Angular-momentum measurement and nonlocality in Bohm's interpretation of quantum theory

C. Dewdney and Z. Malik
Department of Applied Physics, University of Portsmouth, Portsmouth PO1 2DZ, England
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We present the Bohm theory of hydrogenlike atoms, the measurement of orbital angular momentum, and Einstein-Podolsky-Rosen angular momentum correlations. We use the illustrations to discuss the arguments of von Neumann [described in M. Jammer, *The Philosophy of Quantum Mechanics* (Wiley, New York, 1974)] and Kochen and Specker [J. Math. Mech. **17**, 59 (1967)] against hidden variables.

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I. INTRODUCTION

Formally Bohm's causal interpretation [1] or de Broglie's pilot wave [2] interpretation of quantum mechanics arises when the substitution

$$\psi = Re^{iS/\hbar} \tag{1}$$

is made in the Schrödinger equation:

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi = i\hbar\frac{\partial\psi}{\partial t} \tag{2}$$

and the real and imaginary parts are separated, yielding the equations

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0 \tag{3}$$

and

$$-\frac{\partial S}{\partial t} = -\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} + \frac{(\nabla S)^2}{2m} + V\ , \tag{4}$$

where $\rho = |\psi|^2$ and

$$m\mathbf{v} = \nabla S. \tag{5}$$

Equation (3) expresses the conservation of probability density. The particle is assumed to have a definite, but unknown, position with a momentum given by (5). Equation (4) can be interpreted as a Hamilton-Jacobi type of equation with an extra "quantum potential" term Q,

$$Q = -\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}\ . \tag{6}$$

We also have an expression for the "quantum force" due to the quantum potential,

$$\mathbf{F} = -\nabla(Q+V) = -\nabla V_{\text{eff}}\ . \tag{7}$$

The particle trajectories are the integral curves of (5) and since S is determined by (2), the trajectory of an individual particle and the evolution of its dynamical variables are determined by the development of ψ.

In Bohm's approach to quantum mechanics the real existents are the ψ field and the particle position. A given Hamiltonian, initial ψ field, and initial particle position yield a unique individual motion for the particle. All of the predictions of quantum mechanics can be accounted for in terms of this well-defined conceptual scheme and there is no need to introduce "wave-packet collapse," "multiple splitting universes or consciousnesses," "novel epistemologies," "quantum logics," etc. All that one need assume is that the particle actually possesses a definite value of position and given this value, and the wave function, the values of the particle's additional dynamical variables are also sharply determined. We shall demonstrate this in some detail for the case of orbital angular momentum.

There is no contradiction with Heisenberg's uncertainty relations in the assumption of definite values for both the position and the momentum of the particle (or other sets of noncommuting observables), since the uncertainty relations simply refer to the inevitable statistical scatter in the values obtained for complementary variables in an ensemble of measurements. We shall see that in general the values assigned to observables in the Bohm interpretation of quantum mechanics are not limited to the eigenvalues of the appropriate operators but as we demonstrate in this paper, the measurement of a given observable brings about an evolution of its assigned value towards a particular eigenvalue of the associated operator. For Bohm the fundamental interpretation of the wave function is as a new type of field the role of which is to determine the particle's motion. Probability is not inherent in the conceptual structure of the theory but arises as a consequence of the necessary lack of knowledge of the precise initial particle coordinates. Any attempt to determine more accurately the initial coordinates simply alters the wave function of the system.

Many detailed calculations have now been carried out demonstrating exactly how the Bohm approach works in specific cases [3–10]. In particular, in a previous paper [6] the quantum theory of the measurement of a spin component of a spin-half particle was discussed using the

Bohm-Schiller-Tiomno (BST) [11] extension to the Bohm theory. Detailed modeling was used to show exactly how the trajectories and the spin vector become correlated as the particle passes through an inhomogeneous (Stern-Gerlach) magnetic field. The Einstein-Podolsky-Rosen (EPR) experiment in the form given to it by Bohm (two spin-$\frac{1}{2}$ particles in the singlet state) was also modeled. However, the interpretation of the spin in the manner adopted by BST poses some problems, not resolved to the satisfaction of all, in its extension to the many-body case. The problem lies in the provision of a physical interpretation for all of the degrees of freedom in the many-body spinor wave function. Bohm now treats the spin in a rather different manner. According to the extension of his approach to the Dirac equation, spin is not thought to arise from the rotation of an extended body but rather from an additional orbital rotation of a point particle [12,13]. Here we wish to give descriptions of the measurement process and of EPR correlations that are free from these problems and so we discuss in this paper detailed calculations of the measurement of that magnetic moment arising from the orbital motion of an electron in a hydrogenlike atom and the EPR type of correlations between the orbital motions of two such atoms in a correlated angular-momentum state.

The interpretation of orbital angular momentum is straightforward in the Bohm theory. The general outline has been given by Bohm and Hiley [12]; here we complement this work by presenting detailed calculations.

II. STATIONARY STATES OF THE ELECTRON IN A HYDROGENLIKE ATOM

Solving the Schrödinger equation for an electron in a nuclear Coulomb potential in the usual way, we write the simultaneous eigenstates of L^2 and L_z as

$$\psi_{nlm} = R_{nl}(r)P_l^m(\cos\theta)e^{im\phi}, \tag{8}$$

where the R's are the Laguere polynomials, the P_l^m are the spherical harmonics, and n, l, and m are quantum numbers associated with the total energy, the square angular momentum, and its z component. The proton is taken to be the center of force and its motion within the atom is ignored. For the electron we use spherical polar coordinates (r,θ,ϕ) and the associated unit vectors $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi$, with the proton as the origin. We shall refer to these coordinates as the internal coordinates. Using (5) and (8) the momentum is

$$\mathbf{p} = \nabla S = \frac{m\hbar}{r\sin\theta}\mathbf{e}_\phi \ . \tag{9}$$

The angular-momentum vector is

$$\mathbf{L} = \mathbf{r}\times\mathbf{p} = -\frac{m\hbar}{\sin\theta}\mathbf{e}_\theta \ , \tag{10}$$

with rectangular Cartesian components

$$L_z = m\hbar \ , \tag{10a}$$

$$L_y = -m\hbar\cot\theta\sin\phi \ , \tag{10b}$$

$$L_x = -m\hbar\cot\theta\cos\phi \ , \tag{10c}$$

$$L^2 = m^2\hbar^2(1+\cot^2\theta) \neq \langle L^2\rangle_{\rm QM} \ , \tag{11}$$

where $\langle L^2\rangle_{\rm QM}$ is the usual expectation value according to quantum mechanics. A well-defined trajectory is determined for the eigenstates (8) in which the electron is either standing still ($m=0$), or circling the z axis in a clockwise (m positive) or anticlockwise sense (m negative), where ϕ increases in an anticlockwise direction. The velocity of the electron is inversely proportional to its perpendicular distance from the z axis. The projection of the angular-momentum vector on the z axis is always $m\hbar$; while the projections on the x and y axes fluctuate between $m\hbar\cot\theta$ and $-m\hbar\cot\theta$ as the angular-momentum vector traces out a cone of vertex angle θ about the z axis. Notice immediately that the values assigned to L_x, L_y, and L^2 are not eigenvalues.

In order to understand how this, at first sight, rather odd motion is possible in a Coulomb potential, it is necessary to examine the angle-dependent quantum potential; this can be seen to give rise to the necessary torques. From (4) we have

$$Q = E - V - \frac{m^2\hbar^2}{2m_e r^2\sin^2\theta} \tag{12a}$$

or

$$Q = -\frac{m_e e^4}{2\hbar^2 n^2} + \frac{e^2}{r} - \frac{m^2\hbar^2}{2m_e r^2\sin^2\theta} \ , \tag{12b}$$

where e is the electronic charge and n is the principal quantum number. Using (12b) and (7) the quantum force is given by

$$\mathbf{F} = -\frac{m^2\hbar^2}{m_e r^3\sin^2\theta}(\mathbf{e}_r + \mathbf{e}_\theta\cot\theta) \ , \tag{13a}$$

which means there is a resultant force of strength

$$|\mathbf{F}| = \frac{m^2\hbar^2}{m_e r^3\sin^3\theta} \tag{13b}$$

parallel to the x-y plane towards the z axis. Hence it can be seen that the effective force on the electron is just that necessary to maintain the motion described above. It can also be seen that if m=0 there will be no effective force and the electron remains stationary, the quantum potential simply cancels the Coulomb potential. The motion of the electron in the atom is thus susceptible to a deterministic description in terms of well-defined trajectories [14].

If we assume that in an ensemble of atoms the distribution of initial positions is given by $|\psi|^2$ then according to (3) the distribution will remain $|\psi|^2$ thereafter. Some authors have questioned this assumption insisting that the initial distribution must be arbitrary. (For a recent discussion of this point see [15–17].) Bohm and Vigier [18] addressed this question, showing that by introducing a randomly fluctuating background into the theory a kind of Boltzmann H theorem can be demon-

strated. This H theorem accounts for the decay of any arbitrary distribution of initial positions to one governed by $|\psi|^2$ within a certain relaxation time. An electron in an S state and subject to such fluctuations will not be stationary but will execute a random walk in which the probability density approaches $|\psi|^2$. In this paper we will not consider these fluctuations, we simply calculate the mean motion through (5). Of course a completely isolated atom is an extreme abstraction and in general the state of the atom will be some linear combination of the stationary states. This means that the motion of the electron will be rather more complex than that associated with any of the stationary states.

III. MEASUREMENT OF A COMPONENT OF ANGULAR MOMENTUM

The results described thus far for the electron motion and its angular momentum in particular may seem to contradict the results predicted by quantum mechanics. However, it must be emphasized that the actual value assigned to an observable (strictly, following Bell, a beable [19]) in the Bohm theory will not in general be restricted to one of the appropriate eigenvalues unless the particle is uniquely associated with one eigenstate of that observable (see Sec. V). To ensure that this situation pertains is the function of a measurement interaction.

Our description of the process of measurement concerns the measurement of the z component of orbital angular momentum of an electron in a hydrogenlike atom using a Stern-Gerlach apparatus. We ignore the contribution to the magnetic moment arising from the electron spin and also the deflection of the atom perpendicular to the applied magnetic field.

The description is similar to that given for a spin component [6], but here we do not need to deal with spinor wave functions and the physical model is clear. Consider a beam of hydrogenlike atoms whose spatial location is given by a function $\psi_p(x,y,z,t)$, while the "internal" state (of the electron within the atom) is given by one of the stationary states $\psi_{nlm}(r,\theta,\phi)$ given in (8). The complete wave function is then

$$\Psi(x,y,z,t;r,\theta,\phi) = \psi_p(x,y,z,t)\psi_e(r,\theta,\phi) \tag{14}$$

and this product state indicates that there is no correlation between internal and external coordinates. The function of the Stern-Gerlach device is to bring such a correlation about and an observation of the atom's position should enable one to infer the value of the angular-momentum component measured. In this case the apparatus coordinate can be thought of as the atom's position and the measurement brings about a correlation between the apparatus coordinate and the electron's angular momentum. The hydrogenlike atom interacts with the inhomogeneous magnetic field $\mathcal{H}$, of the Stern-Gerlach magnets, whose gradient is along the z direction. The measurement interaction is assumed to be impulsive, leaving ψ_e unchanged. We appreciate a full treatment would produce deflections in directions other than z, but for our purposes such deflections play no essential role. Letting $\widehat{H}_I$ be the interaction Hamiltonian and $\widehat{H}_p$, $\widehat{H}_e$ be the proton and electron Hamiltonians, respectively, we assume that during measurement $\widehat{H}_p$ and $\widehat{H}_e$ are negligible when compared to $\widehat{H}_I$. Therefore before and after interaction with the magnetic field the state of the system is governed by

$$i\hbar\frac{\partial\Psi}{\partial t} = (\widehat{H}_p + \widehat{H}_e)\Psi \tag{15}$$

and during interaction by

$$i\hbar\frac{\partial\Psi}{\partial t} = \widehat{H}_I\Psi\ , \tag{16}$$

where we will assume that the interaction energy is

$$\widehat{H}_I \cong \mu_B(\mathcal{H}_0 + z\mathcal{H}_0')\widehat{L}_z/\hbar \tag{17}$$

and

$$\mu_B = \frac{e\hbar}{2m_e} \tag{18}$$

is the Bohr magneton. Here the magnetic field strength is $\mathcal{H}$ and $\mathcal{H}_0 = (\mathcal{H}_z)_{z=0}$, $\mathcal{H}_0' = \left(\frac{\partial\mathcal{H}_z}{\partial z}\right)_{z=0}$, see [20].

In order to achieve an unambiguous result, the initial form of the wave function must be a localized wave packet. Since we are only interested in the separation of the atomic wave function in the z direction we shall assume that the atom's wave function ψ_p is a function of z and time only. For the purposes of this description the initial spatial function is taken to be Gaussian centered at profile $z = 0$. The initial state of the electron is taken to be a superposition of the simultaneous degenerate eigenstates of $\widehat{L}^2$ and $\widehat{L}_z$ corresponding to $l = 1$.

The initial wave function is taken to be

$$\Psi_0 = \psi_p(z,0)\sum_{j=-1}^{1} c_j\psi_{ej}$$

$$= \frac{1}{2\sqrt{\pi}}\int g(k)e^{ikz}\sum_{j=-1}^{1} c_j\psi_{ej}\,dk\ , \tag{19}$$

where ψ_{ej} are the eigenfunctions of the hydrogen atom Hamiltonian for the case $l = 1$,

$$\psi_{e0} = (32\pi {a_0}^5)^{-1/2} r(\cos\theta)e^{-r/2a_0}\ , \tag{20a}$$

$$\psi_{e\pm1} = \mp(64\pi {a_0}^5)^{-1/2} r(\sin\theta)e^{-r/2a_0}e^{\pm i\phi}\ , \tag{20b}$$

where a_0 is the Bohr radius and $g(k)$ is a normalized Gaussian packet in momentum space, centered around $k = 0$ and given by

$$g(k) = (2\sigma_0^2/\pi)^{-1/4}e^{-k^2\sigma_0^2}, \tag{21}$$

where σ_0 is the initial spatial half-width of the packet.

Now we solve the Schrödinger equation (16) for the impulsive interaction with the Stern-Gerlach apparatus to

find the form of the wave function on leaving the magnets. Thereafter the motion is governed by (15). As we shall see, although no "classical" forces or fields act on the particles after leaving the magnets, there are quantum forces and torques acting on the proton and electron. The wave function of the particle at a time t after leaving the magnetic field can be written

$$\Psi = \sum_{j=-1}^{1} c_j R_j e^{iS_j/\hbar} \psi_{ej} \ , \tag{22}$$

where R and S are real functions given by

$$R_j = \left(\frac{8}{\pi}\right)^{-1/4} (\sigma_0 \sigma_t)^{-1/2} e^{-\frac{(z+jut)^2}{4\sigma_t^2}} \ ,$$

$$S_j = -\hbar j \left[\triangle + (z + jut/2)\, \triangle'\right] - \varepsilon_t/2 + \frac{\hbar t (z+jut)^2}{2\lambda_t^2}$$

and in the above $\triangle = \mu_B \mathcal{H}_0 \Delta t/\hbar$, $\triangle' = \mu_B \mathcal{H}_0' \Delta t/\hbar$, $u = \hbar \triangle'/m_p$,

$$\frac{1}{\lambda_t^2} = \frac{\hbar t}{4 m_p (\sigma_0 \sigma_t)^2} \ ,$$

$$\sigma_t = \sigma_0 \left(1 + \frac{\hbar^2 t^2}{4 m_p^2 \sigma_0^4}\right)^{1/2} \ ,$$

and

$$\varepsilon_t = \hbar \arctan\left(\frac{\hbar t}{2 m_p \sigma_0^2}\right) \ .$$

Now if we write the total wave function as

$$\Psi = R e^{iS/\hbar} = \sum_{j=-1}^{1} \mathcal{R}_j e^{i\mathcal{S}_j/\hbar}, \tag{23}$$

where $\mathcal{R}_j$ and $\mathcal{S}_j$ represent the real and imaginary parts of the three wave packets then

$$R^2 = \sum_{j,k} \mathcal{R}_j \mathcal{R}_k \cos\left(\mathcal{S}_j/\hbar - \mathcal{S}_k/\hbar\right) \tag{24}$$

and

$$S = \hbar \arctan\left(\frac{\sum_j \mathcal{R}_j \sin\left(\mathcal{S}_j/\hbar\right)}{\sum_k \mathcal{R}_k \cos\left(\mathcal{S}_k/\hbar\right)}\right) . \tag{25}$$

From the above it can be seen that in general the initial wave packet (19) evolves into three packets, u_+, u_-, u_0, with amplitudes given by the $|c_j|^2$ which separate along the z direction. To make specific calculations we need to specify the form of the initial wave function (the c_j's in 22); we take

$$c_{+1} = c_{-1} = \frac{1}{2}, \quad c_0 = \frac{1}{\sqrt{2}} \ , \tag{26}$$

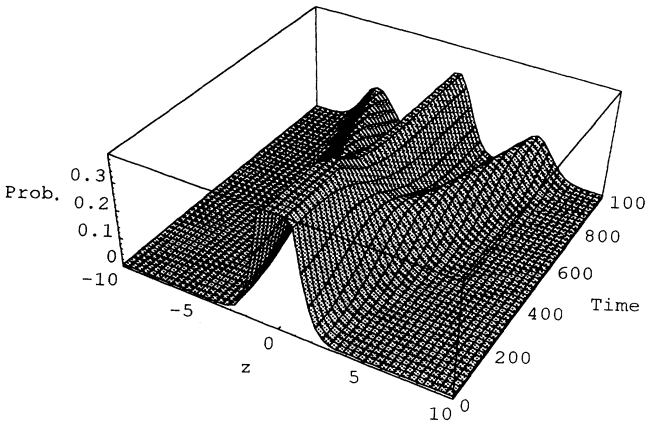

FIG. 1. Evolution in time (arbitrary units) of the atom's position probability density after leaving the Stern-Gerlach field at $t = 0$. The Stern-Gerlach field is in the z direction while the initial state is an eigenstate of $\widehat{L}_x$ with eigenvalue one, $\widehat{L}_x \Psi = +\Psi$.

which corresponds to the choice of an eigenstate of the $\widehat{L}_x$ operator with eigenvalue one. Figure 1 shows the "external" probability distribution of the z coordinate for this wave function. In order to ensure an unambiguous result we need only ensure that the interaction produces packets which completely separate in space, the particle must enter one of the packets, and hence it comes to be associated with a definite value of L_z. In Bohm's approach the manner in which definite results arise in individual experiments is clearly described as a dynamical evolution; individual pointers always have definite positions (and cats are always either dead or alive) even though the system's wave function is a superposition of states.

Given the initial internal (electron) and external (proton) coordinates, as well as the initial wave function, we can calculate the trajectories of the proton and its electron. The trajectories are calculated numerically as the integral curves of

$$v_z = \frac{dz}{dt} = \frac{1}{m_p} \nabla_p S(r, \theta, \phi, z, t) \tag{27}$$

for the atom and

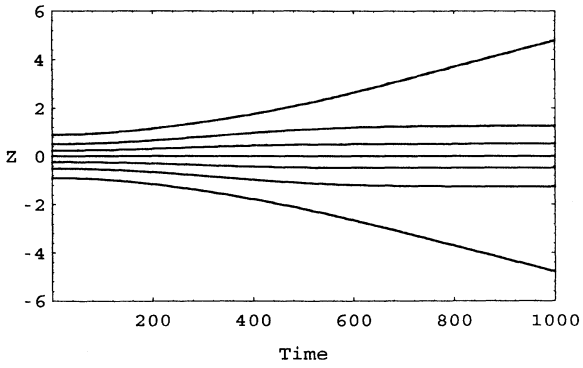

FIG. 2. A set of trajectories corresponding with Fig. 1.

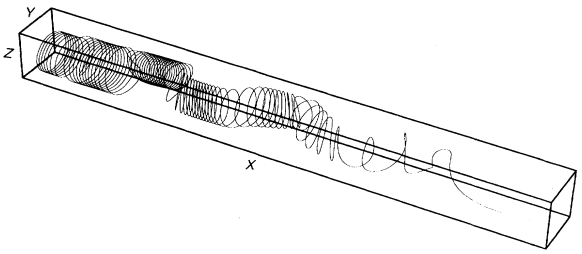

FIG. 3. The motion of the electron as the atom progresses along the path which starts at $z = 0.24$ and ends in the central packet. Initially rotating about the x axis, the electron is finally at rest. Time added parametrically to the x axis.

$$\mathbf{v}_e = \frac{d\mathbf{r}}{dt} = \frac{1}{m_e}\nabla_e S(r,\theta,\phi,z,t) \tag{28}$$

for the electron, where $\mathbf{r}$ is the position vector of the electron relative to the proton. Notice that in general $v_z = v_z(r,\theta,\phi,z,t)$ and $\mathbf{v}_e = \mathbf{v}_e(r,\theta,\phi,z,t)$, showing that the apparatus coordinate z and object are correlated. The components of the electron's velocity, given in polar form, are

$$\dot{r} = \frac{1}{m_e}\frac{\partial S}{\partial r} , \tag{28a}$$

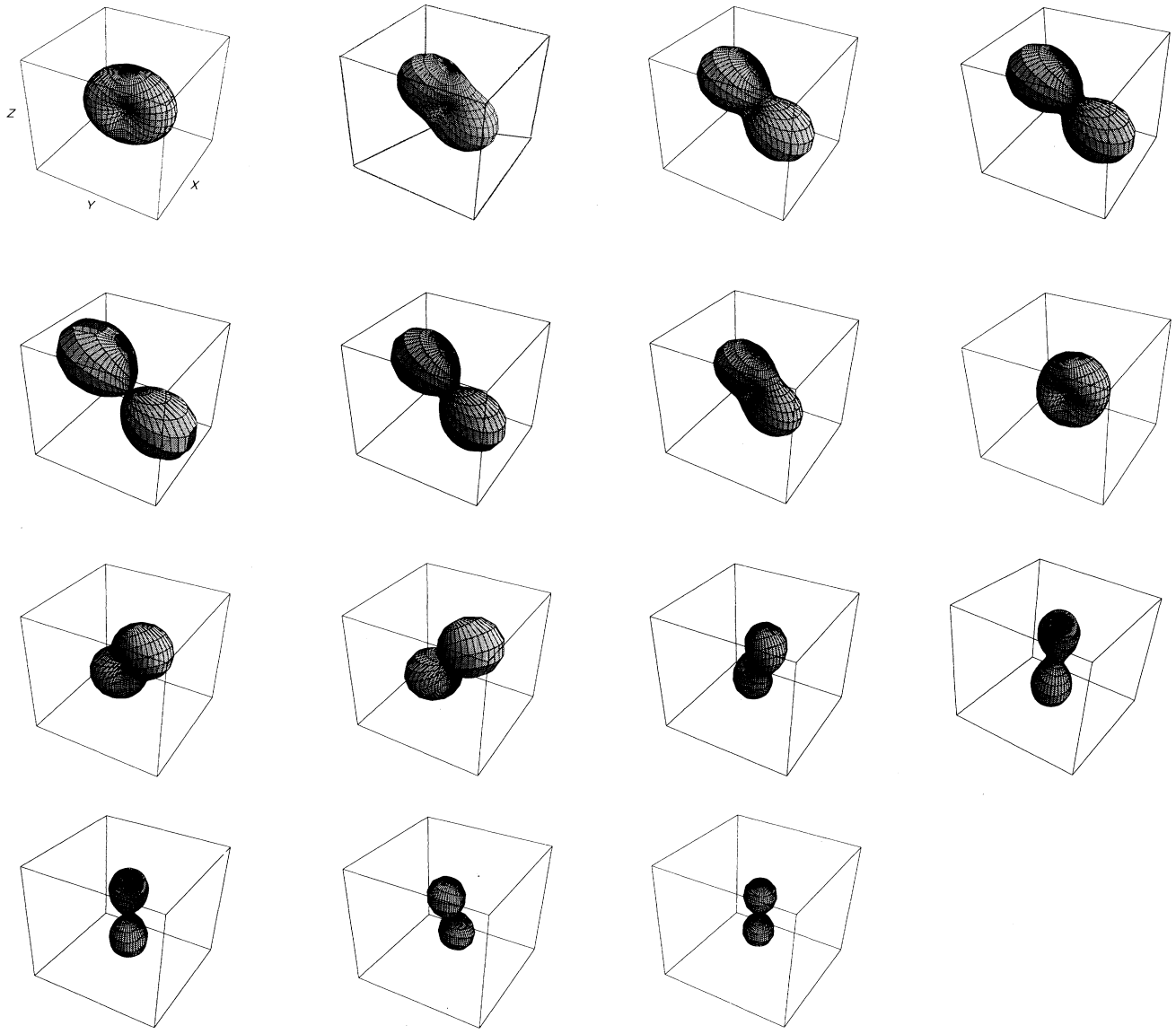

FIG. 4. The evolution of the electron probability density as the atom progresses along the path from $z = 0.24$.

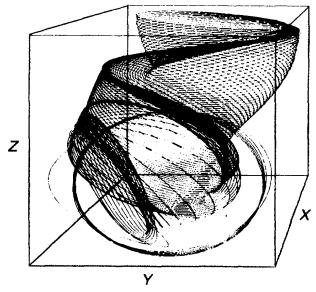

FIG. 5. The motion of the electron as the atom progresses along the path which starts at $z = 0.9$ and ends in the upper packet. Initially rotating about the x axis, the electron is finally rotating about the z axis. Time added parametrically to the x axis.

$$\dot{\theta} = \frac{1}{m_e r^2} \frac{\partial S}{\partial \theta} , \tag{28b}$$

$$\dot{\phi} = \frac{1}{m_e r^2 \sin^2 \theta} \frac{\partial S}{\partial \phi} . \tag{28c}$$

Figure 2 shows a set of representative proton trajectories for this initial state: they correspond to what we associate intuitively with such a measurement of angular momentum. Figure 3 shows the path followed by an electron which ends up in the middle packet (i.e., an $m = 0$ "measurement result" for $\hat{L}_z$). Initially the electron's motion is a rotation around the x axis, as one might expect for an eigenstate in which the x component of angular momentum is $+1$. As time progresses a complex motion ensues as the individual components in the wave func-

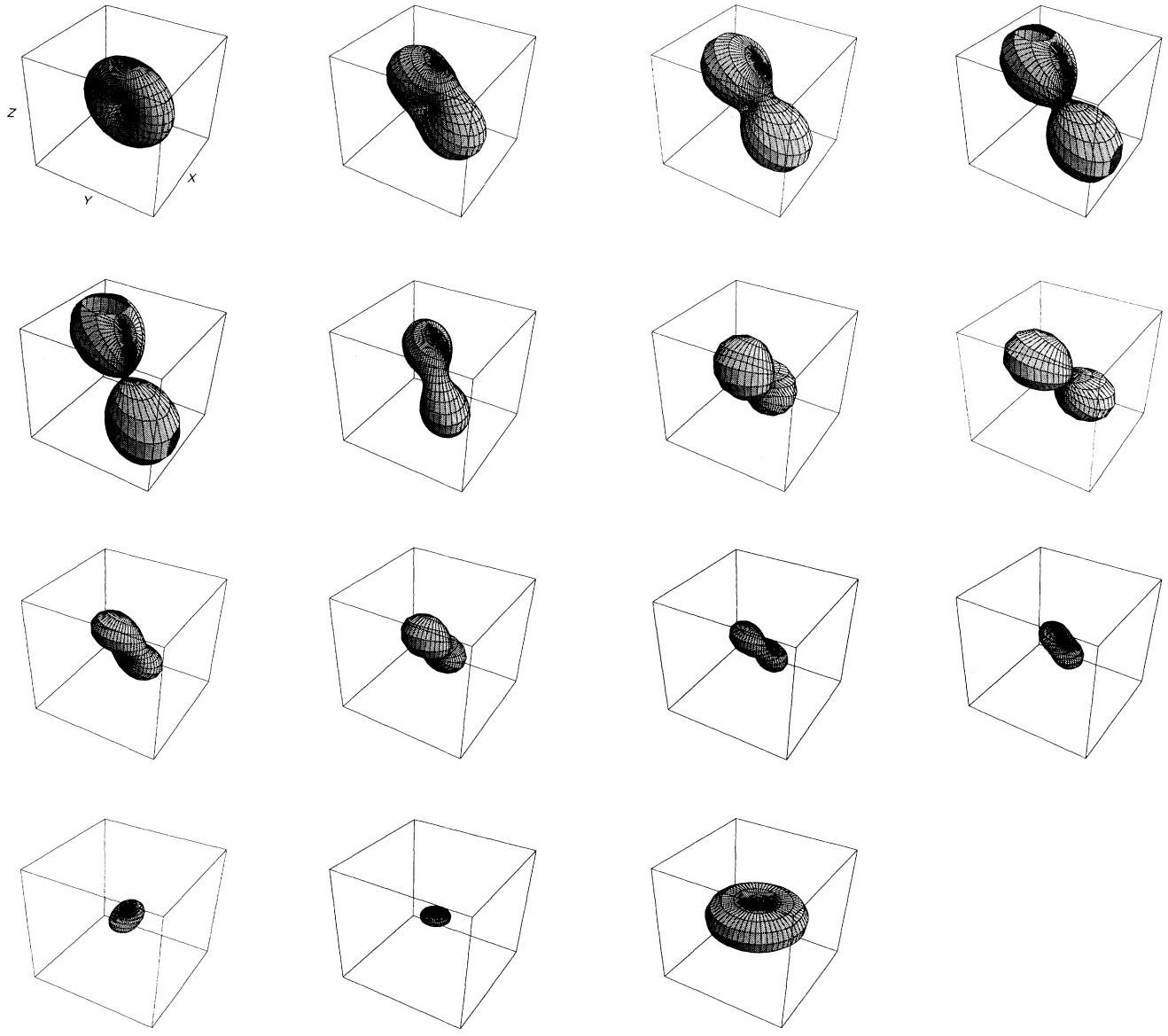

FIG. 6. The evolution of the electron probability density as the atom progresses along the path from $z = 0.9$.

tion corresponding to different z components of angular momentum separate. Eventually the electron no longer rotates about the proton, its motion is correlated with a zero value for the z component of angular momentum. Of course this path is just one of an infinite set of possible paths.

Considering an ensemble of systems, in which the initial value of $\mathbf{r}$ is distributed according to the probability density, we can also show how the internal probability distribution of the electron evolves as the proton proceeds along its path z, t; this is shown in a series of diagrams in Fig. 4 (again for the $m = 0$ case). Each diagram in the set shows the probability distribution of electrons in the ensemble at a particular value of z. This distribution evolves from one associated with the eigenstate of $\widehat{L}_x$ with eigenvalue one to the distribution associated with the eigenstate of $\widehat{L}_z$ with eigenvalue zero. We also show a similar set of figures for a "measurement result" of $m = +1$; the corresponding electron trajectory and internal electron probability distribution are given in Figs. 5 and 6. Figure 5 shows the electron trajectory; the chosen viewpoint clearly shows the turning of the electron's plane of rotation until it lies parallel to the x-y plane. Figure 6 shows how the electron probability density is transformed from that associated with the eigenstate of $\widehat{L}_x$ with eigenvalue one to that associated with the eigenstate of $\widehat{L}_z$ with eigenvalue one as the proton proceeds along its path.

The description given here clearly demonstrates that the Bohm interpretation of quantum mechanics yields a continuous and deterministic description of the process of the measurement of an angular-momentum component. We have considered the case in which the initial wave function is an eigenstate of $\widehat{L}_x$ with eigenvalue one; this is transformed by the measurement process into a superposition of spatially separated eigenstates of $\widehat{L}_z$. During the measurement the system undergoes a transition in which the values assigned to the angular-momentum observables evolve continuously and are not limited to eigenvalues. The function of the measurement interaction is to bring about a motion of the electron which is correlated to the proton's position and hence to the value of the electron angular momentum. In Bohm's description there are no quantum jumps and each system passing through the apparatus emerges with a definite value of L_z even though the system's wave function is a superposition of eigenstates of $\widehat{L}_z$.

It must be remembered that usually nothing can be said about the intrinsic values of observables when the system is not in an eigenstate; the Bohm theory assigns values to observables at all times but these values are not in general eigenvalues. Ultimately the description of any measurement must be given in the configuration space spanned by the system and apparatus coordinates. During the measurement process neither system nor apparatus can be considered separately to have its own wave function. Rather there is one entangled wave function for both. During a measurement the quantum state of the system and the apparatus together evolve into a set of nonoverlapping configuration-space wave packets. Each separate packet consists, after the measurement, of a product of an eigenstate of the observable undergoing measurement and a correlated macroscopically distinguishable apparatus state. The combined system/apparatus coordinate must enter one of these packets and so the value of the observable undergoing measurement is transformed into the corresponding eigenvalue.

IV. EPR

We consider a variation of the EPR [21] argument, which has been considered by a number of authors recently [22], with two H-like atoms, A and B, of angular momentum one in a correlated angular-momentum state. This variation of Bohm's [20] gedanken experiment using two atoms has the advantage of dealing only with orbital angular momentum.

Consider two parallel Stern-Gerlach magnets (SGA and SGB) both aligned to measure the z component of angular momenta, $\widehat{L}_z^A$, $\widehat{L}_z^B$, but spacelike separated with respect to the source. A pair of angular-momentum-one particles is formed at O in a simultaneous eigenstate of the angular-momentum operator in the z direction $(\widehat{L}_z^A + \widehat{L}_z^B)$ and the total angular-momentum operator $(\widehat{L}^A + \widehat{L}^B)^2$ for which both eigenvalues are zero. The particles separate in the x direction and pass through the "Gaussian slits." We consider the measurements to occur simultaneously and the magnets to be identical (nothing of importance is affected by these simplifications; note that we are only dealing with the nonrelativistic case, obviously complications arise when trying to describe simultaneous measurements in the relativistic case). At the entrance to the magnets the initial wave function is

$$\begin{aligned}\Psi_0 &= \psi_p^A(z_A)\psi_p^B(z_B)\frac{1}{\sqrt{3}}\sum_{j=-1}^{1}(-1)^{j+1}\psi_{e+j}^A\psi_{e-j}^B \\ &= \psi_p^A(z_A)\psi_p^B(z_B)\frac{1}{\sqrt{3}}(\psi_{e+1}^A\psi_{e-1}^B + \psi_{e+1}^B\psi_{e-1}^A \\ &\qquad -\psi_{e0}^A\psi_{e0}^B)\ , \end{aligned} \tag{29}$$

where ψ_e^A, ψ_e^B are the electron wave functions of atoms A, B, respectively and ψ_p^A, ψ_p^B are the proton wave functions of atoms A, B respectively. Now we proceed very much as in the single-atom case described above, Fourier analyzing (29) above, solving the Schrödinger equation for the atoms' impulsive interaction with the magnets (this time $\widehat{H}_I = \widehat{H}_I^A + \widehat{H}_I^B$) and then multiplying in the time dependence for the free case when the atoms have left the magnetic fields. We then have

$$\Psi = \sum_{j=-1}^{1}(-1)^{j+1}R_j e^{iS_j/\hbar}\psi_{e+j}^A\psi_{e-j}^B\ , \tag{30}$$

where

$$R_j = \left(\frac{8}{\pi}\right)^{-1/2}\frac{1}{\sigma_0\sigma_t}e^{-\frac{\{(z_A+ju_At)^2+(z_B-ju_Bt)^2\}}{4\sigma_t^2}}\ ,$$

$$S_j = -j\hbar\Big\{\triangle_A - \triangle_B + \left(\frac{z_A + ju_A t}{2}\right)\triangle'_A - \left(z_B - \frac{ju_B t}{2}\right)\triangle'_B\Big\} - \varepsilon_t q + \frac{\hbar t\left[(z_A + ju_A t)^2 + (z_B - ju_B t)^2\right]}{\lambda_t^2},$$

and here the terms $\triangle'_A$, $\triangle'_B$, etc. have the same meanings as before except that now these symbols are labeled with A, B denoting atoms A and B, respectively.

The individual velocities are given by

$$v_{z_A} = \frac{dz_A}{dt} = \frac{1}{m_p}\nabla_p^A S(r_A, \theta_A, \phi_A, z_A, r_B, \theta_B, \phi_B, z_B, t) \tag{31}$$

for atom A and

$$\mathbf{v}_{e_A} = \frac{d\mathbf{r}_A}{dt} = \frac{1}{m_e}\nabla_e^A S(r_A, \theta_A, \phi_A, z_A, r_B, \theta_B, \phi_B, z_B, t), \tag{32}$$

for electron A, with obviously similar expressions for atom B. From these expressions it is clear that in general the motion of any one of the particles depends on the coordinates of all four particles in the system.

Figure 7 shows the initial reduced proton probability distribution (all internal coordinates integrated over) in the two-dimensional configuration space spanned by z_A and z_B; the atoms are initially localized in space around the origin. Initially both electrons are at rest in the state 29. Figure 8 shows how the packets have separated in this configuration space after the measurements have been completed. The three packets correspond to the pairs of values of the observables $(L_z^A = 1, L_z^B = -1)$, $(L_z^A = 0, L_z^B = 0)$, and $(L_z^A = -1, L_z^B = 1)$. The final z positions of the two atoms are correlated. From (30) we see that

$$\frac{\partial S}{\partial \phi_A} = -\frac{\partial S}{\partial \phi_B} \tag{33}$$

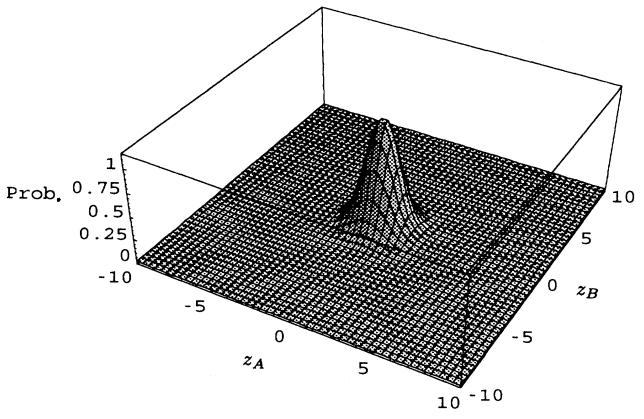

FIG. 7. Atomic position probability distribution, in the configuration space spanned by z_A and z_B, for the initial EPR state.

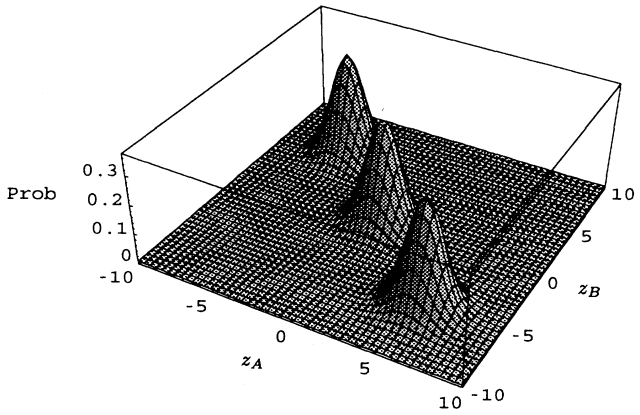

FIG. 8. Atomic probability distribution, in the configuration space spanned by z_A and z_B, after the packets have separated.

showing that the two electrons [see Eq. (10)] either rotate in opposite directions (hence with angular-momentum vectors in opposite directions) or are both stationary. Clearly the motion of the electrons (objects) has become correlated to the motion of the protons (apparatus) as a result of the interaction with the SG fields.

In order to investigate the details of the nonlocality inherent in the Bohm theory we calculate the atomic and electronic trajectories of atoms A and B, keeping the initial values (at the entrance of the magnets) of z_B the same while varying the initial positions of z_A. In all the following calculations we have randomly chosen the initial internal coordinates of the electrons, a better way would, perhaps, be to choose the initial coordinates with some probability distribution. Choosing different initial coordinates for the electrons can affect the path of the atom—the principles of the arguments, however, remain unaltered. Figures 9 and 10 show the trajectories

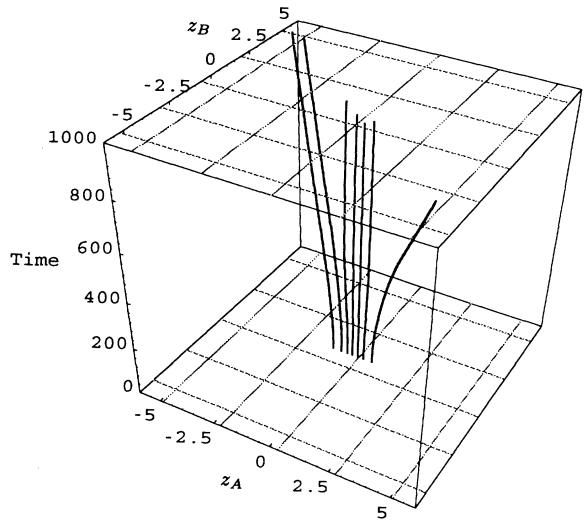

FIG. 9. Configuration-space–time diagram showing the evolution of the atom coordinates. Initially $z_B = 0.24$ while $z_A = 0, \pm 0.24, \pm 0.512, \pm 0.9$.

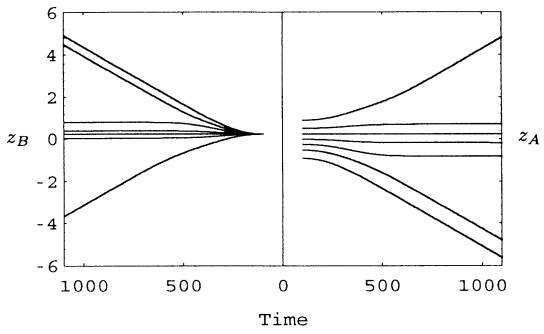

FIG. 10. Individual atomic trajectories associated with Fig. 9.

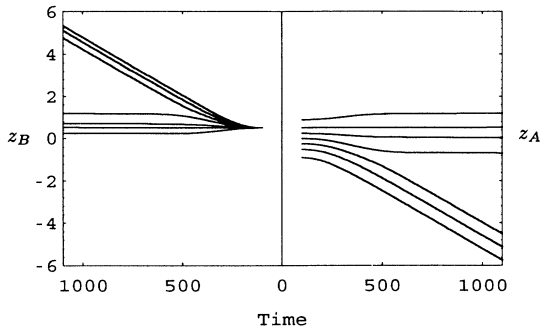

FIG. 12. As Fig. 10 with $z_B = 0.512$.

in a configuration-space–time diagram for those cases in which z_B happens to have the initial value 0.24 (in our arbitrary units). Which packet atom B ends up in depends not only on its own initial z coordinate (which is the same for each configuration-space trajectory shown) but on the initial z coordinate of atom A as well. Figures 11 and 12 show the results of the calculation with an initial value for z_B of 0.512 while the set of values for z_A is as before. We see that for the same initial value of z_A atom A may register a different result for different values of z_B. It is important to note that these initial conditions cannot be controlled and so no signals can be sent using this experiment. Any other attempt by an experimenter to control the hidden parameters will simply alter the system's wave function.

The correlated behavior of atoms A and B evidently follows from the fact that the individual particle velocity fields are nonlocal functions of all the particle coordinates. However often in discussions concerning nonlocality we are concerned not simply with such correlated evolution of pairs of individuals in a given static situation (in which all the experimental settings are constant) but with the effect on a measurement in location A, of changes in the setting of a measuring device at B (at a distant location), just before the measurement is carried out at A. To investigate this type of situation we consider the evolution of the wave function of the system and the consequent individual particle trajectories when the magnet SGB is turned off. We consider the same initial values as used to generate Fig. 9 with $z_B = 0.24$. Figures 13 and 14 show how the trajectories would evolve in the event that the SGB magnet is switched off. The comparison of these sets of figures is interesting since we can infer that what happens to atom A at the exit of the SGA magnet can depend on an arbitrary act of an operator at SGB, i.e., whether or not to switch on the magnets. However, since an observer at A has no access to the position coordinate of atom A, no effect of the switching on or off of the magnet SGB can be detected at SGA and instantaneous signaling is not possible by this means. In Bohm's interpretation, although the individual processes in one region of space-time can be affected by the actions of operators in spacelike separated regions, at the level of the experimentally accessible no such effect can be directly detected.

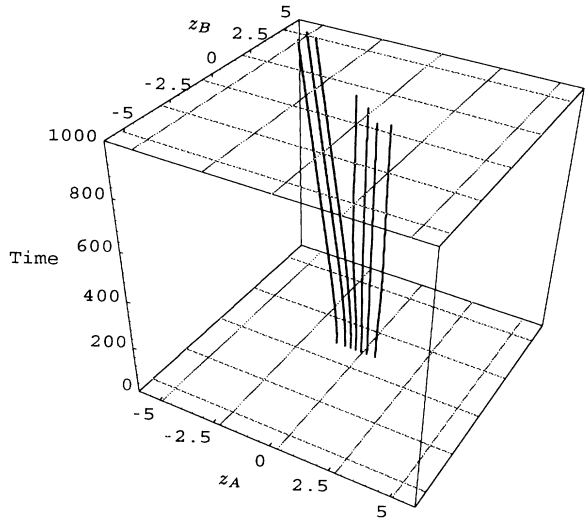

FIG. 11. As Fig. 9 with $z_B = 0.512$.

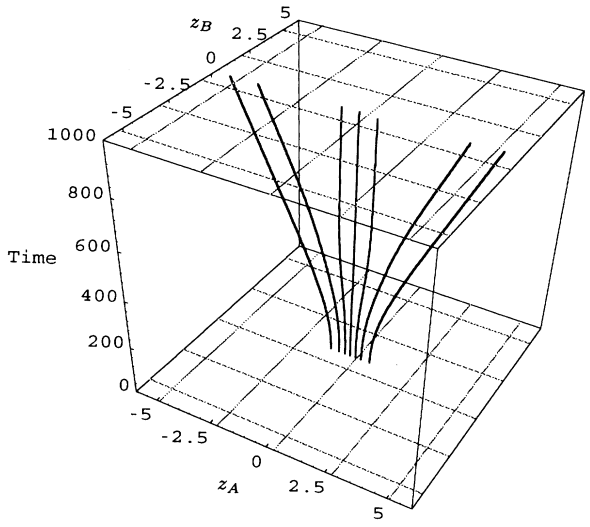

FIG. 13. As Fig. 9 with the SGB magnets switched *off*, i.e., no measurement of B.

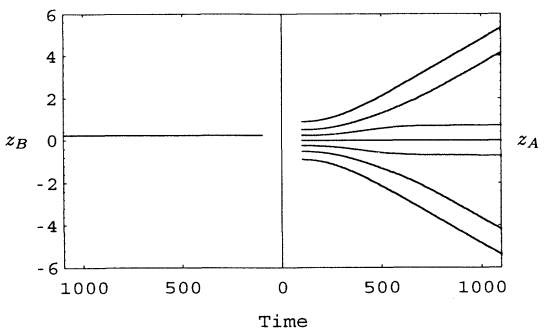

FIG. 14. As Fig. 10 with the SGB magnets switched *off*, i.e., no measurement of B.

V. CONTEXTUALISM AND NO-HIDDEN-VARIABLE THEOREMS

In this section we emphasize the contextual nature of the Bohm theory [23] and show how the theory naturally avoids the major so-called no-hidden-variable theorems. We consider in particular von Neumann's informal version of his argument against hidden variables [25] and the type of argument proposed by Kochen and Specker [24].

Firstly let us summarize the essential features of the process of measurement in the Bohm theory. For any set of dynamical variables the Bohm theory assigns a value to each one, irrespective of the commutation properties of their associated operators. The value assigned to a particular variable will not be, in general, one of the possible eigenvalues of the associated operator. For example, as we have seen above, the values assigned to the orbital-angular-momentum components (and indeed any other dynamical variable) of an electron in a hydrogen atom depend on both the electron's actual position and the quantum state. When the atom is in one of the stationary states given by (8) the values of the angular-momentum components are given by (10). Only L_z is assigned a value that is an eigenvalue, while L_x and L_y fluctuate. When the state is given by (26) L_x is assigned an eigenvalue but in this case L_z and L_y are now inevitably subject to fluctuations. The noncommutation of the operators and the lack of a simultaneous eigenstate of all three angular-momentum components are reflected in the actual motion of the electron. The attempt to make a particular component well defined necessarily results in a disturbance of the motion of the other two components. In Bohm's approach, since it has a dynamical theory of measurement, we are able to give a precise description of measurement interactions and also of the consequent disturbance of the values of observables whose operators do not commute with the operator representing the observable being measured. This state dependence explains the apparent conflict between the values assigned to operators in Bohm's approach and the values allowed (as measurement results) by quantum mechanics. It is also the key feature which allows Bohm's theory to avoid no-hidden-variable theorems of the type proposed by von Neumann and by Kochen and Specker.

In his less formal argument von Neumann argued that dispersion-free ensembles, which must be a consequence of any hidden-variable theory, are incompatible with the predictions of quantum mechanics. Imagine a theory in which there exists a set of hidden parameters, possibly associated with an individual particle, the values of which determine the outcome of any possible measurement or set of measurements. Now consider an initial ensemble of systems with a given distribution of these parameters among the individuals. In such an ensemble the distribution in the results of a measurement of a particular observable could be explained by the existence of the distribution of hidden parameters in the ensemble. If we could select out all those systems with similar values for the hidden parameters, then measurements of any set of observables carried out on this subensemble would always yield the same values and so the ensemble would be dispersion-free. However, dispersion-free ensembles are incompatible with the predictions of quantum mechanics for the results of measurements of observables whose operators are noncommuting.

Let us specialize the discussion, as did von Neumann, to the case that we have described in some detail in this paper and imagine that each atom has some set of parameters that will determine the outcome of the measurement of a component of angular momentum along any direction. By making serial measurements of the angular momentum along different directions and filtering the results we could in principle isolate a set of atoms, with identical hidden parameters and for which the outcome of a measurement of the angular momentum along any of the previously measured directions is known in advance. For instance, we could select all those atoms appearing in the upper beam of a z component of angular-momentum measurement and then pass these through a SG device oriented in the x direction. If we then select once more those atoms appearing in the upper beam along the x direction, we should then have an ensemble in which the hidden parameters are such that a further measurement of the angular momentum in the z direction must yield $+1$. Here we are assuming that the measurement process simply selects those atoms with the appropriate hidden parameter thus revealing a preexisting value possessed by the atom. This is in contradiction with the predictions of quantum mechanics for such a sequence of measurements. According to quantum mechanics a measurement of the z component of angular momentum carried out on a beam of atoms in the eigenstate of L_x with eigenvalue $+1$ will produce three possible outcomes.

We have seen how this works out in Bohm's approach; those atoms which appear in the upper beam of a measurement of L_z on a system in an initial eigenstate of L_x with eigenvalue 1 did not possess the value $L_z = 1$ before the measurement was carried out, in fact the value assigned to L_z before the measurement is fluctuating as the electron executes its orbit around the x axis. Further, the value assigned to L_x (initially one) is disturbed by the measurement interaction. As the wave function evolves so do the values of the angular-momentum components, eventually L_z becomes well defined while L_x either fluctuates or becomes zero. Measurements in general are not

faithful in the Bohm theory, they do not reveal preexisting values. Dispersion-free ensembles, in von Neumann's sense concerning measured outcomes, cannot exist in the Bohm theory either and the reason is simply that one cannot construct simultaneous eigenstates for noncommuting observables.

It is true that, in Bohm's theory, the value that would be obtained for the measurement of any observable is predetermined in a particular instance, with a given set of hidden variables, but this value does not depend solely on the initial value of the hidden parameter (which in any case is uncontrollable), it also depends on the evolution of the quantum state in the system's configuration space. Consequently the predetermined measured value, for a given observable and for a particular individual system depends on that individual's measurement history. For a given atom with given hidden parameters the predetermined outcome for a measurement of L_x depends on whether or not any noncommuting observables (such as L_z) are also measured. Such measurements transform the wave function and hence the values assigned to the system observables. In our example, in the initial state (26), one assigns the value $+1$ for L_x for all of the members of the ensemble, but after a measurement of L_z has been carried out each of the members of the ensemble will have one of three different assigned values for L_x. Just how the state evolves depends on the system's Hamiltonian and hence the value that will be obtained for a particular measurement with a given initial hidden position can depend on which, if any, other observables are measured on the same system. This latter feature has been called context dependence and it is this feature of Bohm's theory that precludes the argument of Kochen and Specker.

The irrelevance of a Kochen and Specker type of argument can be seen by comparing the results predicted by the Bohm theory for particle A when SGB is on and when it is off. The Kochen-Specker type of arguments applied in this context assume that changing the measurement on SGB would have no effect on SGA, clearly not the case in the Bohm theory. The original Kochen-Specker argument has been applied to conceptually simpler situations by Mermin and Peres [26,27]; they are concerned with two spin-$\frac{1}{2}$ particles but their arguments can easily be extended to the case we have discussed of two atoms in correlated angular-momentum-one states. An extended discussion of this point, but dealing with two spin-$\frac{1}{2}$ particles, has been given elsewhere [23,22]. In summary, one of the fundamental assumptions of the Kochen-Specker type of arguments is that the value assigned by the hidden-variable theory for the outcome of a measurement does not depend on which, if any, other variables are measured simultaneously. Thus in the case under discussion the measured value of an observable of particle A say, when the initial state is *completely* specified (i.e., Ψ_0 and z_{0A}, z_{0B}, $\mathbf{r}_{0A}$, $\mathbf{r}_{0B}$ are given) must not depend on what measurement is carried out on B. As we have seen, the Bohm theory violates this assumption. The result of measurement of particle A for a given complete specification of the system's state *does* depend on whether, for example, a measurement is carried out on particle B or not. This feature is what Shimony [29] has called parameter dependence. In a many-particle system contextuality is manifested in a nonlocal manner, since measurements on particles A and B can be carried out at spacelike separations.

The nonlocality inherent in the Bohm theory raises an interesting question concerning the violation of Lorentz invariance for individual processes. Although this question can be discussed only in the context of a relativistic theory [12] (should we be surprised that the nonrelativistic Schrödinger equation entails a non-Lorentz invariant process?), it seems that the feature which causes the breakdown of Lorentz invariance may also persist in a relativistic description. We have seen that the measurement outcome for particle A, for a given complete state, depends on whether or not a simultaneous measurement is carried out on particle B. The problem is evident in the use of the concept of simultaneity, in different frames different outcomes will be calculated even though the same complete state is given [28]. It would seem that any hidden-variable theory which has parameter dependence, which is necessary to provide agreement with quantum mechanics, will violate Lorentz invariance at the level of individual processes. It is an interesting feature of the Bohm theory that although (unobservable) individual processes may not be Lorentz invariant and trajectories calculated in one frame do not Lorentz transform into another, Lorentz invariance *is* recovered at the level of the observable statistical predictions. The full discussion of this problem, giving detailed calculations in a relativistic context, is the subject of a further paper.

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