0 given in Eq. (3) becomes

$$I_0 = [2kr - \sin(2kr)]/(4k). \tag{7}$$

We observe that $I_0 \to r/2$ as $r \to \infty$.

The potential family $\hat{V}$, defined in Eq. (2), becomes

$$\hat{V}(r;\lambda) = \frac{32\,k^2\,\sin^4 kr}{D_0^2} - \frac{8\,k^2 \sin(2kr)}{D_0} \tag{8}$$

with

$$D_0(r;\lambda) = 2kr - \sin(2kr) + 4k\lambda. \tag{9}$$

$\hat{V}$ has a BIC at energy $E_0 = k^2$ with wave function

$$\hat{u}_0(\lambda) = 4k\,\sin kr/D_0. \tag{10}$$

For special values of the parameters k and λ, the potential $\hat{V}$ and its BIC wave functions are shown in Figs. 1(a) and 1(b). The original null potential has now become an oscillatory potential which asymptotically has a $1/r$ envelope. The new wave function at $E_0 = k^2$ also has an additional damping factor of $1/r$ which makes it square integrable. As u_0 appears in the numerator of $\hat{V}$, Eq. (2), every node of $\hat{u}_0$ is associated with a node of $\hat{V}$, but not every node of $\hat{V}$ produces a node of $\hat{u}_0$. The value of the eigenenergy E_0 clearly is above the asymptotic value, zero, of the potential. Evidently, the many oscillations of this potential, none of them able to hold a bound state, conspire in such a way as to keep the particle trapped.

The parameter λ which appears in the denominator function $D_0(r;\lambda)$ plays the role of a damping distance; its magnitude indicates the value of r at which the monotonically growing integral I_0 becomes a significant damping factor, both for the new potential and for the new wave function. This is illustrated graphically in Figs. 1(a) and 1(b), which are drawn for very different values of λ. (Note that the wave functions shown in the figures are not normalized.) The parameter λ must be restricted to values greater than zero in order to avoid infinities in $\hat{V}$ and in the wave functions. In the limit $\lambda \to \infty$, $\hat{V}$ becomes identical to V.

B. Coulomb potential

Starting from the potential $V = Z/r$, for either positive or negative values of Z, one can easily construct the one-parameter family of isospectral potentials possessing a normalizable positive energy wave function. Here the unbound, reduced $l = 0$ wave function satisfies the
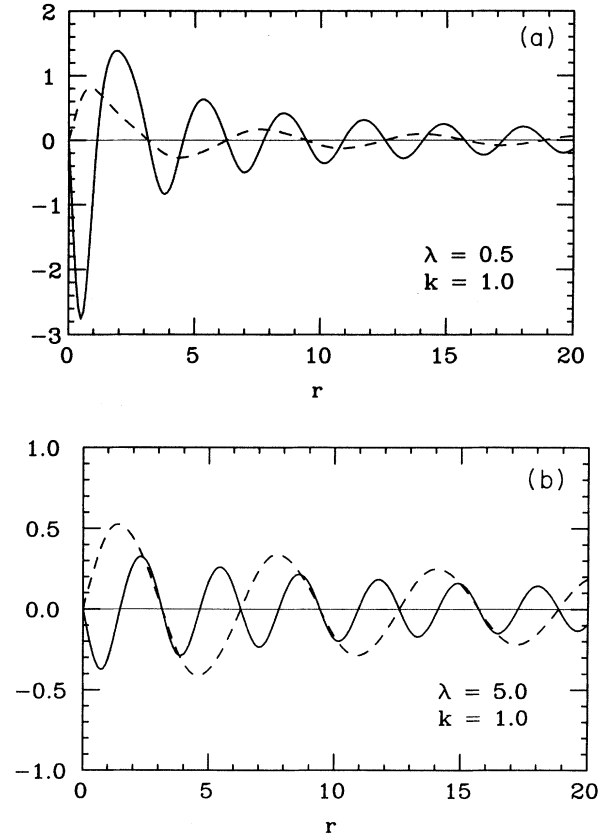

FIG. 1. Two examples of potentials $\hat{V}(r)$ (solid) and the associated BIC wave functions $\hat{u}_0(r)$ (dashed) in the one-parameter family starting from $V(r) = 0$ for $k = 1.0$. (a) is for small λ ($\lambda = 0.5$) and (b) for large λ ($\lambda = 5.0$).

Schrödinger equation Eq. (1), which can be written in standard form

$$u_0'' + \left(1 - 2\frac{\tilde{\eta}}{\rho}\right)u_0 = 0 \tag{11}$$

with $\rho = \sqrt{E}r$ and $\tilde{\eta} = Z/2\sqrt{E}$.

For both positive and negative $\tilde{\eta}$, the solutions involve confluent hypergeometric functions which in the asymptotic limit approach sine waves phase shifted by a logarithmic term. Useful expressions for these solutions in the regions near and far from the origin are available in the literature [19, 20]. Stillinger and Herrick [2], following the method of von Neumann and Wigner [1], have constructed BIC potentials and wave functions for the case of the repulsive Coulomb potential. Here we use our theorem to construct a one-parameter family of isospectral potentials containing a BIC. The procedure is the same for both positive and negative Z, the only difference being in the sign of $\tilde{\eta}$. The formal expressions for the BIC potentials and wave functions have been given above, Eqs. (2) and (4), in terms of u_0.

For both the attractive and the repulsive Coulomb potentials, the positive energy solution of Eq. (11) can be written in the usual form [19, 20, 2] as the real function

$$u_0(\rho) = C_0(\tilde{\eta})\, e^{-i\rho} M(1 - i\tilde{\eta}, 2, 2i\rho), \tag{12}$$

where

$$C_0(\tilde{\eta}) = (e^{-\pi\tilde{\eta}/2})\, |\,\Gamma(1 + i\tilde{\eta})\,| \tag{13}$$

and $M(a, b, z)$ is Kummer's function. Using tabulated expressions for the Coulomb wave functions [20] and doing the integral for I_0 numerically, we have obtained the BIC wave functions for representative values of λ. The corresponding one-parameter family of potentials obtained by the SUSY procedure is given in Eq. (2) with $V_0 = Z/r$.

The results are displayed in Figs. 2 and 3. Figure 2(a) shows the BIC partner to the attractive Coulomb potential for $\lambda = 1$, $k = 1$, and $Z = -2$. Figure 2(b) shows the (unnormalized) wave function of the bound state in the continuum for this potential at $E_0 = k^2$. Figure 3(a) shows the BIC partner potential to a repulsive Coulomb potential for $\lambda = 1$, $k = 1$, and $Z = 6$, while Fig. 3(b) shows the corresponding wave functions. For comparison the original Coulomb potentials and wave functions are also shown by dotted curves. It is seen that the potential which holds a bound state of positive energy shows an oscillatory behavior about the Coulomb potential V_C, as is also evident from the form of Eq. (2) for $\hat{V}$. Since the oscillating component vanishes whenever u_0 vanishes, we have $\hat{V} = V$ at each node of u_0. Compared to the original, unnormalizable wave function, the BIC wave function in both cases shows a damped behavior due to the denominator function. This is also seen in the figures.

A similar behavior is also expected for other radially symmetric potentials with a continuous spectrum of positive eigenvalues. For one-dimensional potentials, the situation is not so clear-cut. Our method works for the Morse potential, which is steeply rising on the negative x axis with correspondingly damped wave functions. It also works for the case of a particle in a constant elec-

tric field for similar reasons. For potentials, such as $V(x) = -V_0\,\text{sech}^2(x)$, the integral I_0, Eq. (3), is not convergent if the starting point is chosen at $-\infty$, and it gets negative contributions if the starting point is selected at finite x values. This leads to a vanishing denominator function in the expressions for some wave functions which makes them unacceptable.

III. THE TWO-PARAMETER FAMILY OF POTENTIALS

In Sec. II, we have seen that a straightforward procedure exists using the SUSY technique for generating a completely isospectral one-parameter family of potentials and that these potentials have a bound state in the continuum if we select as a starting point a positive energy solution of the Schrödinger equation for any potential $V(r)$. We now show how this procedure can be extended to construct two-parameter families which contain two BIC's.

In constructing the new wave functions for the one-parameter family, Eq. (2), we observe that the denominator function given in Eq. (4) was all that was needed
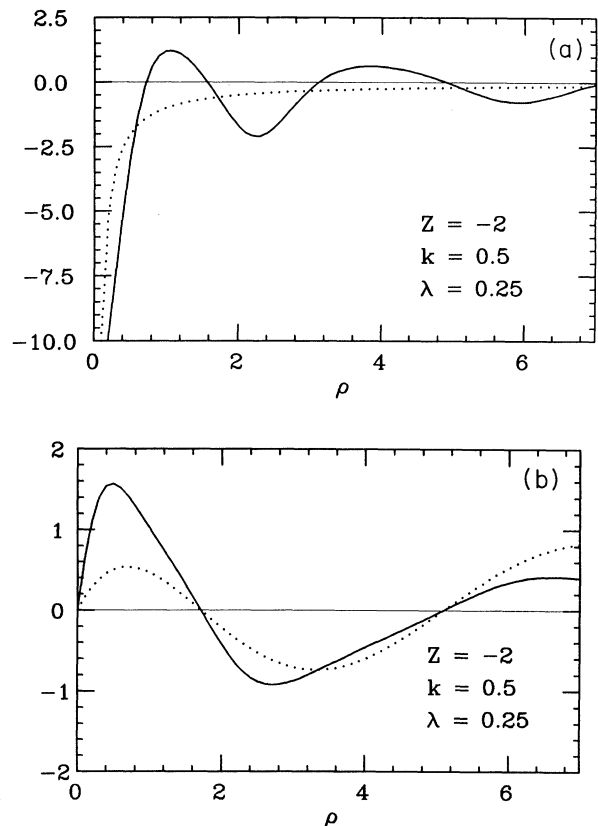

FIG. 2. (a) The BIC potential (solid) derived from the attractive Coulomb potential, which is also shown for comparison (dotted). Observe how the BIC potential oscillates around the original Coulomb potential. (b) The corresponding BIC wave function (solid) and, for comparison, the original Coulomb wave function (dotted). The damping of the BIC wave function, which makes it normalizable, is evident.

to create the BIC, while the operation in Eq. (5) ensured that the wave functions for all the other states, there represented by $\hat{u}_1$, are a solution to the new potential. Note again that there is nothing special about the ordering of the two energy values nor the relative magnitude of E_0 and E_1; therefore we can repeat this procedure by applying the theorem to the wave functions and the potential of the one-parameter family, but this time we transform the state at E_1 into a BIC. The state at E_0, which already is a BIC, is transformed in the step of Eq. (5), suitably modified, to become a solution to the new potential. In this way we obtain the two-parameter family of potentials

$$
\begin{aligned}
\hat{\hat{V}}(r; \lambda, \lambda_1) &= \hat{V} - 2[\ln(\hat{I}_1 + \lambda_1)]'' \\
&= \hat{V} - \frac{4\hat{u}_1 \hat{u}_1'}{\hat{I}_1 + \lambda_1} + \frac{2\hat{u}_1^4}{(\hat{I}_1 + \lambda_1)^2}
\end{aligned} \tag{14}
$$

with the solutions of the corresponding Schrödinger equation

$$
\hat{\hat{u}}_0 = (E_0 - E_1)\hat{u}_0 + \hat{u}_1\, W(\hat{u}_1, \hat{u}_0), \tag{15}
$$

$$
\hat{\hat{u}}_1 = \frac{1}{\hat{I}_1 + \lambda_1}\hat{u}_1, \tag{16}
$$

and

$$
\hat{I}_1 \equiv \int_0^r \hat{u}_1^2(r')dr'. \tag{17}
$$

The precise relationship of the new potential and its wave functions, which are now both BIC's, is illustrated in the last column of Table I.

While the compact form of Eqs. (14)–(16) explicitly shows the method of construction, it is useful to observe that the integral $\hat{I}_1$ can be conveniently recast into a simpler form which contains integrals of the form

$$
I_i = \int_0^r u_i^2(r')dr', \tag{18}
$$

involving the original wave functions only. Making use of Eq. (5) for $\hat{u}_1$, we get

$$
\hat{I}_1 = \int_0^r \left[(E_1 - E_0)^2 u_1^2 + \frac{u_0^2 W^2}{(I_0 + \lambda)^2} \right.
$$
$$
\left. + 2(E_1 - E_0)\frac{u_0 u_1}{(I_0 + \lambda)}W \right] dr'. \tag{19}
$$

The second term is integrated by parts as

$$
\int_0^r \frac{u_0^2}{(I_0 + \lambda)^2}W^2(r')dr' = \frac{-W^2}{I_0 + \lambda}\Big|_0^r + \int_0^r \frac{2WW'}{(I_0 + \lambda)}dr'. \tag{20}
$$

We now use Eq. (6) for the derivative of a Wronskian function of two solutions of the Schrödinger equation to rewrite the second term and observe that it exactly cancels the last term in Eq. (19). We therefore have

$$
\hat{I}_1(r) = \frac{-W^2(r)}{I_0 + \lambda} + (E_1 - E_0)^2\, I_1(r). \tag{21}
$$

Here we have made use of the fact that our boundary conditions imply that $W(0) = 0$.

As an example, we evaluate the two-parameter potential

$$
\hat{\hat{V}} = V - 2\left[\ln\left\{ (I_0 + \lambda)\left((E_1 - E_0)^2 I_1 \right.\right.\right.
$$
$$
\left.\left.\left. - \frac{W^2(r)}{I_0 + \lambda} + \lambda_1 \right) \right\} \right]''. \tag{22}
$$

The argument of the logarithm can be rewritten as

$$
(E_1 - E_0)^2 I_0 I_1 - W^2(r) + \lambda\lambda_1 + \lambda(E_1 - E_0)^2 I_1 + \lambda_1 I_0. \tag{23}
$$

We happen to have transformed first the state at energy E_0 into a BIC and then, in the second step, the state at E_1, which introduced the parameters λ and λ_1. Let us
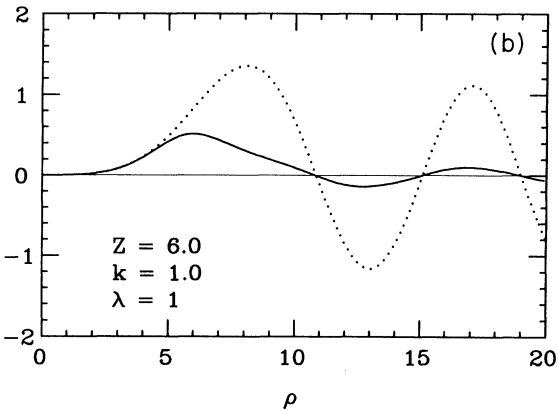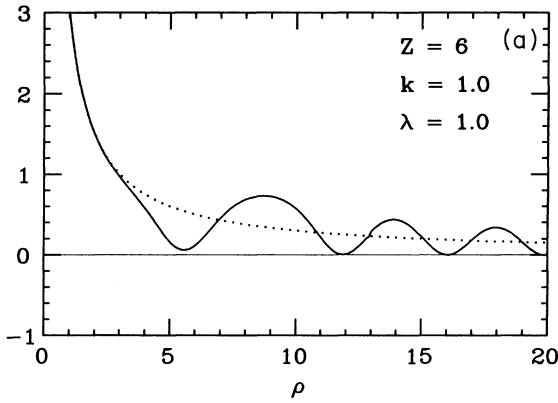
FIG. 3. (a) The BIC potential (solid) derived from the repulsive Coulomb potential, which is also shown for comparison (dotted). Again, the BIC potential oscillates around the original Coulomb potential. (b) The corresponding BIC wave function (solid) and, for comparison, the original Coulomb wave function (dotted). The damping of the BIC wave function, which makes it normalizable, is evident.

now consider applying our procedure in the reverse order, that is, let us first transform the state at energy E_1 into a BIC and then the state at energy E_0, producing the parameters μ and μ_1. For this situation, the argument corresponding to Eq. (23) is

$$(E_1 - E_0)^2 I_0 I_1 - W^2(r) + \mu\mu_1 + \mu_1(E_1 - E_0)^2 I_0 + \mu I_1.$$

$$(24)$$

Clearly, one expects symmetry. This is guaranteed if the parameters are related by

$$\mu = \lambda(E_1 - E_0)^2, \tag{25}$$

$$\mu_1 = \lambda_1/(E_1 - E_0)^2. \tag{26}$$

This also leads to the same two-parameter wave functions. We also note that transforming any state twice by Eq. (4) does not create a second denominator or anything else new, but simply changes the value of the parameter λ as shown in Ref. [12]. Finally, relation (15) ensures that all other eigenstates will be solutions to the new potentials.

Figures 4 and 5 illustrate the appearance of potentials and wave functions for representative choices of λ and λ_1. Clearly, various choices of the parameters λ and λ_1 lead to quite different looking potentials $\hat{V}$. We note in Fig. 5 that, as λ_1 grows large, the two-parameter potential approaches the shape of the one-parameter family shown in Fig. 1 and as discussed analytically above. For easy comparison with other works, we have chosen $k_1 = 2k$ in the figures; however, the ratio of E_1 to E_0 need not be integral.

IV. SUMMARY

We have demonstrated how the SUSY method, originally conceived for discrete spectra, can be generalized for the construction of BIC's. We were able to show how to generate a one-parameter family of potentials which possess a localized positive energy state, starting from
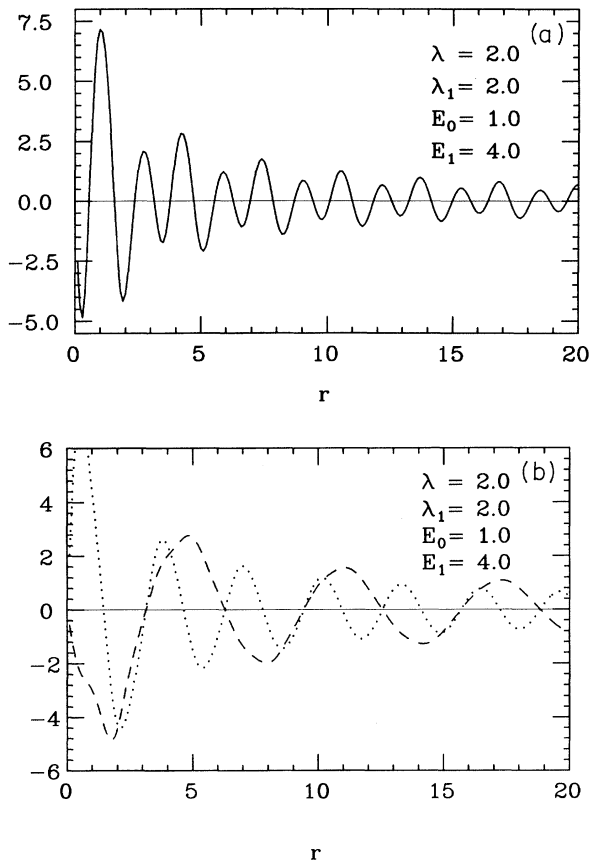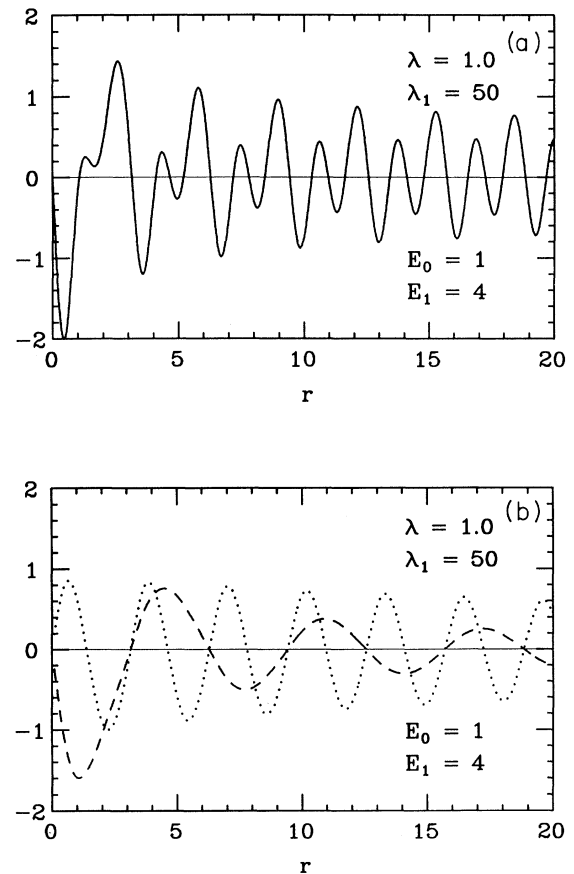

FIG. 4. (a) $\hat{V}$, a typical member of the two-parameter family of BIC potentials obtained from $V(r) = 0$ for $k = 1.0$, $k_1=2.0$, $\lambda = 2.0$, and $\lambda_1 = 2.0$. This potential supports two bound states in the continuum at $E_0 = 1$ and $E_1 = 4$. The associated wave functions are shown in (b), the lower state at E_0 (dashed), the higher one at E_1 (dotted).

FIG. 5. (a) $\hat{V}$, a typical member of the two-parameter family of BIC potentials obtained from $V(r) = 0$ for $k = 1.0$, $k_1=2.0$, $\lambda = 1.0$, and $\lambda_1 = 50$. This potential also supports two bound states in the continuum at $E_0 = 1$ and $E_1 = 4$. The associated wave functions are shown in (b), the lower state at E_0 (dashed), the higher one at E_1 (dotted). Because in this case λ_1 is relatively large, the potential approaches the shape of the one-parameter potential shown in Fig. 1(a). Note that the energy E_1 is above the maximum value of $\hat{V}$.

a potential $V(r)$ which has a continuum of positive energy states. The only requirement that $V(r)$ must satisfy in order that it have such a continuum is that it approaches a constant as $r \to \infty$. Then the solution of the Schrödinger equation with the potential $V(r)$ is oscillatory at large r, which we can take to be of the form $\sin(kr)$. Therefore the integral I_0, Eq. (3), will be of the form of a constant plus $\int_{r_0}^{r} \sin^2(kr')\,dr'$, where r_0 can always be found such that for $r > r_0$ the solution of the Schrödinger equation is approximately proportional to $\sin(kr)$. This means that $I_0 = c_1 + r/2 + \sin(kr)/(4k)$, where c_1 is a constant. Therefore, $\hat{u}_0$, Eq. (4), will vanish at large r as $1/r$, making it a normalizable state. Thus our procedure for constructing a BIC from an initial potential $V(r)$ is actually valid for any spherically symmetric potential which approaches a constant as $r \to \infty$. The situation is more complex for one-dimensional potentials as discussed in the text. The SUSY procedure has in common with the original von Neumann–Wigner

[1] method that it makes the wave functions normalizable by generating a denominator function which grows with r as $r \to \infty$. In the case of $V = 0$, our denominator function, containing I_0, is a special case of the form used by von Neumann and Wigner. We illustrated the one-parameter method for two interesting and analytically solvable cases: $V(r) = 0$, the free particle, and $V(r) = Z/r$, the Coulomb potential. The procedure was readily extended to obtain two-parameter families with two BIC's at arbitrarily selected energies.

ACKNOWLEDGMENTS

[1] J. von Neumann and E. Wigner, Phys. Z. **30**, 465 (1929).

[2] Frank H. Stillinger and David R. Herrick, Phys. Rev. A **11**, 446 (1975).

[3] R. G. Newton, *Scattering Theory of Waves and Particles* (Springer, Berlin, 1982), p. 548ff; B. Simon, Commun. Pure Appl. Math. **22**, 532 (1964).

[4] N. Meyer-Vernet, Am. J. Phys. **50**, 354 (1982).

[5] H. Friedrich and D. Wintgen, Phys. Rev. A **32**, 3231, (1985).

[6] M. Robnik, J. Phys. A **12**, 1175 (1979); **19**, 3845 (1986).

[7] F. Capasso *et al.*, Nature **358**, 5656 (1992).

[8] L. E. Ballentine, *Quantum Mechanics* (Prentice Hall, Englewood Cliffs, NJ, 1990), p. 205.

[9] E. Witten, Nucl. Phys. B **185**, 513 (1981); F. Cooper and B. Freedman, Ann. Phys. (N.Y.) **146**, 262 (1983).

[10] R. Dutt, A. Khare, and U. Sukhatme, Am. J. Phys. **56**, 163 (1988).

[11] Wai-Yee Keung, Uday P. Sukhatme, Qinmou Wang, and Tom Imbo, J. Phys. A **22**, L987 (1989).

[12] A. Khare and U. Sukhatme, J. Phys. A **22**, 2847 (1989).

[13] M. Luban and D. L. Pursey, Phys. Rev. D **33**, 431 (1986); D. L. Pursey, *ibid.* **33**, 1048 (1986); **36**, 1103 (1987).

[14] A. Khare and U. Sukhatme, Phys. Rev. A **40**, 6185 (1989).

[15] G. Darboux, C. R. Acad. Sci. (Paris) **94**, 1456 (1882).

[16] M. M. Crum, Q. J. Math. Oxford **6**, 121 (1955).

[17] M. M. Nieto, Phys. Lett. **145**, 208 (1984).

[18] P. K. Panigrahi and U. Sukhatme, Phys. Lett. A **178**, 251 (1993); A. Jevicki and J. P. Rodrigues, Phys. Lett. B **146**, 55 (1984); J. Casahorran and S. Nam, Int. J. Mod. Phys. A **16**, 2729 (1991); J. Casahorran, Phys. Lett. B **156**, 425 (1991).

[19] *Handbook of Mathematical Functions*, Natl. Bur. Stand. Appl. Math. Ser. No. 55, edited by M. Abramowitz and Irene A. Stegun (U.S. GPO, Washington, D.C., 1964).

[20] *Tables of Coulomb Wave Functions*, Natl. Bur. Stand. Appl. Math. Ser. No. 17 (U.S. GPO, Washington, D.C., 1952).