Relative-state formulation of quantum systems

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(Received 2 April 1993; revised manuscript received 12 July 1993)

A relative-state formulation of quantum systems is presented in terms of relative-coordinate states, relative-number states, and relative-energy states. The relative-coordinate states are used to describe quantum systems in position and momentum representations. The probability distribution is calculated in terms of the relative-coordinate states and is shown to be equivalent to the functional definition of the quantum probability in phase space. It is shown that a quantum-mechanical phase operator can be constructed in terms of the relative-number states without the well-known difficulties. The results are compared with those obtained by the Pegg-Barnett phase-operator formalism and the relations to various other phase-operator methods are also discussed. The energy-measurement and energy-probability distributions are discussed in terms of the relative-energy states. Furthermore, a relative-state formulation is developed in the Liouville space. A phase representation in the Liouville space is introduced to investigate the time evolution of quantum coherence. In the Liouville space a time operator is defined as a canonical conjugate of the time-evolution generator, but not the Hamiltonian energy operator. The relation to the internal time presented by Prigogine and Misra is discussed.

PACS number(s): 03.65.Ca

I. INTRODUCTION

In recent publications [1–7], the author introduced relative-number states to make it possible to investigate the quantum-mechanical properties of phase variables. A relative-number state is defined as follows [1]. Suppose there are two systems A and B which are described by complete orthonormal sets of boson-number eigenstate $\{|m\rangle_A | m \geq 0\}$ and $\{|n\rangle_B | n \geq 0\}$. The relative-number state $|n,m\rangle\rangle$ is defined by

$$|n,m\rangle\rangle = \theta(n)|m+n\rangle_A \otimes |m\rangle_B + \theta(-n-1)|m\rangle_B \otimes |m-n\rangle_B ,$$

where n is an arbitrary integer, m is a non-negative integer, and $\theta(n)$ is defined as $\theta(n)=1$ for $n \geq 0$ and $\theta(n)=0$ for $n<0$. This state is an eigenstate of the number difference (relative number) between the two systems, whose eigenvalue is n. The use of relative-number states overcomes the well-known mathematical difficulties in defining the phase operator [8,9], which stem from the fact that a number operator has a lower bounded spectrum. The phase-operator formalism based on relative number states [1,6,7] is useful for investigating the nonclassical properties of light [10–20]. It is shown that the relative-number-state formulation developed in the Liouville space is also suitable for describing the time evolution of quantum coherence (off-diagonal elements of the density matrix of the system) [1,3–5]. Furthermore, the number-phase quantization in a Josephson junction with ultra-small capacitance can be described in terms of relative-number states [1,2]. In these studies, the use of relative-number states was restricted to problems related to the phase operator. The purpose of this paper is to extend the relative-number-state formalism and to give a method for investigating a wide range of problems to be treated.

This paper is organized as follows. In Sec. II, we consider a quantum system consisting of two subsystems, each of which is described by position and momentum operators with continuous spectra extended over all real values. We introduce several kinds of relative-coordinate states and investigate their properties. Using these states, we calculate and interpret the expectation values of physical quantities. The results reduce to conventional ones if certain conditions are satisfied. Furthermore, we derive the phase-space probability distribution in terms of the relative-coordinate states. The probability distribution thus obtained is compared with the functional definition of quantum probability in phase space given by Aharonov, Albert, and Au [21], O'Connell and Rajagopal [22], Prugovečki [23], and Wódkiewicz [24–26]. In Sec. III, we consider a system consisting of two subsystems described by bosonic annihilation and creation operators, and we develop a phase-operator formalism based on the relative-number states [1,6,7]. The results are compared with those obtained by Pegg-Barnett [27–30]. Furthermore, we discuss the relations to the other phase-operator formalisms obtained by Newton [31], Shapiro [32–34], and Hradil [35,36]. In Sec. IV, we introduce a relative-energy state which corresponds to a continuous version of the relative-number state. We calculate the expectation values of physical quantities by means of the relative-energy states and consider their meaning. In Sec. V, we develop a relative-state formulation in the Liouville space [1,3–5]. We introduce a phase representation to investigate the properties of physical systems in the Liouville space [37–48]. It will be shown that the phase representation is convenient and suitable for describing the time evolution of quantum coherence or phase information of the system. Furthermore, we define a time operator [49–51] in the Liouville space, which is a canonical

conjugate of the time-evolution generator but not the Hamiltonian energy operator, and we discuss its relation to the result obtained by Prigogine, Misra, and Courbage [49,50]. A summary is given in Sec. VI.

II. RELATIVE-COORDINATE REPRESENTATION

A. Relative-coordinate states

In this section, we introduce relative-coordinate states and investigate their properties. For this purpose, we consider a quantum system which consists of two independent subsystems; a relevant system and a reference system. We would like to obtain information only about the relevant system. These systems are assumed to be described by canonical coordinates and momenta, $(\hat{x}_A,\hat{p}_A)$ and $(\hat{x}_B,\hat{p}_B)$, which satisfy the canonical commutation relations $[\hat{x}_A,\hat{p}_A]=i$ and $[\hat{x}_B,\hat{p}_B]=i$ with $\hbar=1$. It is also assumed that these operators have continuous spectra extended over all real values. In this paper, we express quantities of the relevant system as $\hat{O}_A$ and those of the reference system as $\hat{O}_B$.

The relevant and reference systems are described, respectively, by the complete orthonormal sets,

$$S_A=\{|x;A\rangle|\hat{x}_A|x;A\rangle=x|x;A\rangle,x\in\mathbb{R}\}\ , \tag{2.1}$$

$$S_B=\{|x;B\rangle|\hat{x}_B|x;B\rangle=x|x;A\rangle,x\in\mathbb{R}\}\ , \tag{2.2}$$

where $\mathbb{R}$ is the set of all real numbers and we have the following relations:

$$\langle X;x|y;X\rangle=\delta(x-y),\quad \int_{-\infty}^{\infty}dx|x;X\rangle\langle X;x|=\hat{1}_X\ ,$$
$$\text{for } X=A,B\ , \tag{2.3}$$

where $\hat{1}_A$ and $\hat{1}_B$ are unit operators of the relevant and reference systems. Thus, the complete orthonormal basis of the total system is given by

$$S_{A+B}=\{|x,y\rangle=|x;A\rangle\otimes|y,B\rangle|x,y\in\mathbb{R}\}\ , \tag{2.4}$$

and the state $|x,y\rangle$ satisfies the following relations:

$$\langle y_1,x_1|x_2,y_2\rangle=\delta(x_1-x_2)\delta(y_1-y_2)\ ,$$
$$\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy|x,y\rangle\langle y,x|=\hat{1}\ , \tag{2.5}$$

where $\hat{1}=\hat{1}_A\otimes\hat{1}_B$ is a unit operator of the total system.

Now, we consider a state $|r,X\rangle\rangle$ of the total system defined by

$$|r,X\rangle\rangle\equiv|r+X,X\rangle=|r+X;A\rangle\otimes|X,B\rangle\ . \tag{2.6}$$

It is easily found that the state $|r,X\rangle\rangle$ satisfies the relations,

$$\hat{x}|r,X\rangle\rangle=r|r,X\rangle\rangle,\quad \hat{x}_B|r,X\rangle\rangle=X|r,X\rangle\rangle\ , \tag{2.7}$$

where $\hat{x}$ is defined by $\hat{x}=\hat{x}_A-\hat{x}_B$. A set given by $S_R=\{|r,X\rangle\rangle|r,X\in\mathbb{R}\}$ becomes a complete orthonormal set of the total system. The state $|r,X\rangle\rangle$ satisfies

$$\langle\langle X_1,r_1|r_2,X_2\rangle\rangle=\delta(r_1-r_2)\delta(X_1-X_2)\ ,$$
$$\int_{-\infty}^{\infty}dX\int_{-\infty}^{\infty}dr|r,X\rangle\rangle\langle\langle X,r|=\hat{1}\ . \tag{2.8}$$

It should be noted that the operator $\hat{x}=\hat{x}_A-\hat{x}_B$ represents the relative position of the relevant system with respect to the position of the reference system. Thus, we call $|r,X\rangle\rangle$ the relative-position state. The relative-position state $|r,X\rangle\rangle$ is obtained from the state $|r,X\rangle=|r;A\rangle\otimes|X,B\rangle$ by a unitary transformation $\hat{U}=e^{-i\hat{x}_B\hat{p}_A}$.

Now, we define a unitary operator $\hat{D}(s)$ in terms of the relative-position states $\{|r,X\rangle\rangle\}$ through the following relations:

$$\hat{D}(s)|r,X\rangle\rangle=|r-s,X\rangle\rangle\ ,$$
$$\hat{D}^{\dagger}(s)|r,X\rangle\rangle=|r+s,X\rangle\rangle\ . \tag{2.9}$$

The operator $\hat{D}(s)$ is a displacement operator for the relative position r of the relevant system. It is easily found from the definition that the commutation relation between the displacement operator $\hat{D}(s)$ and the relative-position operator $\hat{x}$ is given by

$$[\hat{D}(s),\hat{x}]=s\hat{D}(s)\ . \tag{2.10}$$

It is also seen from (2.9) that the following relations are established:

$$\hat{D}(s_1)\hat{D}(s_2)=\hat{D}(s_1+s_2)\ ,$$
$$\hat{D}(s_1)\hat{D}^{\dagger}(s_2)=\hat{D}(s_1-s_2)\ , \tag{2.11}$$
$$\hat{D}(0)=1\ .$$

These relations show that a set defined by $S_D=\{\hat{D}(s)|s\in\mathbb{R}\}$ becomes a one-parameter unitary group, and $\hat{D}(s)$ is strongly continuous. Therefore Stone's theorem [52] gives a Hermitian operator $\hat{p}$ which satisfies the relation,

$$\hat{D}(s)=\exp[is\hat{p}]\ . \tag{2.12}$$

Towards the limit as $s\to0$ in (2.10), we obtain the commutation relation,

$$[\hat{x},\hat{p}]=i\ . \tag{2.13}$$

This indicates that the Hermitian operator $\hat{p}$ is a canonical conjugate of the relative-position operator $\hat{x}$.

It is found from (2.9) and (2.13) that the operations of $\hat{p}$ on $|r,X\rangle\rangle$ and $\langle\langle X,r|$ are represented, respectively, by $\hat{p}|r,X\rangle\rangle=i(\partial/\partial r)|r,X\rangle\rangle$ and $\langle\langle X,r|\hat{p}=-i(\partial/\partial t)\langle\langle X,r|$. It is easily seen that $\hat{p}$ and $\hat{D}(s)$ transform a direct product state $|\psi;A\rangle\otimes|\phi,B\rangle$ into another direct product state $|\psi';A\rangle\otimes|\phi';B\rangle$. Thus, if a state $|\Psi\rangle$ is factorizable, that is $|\Psi\rangle=|\psi;A\rangle\otimes|\phi;B\rangle$, the operators $\hat{p}$ and $\hat{D}(s)$ preserve the factorizability of the state. It will be found in the following sections that quantum-mechanical phase and time operators do not have such a property.

The eigenstate of $\hat{p}$ is expressed in terms of the relative-position states $\{|r,X\rangle\rangle\}$,

$$|p,X\rangle\rangle=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dr|r,X\rangle\rangle e^{ipr}\ , \tag{2.14}$$

where $\hat{p}|p,X\rangle\rangle=p|p,X\rangle\rangle$. A set given by $S_p=\{|p,X\rangle\rangle|p,X\in\mathbb{R}\}$ becomes a complete orthonormal set for the total system,

$$\langle\langle X_1,p_1|p_2,X_2\rangle\rangle=\delta(p_1-p_2)\delta(X_1-X_2)\ ,$$
$$\int_{-\infty}^{\infty}dX\int_{-\infty}^{\infty}dp|p,X\rangle\rangle\langle\langle X,p|=\hat{1}\ . \tag{2.15}$$

From (2.6) and (2.14), the state $|p,X\rangle\rangle$ can be expressed by a direct product of the momentum eigenstate of the relevant system $|p;A\rangle$ and the position eigenstate of the reference system $|X;B\rangle$ with a phase factor of e^{-ipX}, so that $|p;X\rangle\rangle=|p;A\rangle\otimes|X;B\rangle e^{-ipX}$. Furthermore, we define another relative-coordinate state which is the Fourier transform of $|r,X\rangle\rangle$ with respect to variable X,

$$|r,k\rangle\rangle=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dX|r,X\rangle\rangle e^{ikX}\ . \tag{2.16}$$

This state plays an important role when we consider phase-space distribution functions (see Sec. II C). It will be found that $\{|r,k\rangle\rangle\}$ are closely related to the functional definition of the quantum probability distribution in phase space. It is easily seen that a set given by $S_k=\{|r,k\rangle\rangle|r,k\in\mathbb{R}\}$ becomes a complete orthonormal basis of the total system, satisfying the following relations:

$$\hat{x}|r,k\rangle\rangle=r|r,X\rangle\rangle,\quad \hat{x}_B|r,k\rangle\rangle=-i\frac{\partial}{\partial k}|r,k\rangle\rangle\ , \tag{2.17}$$

$$\langle\langle k_1,r_1|r_2,k_2\rangle\rangle=\delta(k_1-k_2)\delta(r_1-r_2)\ ,$$
$$\int_{-\infty}^{\infty}dk\int_{-\infty}^{\infty}dr|r,k\rangle\rangle\langle\langle k,r|=\hat{1}\ . \tag{2.18}$$

Before proceeding further, we will briefly mention the relative state theory proposed by Everett [53]. Consider a direct product state given by $|\Psi\rangle=|\psi;A\rangle\otimes|\phi,B\rangle$. The scalar product with the relative-position state $|r,X\rangle\rangle$ leads to $\Psi(r,X)=\psi_A(r+X)\phi_B(X)$ with $\psi_A(x)=\langle A;x|\psi;A\rangle$ and $\phi_B(x)=\langle B;x|\phi;B\rangle$. This wave function is the simplest considered by Everett. According to Everett, the wave function $\Psi(r,X)$ is interpreted as follows. When one subsystem is in state $\phi_B(x)$ while the total system is in state $\Psi(r,X)$, then the corresponding relative state of another subsystem is given by $\psi_A(r+X)$. It is stressed that the state of one subsystem cannot be determined independent of the state of another subsystem. Indeed, $\psi_A(r+X)$ depends on the position of another subsystem. By developing such ideas, the measurement process in quantum mechanics was discussed [53].

B. Expectation values of physical quantities

We consider the expectation values of physical quantities of the relevant system in terms of the relative-coordinate states introduced in the previous section. It is postulated in our formulation that the physical quantities of the relevant system are measured as differences from the corresponding quantities of the reference system. In other words, only variations from the reference values determined by the state of the reference system are observed. When we would like to know a quantity $\hat{O}_A$ of the relevant system, we measure the relative quantity given by $\hat{O}=\hat{O}_A-\hat{O}_B$, where $\hat{O}_B$ is the quantity of the reference system corresponding to $\hat{O}_A$. For example, in a position measurement, the quantity $\hat{x}=\hat{x}_A-\hat{x}_B$ is observed, and so the observable quantity is the relative position of the relevant system with respect to that of the reference system. Furthermore, we assume that the physical state $|\Psi\rangle$ of the total system, is given by the direct product of states of the subsystems,

$$|\Psi\rangle=|\psi;A\rangle\otimes|\phi;B\rangle\ , \tag{2.19}$$

where $|\psi;A\rangle$ is the state of the relevant system and $|\phi;B\rangle$ is that of the reference system.

Consider an arbitrary observable quantity $A(\hat{x},\hat{p})$ of the relevant system expressed in terms of the relative position $\hat{x}$ and its canonical conjugate $\hat{p}$,

$$A(\hat{x},\hat{p})=\sum_m\sum_n A_{mn}\hat{x}^m\hat{p}^n\ . \tag{2.20}$$

Using the commutation relation (2.13), any analytic function of $\hat{x}$ and $\hat{p}$ can be written in this form. Now we calculate the expectation value of $A(\hat{x},\hat{p})$ with the $|\Psi\rangle$ given by (2.19). Using the fact that the relative position states $\{|r,X\rangle\rangle\}$ become a complete orthonormal basis and using the relations $\langle\langle X,r|\hat{x}=r\langle\langle X,r|$ and $\langle\langle X,r|\hat{p}=-i(\partial/\partial r)\langle\langle X,r|$, we can calculate the expectation value as

$$\langle\Psi|A(\hat{x},\hat{p})|\Psi\rangle=\int_{-\infty}^{\infty}dy\int_{-\infty}^{\infty}dx\,|\phi_B(y)|^2\psi_A^*(x)A(x-y,-i\partial/\partial x)\psi_A(x)\ , \tag{2.21}$$

where $\psi_A(x)=\langle A;x|\psi;A\rangle$ and $\phi_B(x)=\langle B;x|\phi;B\rangle$ are the wave functions of the relevant and reference systems. Then we finally obtain the following expression for the expectation value of $A(\hat{x},\hat{p})$:

$$\langle\Psi|A(\hat{x},\hat{p})|\Psi\rangle=\int_{-\infty}^{\infty}dy\langle A:\psi|A(\hat{x}_A-y,\hat{p}_A)|\psi;A\rangle|\phi_B(y)|^2\ . \tag{2.22}$$

On the right-hand side of (2.22), $\langle A;\psi|A(\hat{x}_A-y,\hat{p}_A)|\psi;A\rangle$ represents the expectation value of only the relevant system, in which the position of the relevant system is measured from the position y at which the reference system is placed, and $|\phi_B(y)|^2dy$ is the probability that the reference system is placed at a position between y and $y+dy$. This shows that the quantity $\langle\Psi|A(\hat{x},\hat{p})|\Psi\rangle$ can be calculated as follows. First, the expectation value of the relevant system with a fixed reference point is calculated as $\langle A;\psi|A(\hat{x}_A-y,\hat{p}_A)|\psi;A\rangle$, and then the average over all possible reference points is taken. Thus, (2.22) can be expressed as

$$\langle\Psi|A(\hat{x},\hat{p})|\Psi\rangle=\overline{\langle A;\psi|A(\hat{x}_A-y,\hat{p}_A)|\psi;A\rangle}\ , \tag{2.23}$$

where $\overline{(\ )}$ means the average over the possible reference

points. If the reference system is treated as a classical system, its position can be fixed with great accuracy. Consequently, when the position of the reference system is chosen as the origin of the coordinate system ($y=0$), then (2.23) can be approximated by

$$\langle \Psi | A(\hat{x},\hat{p}) | \Psi \rangle \approx \langle A;\psi | A(\hat{x}_A,\hat{p}_A) | \psi;A \rangle . \quad (2.24)$$

The right-hand side of (2.24) is the conventional expression for the expectation value of a relevant system. Therefore, our formulation reduces to the conventional case if the reference system is assumed to be a classical object.

Using the same method as that used to derive (2.22), we find that in the momentum representation the expectation value of $A(\hat{q},\hat{\pi})$ can be expressed as

$$\langle \Psi | A(\hat{q},\hat{\pi}) \Psi \rangle = \int_{-\infty}^{\infty} dk \langle A;\psi | A(\hat{x}_A,\hat{p}_A-k) | \psi;A \rangle |\tilde{\phi}_B(k)|^2 , \quad (2.25)$$

where $\hat{\pi}=\hat{p}_A-\hat{p}_B$ is the relative momentum, and $\hat{q}$ is the canonical conjugate such that $[\hat{q},\hat{\pi}]=i$, and $\tilde{\phi}_B(k)$ is the momentum representation of the wave function of the reference system, which is the Fourier transform of $\phi_B(x)$. On the right-hand side of (2.25), $\langle A;\psi | A(\hat{x}_A,\hat{p}_A-k) | \psi;A \rangle$ represents the average value of only the relevant system, of which the momentum is measured from the reference value k, and $|\tilde{\phi}_B(k)|^2 dk$ is the probability that the reference system has a momentum between k and $k+dk$. Thus, $\langle \Psi | A(\hat{q},\hat{\pi}) | \Psi \rangle$ is expressed as

$$\langle \Psi | A(\hat{q},\hat{\pi}) | \Psi \rangle = \overline{\langle A;\psi | A(\hat{x}_A,\hat{p}_A-k) | \psi;A \rangle} , \quad (2.26)$$

where ( ) is the average taken over all possible reference values k. When the reference system has zero momentum with great accuracy, (2.25) can be approximated by

$$\langle \Psi | A(\hat{q},\hat{\pi}) | \Psi \rangle \approx \langle A;\psi | A(\hat{x}_A,\hat{p}_A) | \psi;A \rangle . \quad (2.27)$$

The right-hand side of (2.27) is the conventional expression for the average value of the relevant system.

It is seen from (2.22)–(2.27) that our formulation gives the conventional expressions for the expectation values of physical quantities when the reference system satisfies certain conditions. It seems to be sufficient, at least, that the reference system satisfies the semiclassical condition.

C. Probability distribution

In this section, we consider the probability distribution in terms of the relative coordinate states $\{|r,k\rangle\rangle\}$ defined by (2.16). For this purpose, it is convenient to introduce a probability operator measure defined by

$$\hat{\Pi}[\Delta_r,\Delta_k] = \int_{r\in\Delta_r} dr \int_{k\in\Delta_k} dk \, |r,k\rangle\rangle\langle\langle k,r| , \quad (2.28)$$

where Δ_r and Δ_k are subsets of $\mathbb{R}$. Using the probability operator measures (2.28), the probability distribution that the relative position r takes a value between r and $r+dr$ and the parameter k takes a value between k and $k+dk$ in the state $|\Psi\rangle$ is calculated as

$$P(r,k)dr\,dk = \langle \Psi | \hat{\Pi}[\Delta_{r+dr},\Delta_{k+dk}] | \Psi \rangle , \quad (2.29)$$

where $\Delta_{r+dr}=[r,r+dr)$ and $\Delta_{k+dk}=[k,k+dk)$. When the state of the total system is given by (2.19), it is found from (2.16) that (2.29) becomes

$$P(r,k) = \frac{1}{2\pi} \left| \int_{-\infty}^{\infty} dX \, \psi_A(r+X)\phi_B(X)e^{-ikX} \right|^2 , \quad (2.30)$$

where $\psi_A(x)=\langle A;x|\psi;A\rangle$ and $\phi_B(x)=\langle B;x|\phi;B\rangle$. From (2.30), the marginal probability distribution $P(r)=\int_{-\infty}^{\infty} dk\, P(r,k)$ that the relative position r takes a value between r and $r+dr$ in the state $|\Psi\rangle$ reduces to

$$P(r) = \int_{-\infty}^{\infty} dX |\psi_A(r+X)|^2 |\phi_B(X)|^2 . \quad (2.31)$$

Let us consider two systems characterized by the wave functions $\psi(x)$ and $\phi(x)$ in order to compare the above result with those given in Refs. [21–26]. The Wigner functions of these systems are given by

$$W_\psi(q,p) = \frac{1}{\pi} \int_{-\infty}^{\infty} dx \, \psi^*(q+x)\psi(q-x)e^{2ipx} , \quad (2.32)$$

$$W_\phi(q,p) = \frac{1}{\pi} \int_{-\infty}^{\infty} dx \, \phi^*(q+x)\phi(q-x)e^{2ipx} , \quad (2.33)$$

respectively. It is seen that $W_\psi(q+r,p+k)W_\phi(q,p)$ is the quasiprobability density that one system in state $|\psi\rangle$ has momentum $p+k$ and position $q+r$ are the other system in state $|\phi\rangle$ has momentum p and position q. Thus the propensity of these two systems to have momenta and positions differing by amounts k and r, respectively, is given by

$$P_W(r,k) = \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dq \, W_\psi(q+r,p+k)W_\phi(q,p) = \frac{1}{2\pi} \left| \int_{-\infty}^{\infty} dx \, \psi(x+r)\phi^*(x)e^{-ikx} \right|^2 . \quad (2.34)$$

This is clearly equivalent to (2.30) if we chose $\psi(x)=\psi_A(x)$ and $\phi(x)=\phi_B^*(x)$. Note that (2.34) is formally written as

$$P_W(r,k) = \frac{1}{2\pi} |\langle \phi | \hat{D}(r,k) | \psi \rangle|^2 , \quad \hat{D}(r,k) = \exp[i(r\hat{p}-k\hat{q})] , \quad (2.35)$$

where $\hat{q}$ and $\hat{p}$ are position and momentum operators and $\hat{D}(r,k)$ is a displacement operator in phase space. The meaning of (2.34) was first considered by Aharonov, Albert, and Au [21].

Next we consider another situation treated by Wódkiewicz [25]. The probability distribution that the system described by the density matrix $\hat{\rho}$ is in the state $|\phi\rangle$ is usually given by $P=\mathrm{Tr}[\hat{\rho}\hat{P}_\phi]$, where $\hat{P}_\phi=|\phi\rangle\langle\phi|$ is a projection operator. According to Wódkiewicz [25], in order to physically compare the state of the relevant system with that of the measurement apparatus in a realistic laboratory arrangement, we have to bring the measurement apparatus towards the relevant system to be measured. This indicates that P should be modified as follows:

$$P_g = \frac{1}{Z}\mathrm{Tr}[\hat{\rho}\hat{\Pi}_\phi(g)], \quad \hat{\Pi}_\phi(g) = \hat{U}^\dagger(g)\hat{P}_\phi\hat{U}(g) , \quad (2.36)$$

where $\hat{U}(g)$ is a unitary operator which represents the displacement of the measurement apparatus and g is an element of the transformation group to express such a movement. The normalization constant Z is given by $Z=\sum_g \mathrm{Tr}[\hat{\rho}\hat{\Pi}_\phi(g)]$, where $\sum_g$ means the summation (or integration) over all possible transformations. When the relevant system is in a pure state $\hat{\rho}=|\psi\rangle\langle\psi|$, (2.36) becomes

$$P_g=\frac{1}{Z}|\langle\phi|\hat{U}(g)|\psi\rangle|^2 . \tag{2.37}$$

If the unitary transformation $\hat{U}(g)$ is given by a spatial displacement in phase space, such that $\hat{U}(g)=\hat{D}(r,k)$, (2.37) reduces to (2.35) and so to (2.30).

Therefore, it is found that the probability distribution calculated by the probability operator measure constructed in terms of the relative-coordinate states $\{|r,k\rangle\rangle\}$ is equivalent to the operational phase-space probability distribution or the propensity in phase space considered by several authors [21–26].

III. RELATIVE-NUMBER STATE AND PHASE ORDER

A. Relative-number state

In this section, we consider a system which consists of two independent subsystems described in terms of bosonic annihilation and creation operators. We denote the annihilation and creation operators of the relevant subsystem as $(a,a^\dagger)$ and those of the reference subsystem as $(b,b^\dagger)$, where $[a,a^\dagger]=[b,b^\dagger]=1$. For simplicity, we assume a single-mode boson for each subsystem. As we did in the previous section, we assume that the physical quantity of the relevant system is measured as a difference from the corresponding quantity of the reference system. Thus, in the boson-number measurement the observable quantity is

$$\hat{N}=\hat{N}_A-\hat{N}_B , \tag{3.1}$$

where $\hat{N}_A=a^\dagger a$ and $\hat{N}_B=b^\dagger b$. In the following, we first introduce a relative-number state [1,6] and then we consider boson-number measurement. Using the relative-number states, we can define a phase operator and a phase probability distribution [7].

The relevant and reference systems now considered are assumed to be described by complete orthonormal sets,

$$S_A=\{|n;A\rangle|\hat{N}_A|n;A\rangle=n|n;A\rangle,n\in\mathbb{Z}_+\} , \tag{3.2}$$

$$S_B=\{|n;B\rangle|\hat{N}_B|n;B\rangle=n|n;B\rangle,n\in\mathbb{Z}_+\} , \tag{3.3}$$

which satisfy the following relations:

$$\langle X;m|n;X\rangle=\delta_{mn},\quad \sum_{n=0}^{\infty}|n;X\rangle\langle X;n|=\hat{1}_x ,$$
$$\text{for } X=A,B . \tag{3.4}$$

Here, $\mathbb{Z}_+$ is the set of non-negative integers and $\hat{1}_A$ ($\hat{1}_B$) is the unit operator of the relevant (reference) system. Thus, the complete orthonormal basis of the whole system is given by

$$S_{A+B}=\{|m,n\rangle=|m;A\rangle\otimes|n,B\rangle|m,n\in\mathbb{Z}_+\} , \tag{3.5}$$

which satisfies the following relations:

$$\langle n_1,m_1|m_2,n_2\rangle=\delta_{m_1m_2}\delta_{n_1n_2} ,$$
$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|m,n\rangle\langle n,m|=\hat{1} , \tag{3.6}$$

with $\hat{1}=\hat{1}_A\otimes\hat{1}_B$.

We define a relative-number state (RNS) as follows [1,6,7]:

$$|n,m\rangle\rangle=\theta(n)|m+n,m\rangle+\theta(-1-n)|m,n-n\rangle , \tag{3.7}$$

where $\theta(n)=1$ for $n\geq 0$ and $\theta(n)=0$ for $n<0$. Note that in (3.7), n can be any integer while m is a non-negative integer . Since the quantum number n in $|n,A\rangle$ or $|n,B\rangle$ cannot take a negative integer, we introduce the $\theta(n)$ function and express $|n,m\rangle\rangle$ as a superposition of two terms: one with $n\geq 0$ and the other with $n<0$. It is easily seen from the definition that n represents the boson-number difference between the relevant and reference systems and that $|n,m\rangle\rangle$ is an eigenstate of $\hat{N}$ defined by (3.1), $\hat{N}|n,m\rangle\rangle=n|n,m\rangle\rangle$. The set of relative-number states, $S_R=\{|n,m\rangle\rangle|m\in\mathbb{Z}_+,n\in\mathbb{Z}\}$, becomes a complete orthonormal basis of the whole system, which satisfies the following relations:

$$\langle\langle m_1,n_1|n_2,m_2\rangle\rangle=\delta_{m_1m_2}\delta_{n_1n_2} ,$$
$$\sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty}|n,m\rangle\rangle\langle\langle m,n|=\hat{1} . \tag{3.8}$$

Here $\mathbb{Z}$ is the set of all integers.

Next we introduce the following projection operator in order to consider a boson-number probability:

$$\hat{\Pi}_n=\sum_{m=0}^{\infty}|n,m\rangle\rangle\langle\langle m,n| . \tag{3.9}$$

This projection operator maps the whole space into a subspace with a fixed boson-number difference n, which satisfies the following relations:

$$\hat{\Pi}_n^\dagger=\hat{\Pi}_n,\quad \hat{\Pi}_n\hat{\Pi}_{n'}=\delta_{nn'}\hat{\Pi}_n,\quad \hat{\Pi}_n\geq 0,\quad \sum_{n=-\infty}^{\infty}\hat{\Pi}_n=\hat{1} . \tag{3.10}$$

Using (3.9), we can obtain the probability distribution $P(n)$ that the boson-number difference between the relevant system and the reference system in state $|\Psi\rangle$ is n,

$$P(n)=\langle\Psi|\hat{\Pi}_n|\Psi\rangle . \tag{3.11}$$

If a state of the whole system is given by $|\Psi\rangle=|\psi;A\rangle\otimes|\phi;B\rangle$, $P(n)$ is calculated to be

$$P(n)=\theta(n)\sum_{m=0}^{\infty}|\psi_A(m+n)|^2|\phi_B(m)|^2$$
$$+\theta(-1-n)\sum_{m=0}^{\infty}|\psi_A(m)|^2|\phi_B(m-n)|^2 , \tag{3.12}$$

where the wave functions $\psi_A(n)$ and $\phi_B(n)$ are defined,

respectively, by $\psi_A(n)=\langle A;n|\psi;A\rangle$ and $\phi_B(n)=\langle B;n|\phi;B\rangle$. This result corresponds to (2.31) in the relative-coordinate state. In particular, when the reference system is in the vacuum state $|\phi;B\rangle=|0;B\rangle$, $P(n)$ reduces to $P(n)=\theta(n)|\psi_A(n)|^2$. This is the conventional expression for the probability distribution of boson number.

Remember that it is postulated in our formulation that the boson number of the relevant system should be measured as a difference from the boson number of the reference system and that the basic observable quantity is $\hat{N}=\hat{N}_A-\hat{N}_B$. It is found from (3.12) that the expectation value of any analytic function of $\hat{N}$, $f(\hat{N})$, is calculated as

$$\langle\Psi|f(\hat{N})|\Psi\rangle=\sum_{n=0}^{\infty}\langle A;\psi|f(\hat{N}_A-n)|[\psi;A\rangle|\phi_B(n)|^2$$
$$=\sum_{n=-\infty}^{\infty}f(n)P(n)\ . \tag{3.13}$$

Furthermore, (3.13) is expressed as follows:

$$\langle\Psi|f(\hat{N})|\Psi\rangle=\overline{\langle A;\psi|f(\hat{N}_A-n)|\psi;A\rangle}\ , \tag{3.14}$$

where $\overline{(\ )}$ means the average taken over the boson numbers of the reference system. This expression is the same as (2.23) and (2.26). If the reference system is in the vacuum state, (3.13) reduces to the conventional expression of the expectation value,

$$\langle\Psi|f(\hat{N})|\Psi\rangle=\langle A;\psi|f(\hat{N}_A)|\psi;A\rangle$$
$$=\sum_{n=0}^{\infty}f(n)P(n)\ . \tag{3.15}$$

Therefore, it is clear from (3.13) and (3.15) that our formulation based on the relative-number states $\{|n,m\rangle\rangle\}$ gives the conventional results when the reference system is in the vacuum state.

B. Phase operator

Now we consider a quantum-mechanical phase operator in terms of the relative-number states $\{|n,m\rangle\rangle\}$. Recently, the phase operator has been extensively used in quantum optics by many authors to investigate the nonclassical properties of light [10–20]. First, using the relative-number states, we introduce a unitary operator $\hat{D}$,

$$\hat{D}=\sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty}|n,m\rangle\rangle\langle\langle m,n+1|\ . \tag{3.16}$$

From the definition, $\hat{D}$ is a displacement operator for the boson-number difference between the relevant and reference systems, which is unitary operator satisfying

$$\hat{D}|n,m\rangle\rangle=|n-1,m\rangle\rangle\ ,$$
$$\hat{D}^{\dagger}|n,m\rangle\rangle=|n+1,m\rangle\rangle\ . \tag{3.17}$$

It should be noted that since n is an arbitrary integer, $\hat{D}$ becomes a unitary operator. It is easily seen that the commutation between $\hat{D}$ and $\hat{N}$ is given by

$$[\hat{D},\hat{N}]=\hat{D}\ . \tag{3.18}$$

Using the completeness relations $\sum_{n=0}^{\infty}|n;A\rangle\langle A;n|=\hat{1}_A$ and $\sum_{n=0}^{\infty}|n;B\rangle\langle B;n|=\hat{1}_B$ in the Hilbert spaces of the relevant and reference systems, we can formally rewrite (3.16) into the following form:

$$\hat{D}=\sum_{n=0}^{\infty}(\hat{e}_A^{\dagger})^n(\hat{e}_A)^{n+1}\otimes|n,B\rangle\langle B;n|$$
$$+\sum_{n=0}^{\infty}|n;A\rangle\langle A:n|\otimes(\hat{e}_B^{\dagger})^{n+1}(\hat{e}_B)^n$$
$$=(\hat{e}_A^{\dagger})^{\hat{N}_B}(\hat{e}_A)^{\hat{N}_B+1}+(\hat{e}_B^{\dagger})^{\hat{N}_A+1}(\hat{e}_B)^{\hat{N}_A}\ , \tag{3.19}$$

where $\hat{e}_A$ and $\hat{e}_B$ are the Susskind-Glogower phase operators [8,9] of the relevant and reference systems, which are defined by $\hat{e}_A=(aa^{\dagger})^{-1/2}a$ and $\hat{e}_B=(bb^{\dagger})^{-1/2}b$. Since a boson-number operator has a lower bounded spectrum, the Susskind-Glogower phase operator is isometry, but not unitary. In the following, the displacement operator $\hat{D}$ defined by (3.16) is called the RNS phase operator. It is easily seen from the definition that by restricting the domain of the RNS phase operator $\hat{D}$ from $\mathcal{H}_A\otimes\mathcal{H}_B$ to $\mathcal{H}_A\otimes\{|0;B\rangle\}$, where $\mathcal{H}_A$ and $\mathcal{H}_B$ are the Fock spaces of the relevant and reference systems, the Susskind-Glogower phase operator of the relevant system is obtained from the RNS phase operator, $\langle B;0|\hat{D}|0;B\rangle=\hat{e}_A$.

It is important to note that the RNS phase operator $\hat{D}$ does not preserve the factorizability of state. This indicates that $\hat{D}|\Psi\rangle$ cannot be expressed by a direct product state such as $|\psi;A\rangle\otimes|\phi;B\rangle$, when $|\Psi\rangle$ is factorizable, $|\Psi\rangle=|\psi_0;A\rangle\otimes|\phi_0;B\rangle$. This contrasts with the case in which we considered the relative-coordinate states, where $\hat{D}(s)$ defined by (2.12) does preserve the factorizability of state. It is considered that the nonfactorizability of $\hat{D}$ requires the introduction of the reference system to define the phase operator. The eigenstate of the RNS phase operator is given by

$$|\phi,m\rangle\rangle=\frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty}|n,m\rangle\rangle e^{-in\phi}\ , \tag{3.20}$$

with $\phi\in[-\pi+\phi_0,\pi+\phi_0)$. Here, ϕ_0 is a phase-reference value which can take any real value and determines the 2π-phase window. The properties of the RNS phase operator $\hat{D}$, its eigenstate $|\phi,m\rangle\rangle$, and the phase probability distribution $P(\phi)$ have been investigated in the previous papers [1–7].

C. Relations to other phase operators

In quantum optics, the Pegg-Barnett (PB) Hermitian phase operator [27–30] is frequently used. Hence, in the following, we will show that in the physical state, the PB phase operator and the RNS phase operator give the same physical results in spite of their mathematical structures being quite different. In the PB phase-operator formalism, all physical quantities are calculated in $(s+1)$-dimensional space, where s is arbitrarily large but finite, and after the all calculations are completed, s is made infinite ($s\to\infty$). The PB phase operator and its eigenstate of the relevant system in $(s+1)$-dimensional space

are defined by

$$\hat{\Phi}^{s}_{\mathrm{PB}}=\sum_{m=0}^{s}|\phi^{s}_{m};A\rangle\phi^{s}_{m}\langle A;\phi^{s}_{m}|\ , \tag{3.21}$$

where $|\phi^{s}_{m};A\rangle$ is the PB phase eigenstate defined by

$$|\phi^{s}_{m};A\rangle=\frac{1}{\sqrt{s+1}}\sum_{n=0}^{s}e^{-in\phi^{s}_{m}}|n;A\rangle\ , \tag{3.22}$$

where ϕ^{s}_{m} is given by $\phi^{s}_{m}=\phi_0-\pi+[2\pi m/(s+1)]$ $(m=0,1,2,\ldots,s)$. From (3.21) and (3.22), $\hat{D}^{s}_{\mathrm{PB}}=\exp[-i\hat{\Phi}^{s}_{\mathrm{PB}}]$ is expressed as

$$\hat{D}^{s}_{\mathrm{PB}}=\sum_{n=0}^{s-1}|n;A\rangle\langle A;n+1|+e^{i(s+1)\phi_0}|s;A\rangle\langle A;0|\ . \tag{3.23}$$

In $(s+1)$-dimensional space, the physical state of the relevant system is expressed as

$$|\psi^{s};A\rangle=\sum_{n=0}^{s}a_n|n;A\rangle\ , \tag{3.24}$$

which is assumed to satisfy the condition

$$\lim_{s\to\infty}\sum_{n=0}^{s}|a_n|^2n^{\mu}=(\text{finite})\ , \tag{3.25}$$

for an arbitrary non-negative number μ [27,28]. Thus, in the PB phase-operator formalism, the average value of a phase quantity can be calculated as follows:

$$\langle F(\phi)\rangle=\lim_{s\to\infty}\langle A;\psi^{s}|F(\hat{\Phi}^{s}_{\mathrm{PB}})|\psi^{s};A\rangle\ . \tag{3.26}$$

Now we show that when the reference system is in the vacuum state $|0;B\rangle$, the RNS phase-operator formalism gives the same average values of physical quantities as those obtained by the Pegg-Barnett formalism. To do this, we first calculate the kth-order moment of the phase operators. For the RNS phase operator, we obtain

$$\langle\Psi|\hat{D}^{k}|\Psi\rangle=\sum_{n=0}^{\infty}\langle A;\psi|n;A\rangle\langle A;n+k|\psi;A\rangle\ , \tag{3.27}$$

where $|\Psi\rangle=|\psi;A\rangle\otimes|0;B\rangle$. On the other hand, for the PB phase operator, we have

$$\lim_{s\to\infty}\langle A;\psi^{s}|(\hat{D}^{s}_{\mathrm{PB}})^{k}|\psi^{s};A\rangle=\lim_{s\to\infty}\sum_{n=0}^{s-k}\langle A;\psi^{s}|n;A\rangle\langle A;n+k|\psi^{s};A\rangle+\lim_{s\to\infty}e^{i(s+1)\phi_0}\sum_{n=0}^{k-1}a^{*}_{s-n}a_{k-n-1}\ , \tag{3.28}$$

where we have used (3.21)–(3.24). The first term on the right-hand side of (3.28) is equal to (3.27) and the second term vanishes because of the condition (3.25). Thus we obtain

$$\langle\Psi|\hat{D}^{k}|\Psi\rangle=\lim_{s\to\infty}\langle A;\psi^{s}|(\hat{D}^{s}_{\mathrm{PB}})^{k}|\psi^{s};A\rangle\ . \tag{3.29}$$

Similarly, we can get

$$\langle\Psi|\hat{D}^{\dagger k}|\Psi\rangle=\lim_{s\to\infty}\langle A;\psi^{s}|(\hat{D}^{s\dagger}_{\mathrm{PB}})^{k}|\psi^{s};A\rangle\ . \tag{3.30}$$

Let $f(x,y)$ be an arbitrary polynomial of x and y expressed as

$$f(x,y)=\sum_{m=0}^{M}\sum_{n=0}^{N}a_{mn}x^{m}y^{n}\ . \tag{3.31}$$

Since both $\hat{D}$ and $\hat{D}^{s}_{\mathrm{PB}}$ are unitary operators, we have

$$f(\hat{D}^{\dagger},\hat{D})=\sum_{m=0}^{M}\sum_{n=0}^{N}a_{mn}\hat{D}^{n-m}=\sum_{m=0}^{M}\sum_{n=0}^{N}a_{mn}(\hat{D}^{\dagger})^{m-n}\ , \tag{3.32}$$

$$f(\hat{D}^{s\dagger}_{\mathrm{PB}},\hat{D}^{s}_{\mathrm{PB}})=\sum_{m=0}^{M}\sum_{n=0}^{N}a_{mn}(\hat{D}^{s}_{\mathrm{PB}})^{n-m}=\sum_{m=0}^{M}\sum_{n=0}^{N}a_{mn}(\hat{D}^{s\dagger}_{\mathrm{PB}})^{m-n}\ . \tag{3.33}$$

It is found from (3.29)–(3.33) that we obtain

$$\langle\Psi|f(\hat{D}^{\dagger},\hat{D})|\Psi\rangle=\lim_{s\to\infty}\langle A;\psi^{s}|f(\hat{D}^{s\dagger}_{\mathrm{PB}},\hat{D}^{s}_{\mathrm{PB}})|\psi^{s};A\rangle\ . \tag{3.34}$$

In particular, when we put $f(x,y)=[(x+y)/2]^{n}$ and $f(x,y)=[i(x-y)/2]^{n}$, we obtain the following relations:

$$\langle\Psi|\hat{S}^{n}|\Psi\rangle=\lim_{s\to\infty}\langle A;\psi^{s}|(\hat{S}^{s}_{\mathrm{PB}})^{n}|\psi^{2};A\rangle\ , \tag{3.35}$$

$$\langle\Psi|\hat{C}^{n}|\Psi\rangle=\lim_{s\to\infty}\langle A;\psi^{s}|(\hat{C}^{s}_{\mathrm{PB}})^{n}|\psi^{2};A\rangle\ , \tag{3.36}$$

where $\hat{S}$, $\hat{C}$, $\hat{S}^{s}_{\mathrm{PB}}$, and $\hat{C}^{s}_{\mathrm{PB}}$ are sine and cosine operators in the RNS phase-operator formalism and in the PB formalism.

Therefore it is found that when we calculate the average value with the state vector $|\Psi\rangle=|\psi;A\rangle\otimes|0;B\rangle$, the RNS phase-operator formalism gives results equivalent to those obtained by the PB formalism. However, the mathematical structures of the formalisms are quite different. In the PB formalism, the commutation relation between the annihilation and creation operators becomes $[a,a^{\dagger}]=1-|s\rangle(s+1)\langle s|$, and the PB phase operator includes the term $|s\rangle\langle 0|$ due to the finite dimensionality of the formalism. As pointed out by Collett [54], these extra terms give unphysical dynamics when the time evolution caused by a certain interaction Hamiltonian is considered. On the other hand, we have $[a,a^{\dagger}]=[b,b^{\dagger}]=1$ in the RNS phase-operator formalism. It will be shown in the next section that the RNS phase-operator formalism can be constructed in the Liouville space. The mathematical structure of the formalism is quite similar to thermofield dynamics [43], and so the useful method in thermofield dynamics can be used in the RNS phase-operator formalism. Thus, the RNS phase-operator formalism can be applied to investigate many kinds of physi-

cal systems besides quantum optical systems.

We have found that by using the state belonging to the subspace $\mathcal{H}_A \otimes \{|0;B\rangle\}$ of $\mathcal{H}_A \otimes \mathcal{H}_B$, the RNS phase-operator formalism gives results equivalent to those obtained using the Pegg-Barnett phase-operator method. The projection on the subspace $\mathcal{H}_A \otimes \{|0;B\rangle\}$ is expressed as $\hat{P} = \hat{1}_A \otimes |0;B\rangle\langle B;0|$. Now, let us consider another projection given by

$$\hat{\Pi} = |0;A\rangle\langle A;0| \otimes |0;B\rangle\langle B;0| + \sum_{n=1}^{\infty} \{|n;A\rangle\langle A;n| \otimes |0;B\rangle\langle B;0| + |0;A\rangle\langle A;0| \otimes |n;B\rangle\langle B;n|\} , \tag{3.37}$$

which transformations the RNS phase operator $\hat{D}$ into the following form:

$$\hat{D} \rightarrow \hat{Y} = \hat{\Pi}^{\dagger} \hat{D} \hat{\Pi} = \hat{e}_A \otimes |0;B\rangle\langle B;0| + |0;A\rangle\langle A;0| \otimes \hat{e}_B^{\dagger} , \tag{3.38}$$

where $\hat{e}_A$ and $\hat{e}_B$ are the Susskind-Glogower phase operators. This unitary operator $\hat{Y}$ is identical to the extended Susskind-Glogower phase operator considered by Shapiro *et al.* [32–34] who developed the phase-measurement theory within the framework of quantum detection theory [55,56]. Recently, Hradil derived the relation between the RNS phase operator and the description of the realizable Shapiro-Wagner phase [36,37]. The Shapiro-Wagner phase is treated within the framework of heterodyne detection and is expressed as a unitary operator as $\hat{R} = \sqrt{(\hat{a} + \hat{b}^{\dagger})/(\hat{a}^{\dagger} + \hat{b})}$. The commutation relation between $\hat{R}$ and $\hat{N} = a^{\dagger}a - b^{\dagger}b$ is given by $[\hat{R}, \hat{N}] = \hat{R}$, which is the same commutation relation as between $\hat{D}$ and $\hat{N}$. Hradil has shown the relation, $\hat{D} = \hat{U}\hat{R}\hat{U}^{\dagger}$, where $\hat{U}$ is a nonunitary operator. This relation has been derived by means of the Linblad-Nagel basis of su(1,1) Lie algebra [58]. The RNS phase operator is thus closely related to the heterodyne detection. The details are given in Refs. [35,36].

In this section, we have used the relative-number states $\{|n,m\rangle\rangle\}$ in order to remove the difficulties in defining the phase operator, which are caused by the lower bounded spectrum of the number operator. For the same purpose, Newton considered a two-valued spinlike variable [31], and he expressed a state vector of the relevant system as $|\Phi\rangle\rangle = \begin{pmatrix} |a\rangle\rangle \\ |b\rangle\rangle \end{pmatrix}$, where the upper component corresponds to the state with the up spin and the lower component to the state with down spin. In order to define the phase operator, Newton introduced a complete orthonormal set of the relevant system as follows:

$$S_N = \{|n\rangle\rangle \,|\, -\infty < n < \infty, n \in \mathbb{Z}\} \tag{3.39}$$

with

$$|n\rangle\rangle = \theta(n) \begin{pmatrix} |n\rangle \\ 0 \end{pmatrix} + \theta(-1-n) \begin{pmatrix} 0 \\ |-n-1\rangle \end{pmatrix} . \tag{3.40}$$

Then, using the states $\{|n\rangle\rangle\}$, the unitary phase operator is defined as

$$\hat{D}_N = \sum_{n=-\infty}^{\infty} |n\rangle\rangle\langle\langle n+1| . \tag{3.41}$$

In Newton's formalism, only the subspace with up spin has physical meaning. The projection on the subspace with up spin corresponds to that the state of the reference system is assumed to be vacuum in the relative-number-state formalism.

IV. ENERGY MEASUREMENT AND RELATIVE-ENERGY STATE

In this section, we apply the method developed in Sec. III to physical systems which have continuous energy spectra. We consider the measurement of energy in the relevant system and relative-energy states. Let us first consider the measurement of energy in the relevant system. In our formalism, it is postulated that the energy of the relevant system is measured as a difference from the energy of the reference system. The observable quantity is the relative energy $\hat{H} = \hat{H}_A - \hat{H}_B$, where $\hat{H}_A$ and $\hat{H}_B$ are the Hamiltonians of the relevant and reference systems, respectively. It is important to notice that $\hat{H}_A$ and $\hat{H}_B$ have lower bounded spectra where the lowest energies are assumed to be zero, while $\hat{H}$ has unbounded spectrum extended over all real values. We also assume that the physical state of the whole system is given by $|\Psi\rangle = |\psi, A\rangle \otimes |\phi, B\rangle$.

Under these assumptions, the average value of an analytic function of $\hat{H}$, $f(\hat{H})$, is calculated to be

$$\langle \Psi | f(\hat{H}) | \Psi \rangle = \int_0^{\infty} dE \langle A;\psi | f(\hat{H}_A - E) | \psi; A \rangle |\phi_B(E)|^2 , \tag{4.1}$$

where $\phi_B(E) = \langle B;E|\phi;B\rangle$ and $|E;B\rangle$ is an eigenstate of $\hat{H}_B$ with eigenvalue E. On the right-hand side of (4.1), it is seen that $\langle A;\psi|f(\hat{H}_A - E)|\psi;A\rangle$ is the average value of the relevant system, in which the energy of the relevant system is measured from the energy of the reference system E, and that $|\phi_B(E)|^2 dE$ is the probability that the reference system has an energy between E and $E + dE$. In particular, when the reference system has the lowest energy $E = 0$ with great accuracy, (4.1) can be approximated by

$$\langle \Psi | f(\hat{H}) | \Psi \rangle \approx \langle A;\psi | f(\hat{H}_A) | \psi; A \rangle . \tag{4.2}$$

The right-hand side of (4.4) is the conventional expression for the average value of the relevant system.

If the reference system has a discrete energy spectrum, (4.1) is modified as follows:

$$\langle \Psi | f(\hat{H}) | \Psi \rangle = \sum_{n=0}^{\infty} \langle A;\psi | f(\hat{H}_A - E_n) | \psi; A \rangle |\phi_B(E_n)|^2 , \tag{4.3}$$

where $\phi_B(E_n)=\langle B;E_n|\phi;B\rangle$ and $|E_n;B\rangle$ is an eigenstate of $\hat{H}_B$ with eigenvalue E_n. In particular, when the reference system is in the lowest-energy state ($E_0=0$), (4.3) becomes

$$\langle\Psi|f(\hat{H})|\Psi\rangle=\langle A;\psi|f(\hat{H}_A)|\psi;A\rangle\ . \tag{4.4}$$

Thus, we also obtain the conventional expression for the average value of the relevant system. It should be noted that both (4.1) and (4.3) can be expressed as

$$\langle\Psi|f(\hat{H})|\Psi\rangle=\overline{\langle A;\psi|f(\hat{H}_A-E)|\psi;A\rangle}\ , \tag{4.5}$$

where $\overline{(\ )}$ means the average taken over the possible energy of the reference system. This expression is the same as (2.23), (2.26), and (3.14).

Now we introduce a relative-energy state and consider its properties. To do this, we first suppose that the relevant and reference systems are described, respectively, by complete orthonormal sets,

$$S_A=\{|E;A\rangle|\hat{H}_A|E;A\rangle=E|E;A\rangle,E\in\mathbb{R}_+\}\ , \tag{4.6}$$

$$S_B=\{|E;B\rangle|\hat{H}_B|E;B\rangle=E|E;B\rangle,E\in\mathbb{R}_+\}\ , \tag{4.7}$$

where $\mathbb{R}_+$ is the set of non-negative real numbers, and $|E;A\rangle$ and $|E;B\rangle$ satisfy the following relations:

$$\langle X;E_1|E_2;X\rangle=\delta(E_1-E_2),\quad \int_0^\infty dE|E;X\rangle\langle X;E|=\hat{1}_X\ ,$$
$$\text{for } X=A,B\ , \tag{4.8}$$

where $\hat{1}_A$ and $\hat{1}_B$ are unit operators acting on the respective Hilbert space. Thus, the complete orthonormal basis of the whole system becomes

$$S=\{|E_1,E_2\rangle=|E_1;A\rangle\otimes|E_2;B\rangle|E_1,E_2\in\mathbb{R}_+\}\ , \tag{4.9}$$

where $|E_1,E_2\rangle$ satisfies

$$\begin{aligned}&\langle E_2,E_1|E_1',E_2'\rangle=\delta(E_1-E_1')\delta(E_2-E_2')\ ,\\ &\int_0^\infty dE_1\int_0^\infty dE_2|E_1,E_2\rangle\langle E_2,E_1|=\hat{1}\ ,\end{aligned} \tag{4.10}$$

with $\hat{1}=\hat{1}_A\otimes\hat{1}_B$.

Now, we define a relative-energy state $|E,\varepsilon\rangle\!\rangle$ as follows:

$$|E,\varepsilon\rangle\!\rangle=\Theta(E)|\varepsilon+E,\varepsilon\rangle+\Theta(-E)|\varepsilon,\varepsilon-E\rangle\ , \tag{4.11}$$

where $\Theta(E)$ is the usual step function. Note that E takes an arbitrary real value while ε is non-negative. The relative-energy state $|E,\varepsilon\rangle\!\rangle$ is an eigenstate of $\hat{H}=\hat{H}_A-\hat{H}_B$, so that $\hat{H}|E,\varepsilon\rangle\!\rangle=E|E,\varepsilon\rangle\!\rangle$. It is seen from (4.10) that the set $S_R=\{|E,\varepsilon\rangle\!\rangle|E\in\mathbb{R},\varepsilon\in\mathbb{R}_+\}$ becomes a complete orthonormal basis of the whole system, which satisfies

$$\begin{aligned}&\langle\!\langle\varepsilon_1,E_1|E_2,\varepsilon_2\rangle\!\rangle=\delta(E_1-E_2)\delta(\varepsilon_1-\varepsilon_2)\ ,\\ &\int_0^\infty d\varepsilon\int_{-\infty}^\infty dE|E,\varepsilon\rangle\!\rangle\langle\!\langle\varepsilon,E|=\hat{1}\ .\end{aligned} \tag{4.12}$$

Next, we consider the probability distribution for the energy measurement. To do this, we define a probability operator measure in terms of the relative-energy states $\{|E,\varepsilon\rangle\!\rangle\}$ as follows:

$$\hat{\Pi}(\Delta)=\int_E^{E+\Delta}dE'\int_0^\infty d\varepsilon|E',\varepsilon\rangle\!\rangle\langle\!\langle\varepsilon,E'|\ . \tag{4.13}$$

It is easily verified that $\hat{\Pi}(\Delta)$ satisfies the properties of probability operator measure. From (4.13), the probability distribution $P(E)dE$ that the energy difference between the relevant and reference systems takes a value between E and $E+dE$ is given by

$$P(E)dE=\langle\Psi|\hat{\Pi}(dE)|\Psi\rangle\ , \tag{4.14}$$

where $|\Psi\rangle$ is a physical state of the whole system. Using $|\Psi\rangle=|\psi;A\rangle\otimes|\phi,B\rangle$, (4.14) is calculated to be

$$\begin{aligned}P(E)=&\Theta(E)\int_0^\infty d\varepsilon|\psi_A(\varepsilon+E)|^2|\phi_B(\varepsilon)|^2\\ &+\Theta(-E)\int_0^\infty d\varepsilon|\psi_A(\varepsilon)|^2|\phi_B(\varepsilon-E)|^2\ ,\end{aligned} \tag{4.15}$$

where $\psi_A(E)=\langle A;E|\psi;A\rangle$ and $\phi_B(E)=\langle B;E|\phi;B\rangle$. It is found from (4.15) that the average value of $f(\hat{H})$ is

$$\langle\Psi|f(\hat{H})|\Psi\rangle=\int_{-\infty}^\infty dE\,f(E)P(E)\ . \tag{4.16}$$

In particular, when the reference system has energy $E=0$ with great accuracy, (4.15) can be approximated by $P(E)\approx\theta(E)|\psi_A(E)|^2$. This is the conventional expression for the probability density of the relevant system.

V. PHASE REPRESENTATION IN THE LIOUVILLE SPACE

A. Liouville space formulation

In the previous sections, we have considered the relative-coordinate states, the relative-number state, and the relative-energy state for a quantum system consisting of two subsystems: the relevant system and the reference system. In the following, we will consider the relative-number state and the relative-energy state for a single quantum system in the Liouville space. In this section, we briefly summarize the Liouville space formulation [37–41,45–48] and thermofield field dynamics [42–44] The Liouville space $\mathcal{L}$ can be constructed as a direct product of two Hilbert spaces, $\mathcal{L}=\mathcal{H}\otimes\tilde{\mathcal{H}}$. Here, $\mathcal{H}$ and $\tilde{\mathcal{H}}$ are the ordinary Hilbert spaces. The term A denotes an arbitrary operator acting on any vector in $\mathcal{H}$, and the corresponding operator $\tilde{A}$ acting on vectors in $\tilde{\mathcal{H}}$ is given by the tilde conjunction of A [42–44]. The tilde conjunction is defined by

$$\begin{aligned}&(A_1A_2)^\sim=\tilde{A}_1\tilde{A}_2,\quad(A^\dagger)^\sim=(\tilde{A})^\dagger\ ,\\ &(a_1A_1+a_2A_2)^\sim=a_1^*\tilde{A}_1+a_2^*\tilde{A}_2,\quad\tilde{\tilde{A}}=\sigma A\ ,\end{aligned} \tag{5.1}$$

where A, A_1, and A_2 are arbitrary operators, and a_1 and a_2 are c numbers. In the last relation of (5.1), $\sigma=1$ for a bosonic operator A and $\sigma=-1$ for a fermionic operator A. We consider a single-mode bosonic system, for simplicity. In this case, the Liouville space $\mathcal{L}$ is spanned by vectors belonging to a complete orthonormal set,

$$S_L=\{|m,n\rangle=|m\rangle\otimes|\tilde{n}\rangle\|m\rangle\in\mathcal{H},|\tilde{n}\rangle\in\tilde{\mathcal{H}},m,n\in\mathbb{Z}_+\}\ , \tag{5.2}$$

that satisfies the following relations:

$$\langle n_1,m_1|m_2,n_2\rangle=\delta_{m_1m_2}\delta_{n_1n_2}\ ,$$
$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|m,n\rangle\langle n,m|=1\ , \tag{5.3}$$

where $|m\rangle$ is the eigenstate of the number operator N in $\mathcal{H}$, $|\tilde{n}\rangle$ is the eigenstate of $\tilde{N}$ in $\tilde{\mathcal{H}}$, where $\tilde{N}$ is the tilde conjugate of N.

In the Liouville space $\mathcal{L}$, we introduce a state vector $|1\rangle$ [45–47] defined by

$$|1\rangle=\sum_{n=0}^{\infty}|n,n\rangle\ , \tag{5.4}$$

which satisfies the following relations:

$$a|1\rangle=\tilde{a}^{\dagger}|1\rangle,\quad a^{\dagger}|1\rangle=\tilde{a}|1\rangle,\quad |\tilde{1}\rangle=|1\rangle\ , \tag{5.5}$$

where a and $a^{\dagger}$ are bosonic annihilation and creation operators, and where $\tilde{a}$ and $\tilde{a}^{\dagger}$ are their tilde conjugates. These operators are defined by

$$a|m,n\rangle=\sqrt{m}\,|m-1,n\rangle\ ,$$
$$a^{\dagger}|m,n\rangle=\sqrt{m+1}|m+1,n\rangle\ , \tag{5.6}$$

$$\tilde{a}|m,n\rangle=\sqrt{n}\,|m,n-1\rangle\ ,$$
$$\tilde{a}^{\dagger}|m,n\rangle=\sqrt{n+1}|m,n+1\rangle\ , \tag{5.7}$$

with $a|0,n\rangle=\tilde{a}|n,0\rangle=0$ for all n. The relation (5.5) means that $|1\rangle$ is a tilde invariant state. Any state vector in the Liouville space $\mathcal{L}$ corresponds to an operator in the Hilbert space $\mathcal{H}$ [40,46]. For example, state vector $|m,n\rangle$ in $\mathcal{L}$ is equivalent to operator $|m\rangle\langle n|$ in $\mathcal{H}$. The correspondence between a state vector in $\mathcal{L}$ and an operator $\mathcal{H}$ is derived from the following rules:

$$a|m,n\rangle\Longleftrightarrow a|m\rangle\langle n|,\quad a^{\dagger}|m,n\rangle\Longleftrightarrow a^{\dagger}|m\rangle\langle n|\ , \tag{5.8}$$

$$\tilde{a}|m,n\rangle\Longleftrightarrow|m\rangle\langle n|a^{\dagger},\quad \tilde{a}^{\dagger}|m,n\rangle\Longleftrightarrow|m\rangle\langle n|a\ . \tag{5.9}$$

In general, we have

$$|A\rangle=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A_{mn}|m,n\rangle\Longleftrightarrow A=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|m\rangle A_{mn}\langle n|\ . \tag{5.10}$$

An operator acting on state vector $|A\rangle$ in $\mathcal{L}$ is equivalent to a superoperator acting on operator $\mathcal{A}$ in $\mathcal{H}$ [40,46].

An arbitrary state $|\Psi\rangle$ in $\mathcal{L}$ can be expanded as follows:

$$|\Psi\rangle=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}f_{mn}|m,n\rangle\ . \tag{5.11}$$

It is easily found from (5.4) that for any operator expressed as $A=A(a,a^{\dagger})$, the relation

$$\langle 1|A(a,a^{\dagger})|\Psi\rangle=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\langle n|A(a,a^{\dagger})|m\rangle f_{mn}\ , \tag{5.12}$$

is established. When an operator F acts on a vector in $\mathcal{H}$, whose matrix element is $f_{mn}=\langle m|F|n\rangle$, then we have $\langle 1|A(a,a^{\dagger})|\Psi\rangle=\mathrm{Tr}[A(a,a^{\dagger})F]$, where Tr indicates the trace operation on $\mathcal{H}$. This indicates that in $\mathcal{L}$ a scalar product with state vector $\langle 1|$ is equivalent to the trace operation in $\mathcal{H}$. Thus, if we put $F=\rho$, where ρ is a statistical operator of the system, we find that the quantum statistical average of $A=A(a,a^{\dagger})$ is calculated by the following matrix element in $\mathcal{L}$:

$$\langle A(a,a^{\dagger})\rangle=\langle 1|A(a,a^{\dagger})|\rho\rangle\ , \tag{5.13}$$

where

$$|\rho\rangle=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\rho_{mn}|m,n\rangle=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|m,n\rangle\langle m|\rho|n\rangle\ .$$

It is found that since ρ is a Hermitian operator, $|\rho\rangle$ is a tilde invariant state, $(|\rho\rangle)^{\sim}=|\rho\rangle$. For example, the thermal average is obtained if we substitute $f_{mn}=\delta_{mn}\bar{n}^{\,n}/(1+\bar{n})^{n+1}$ into (5.12). Here $\bar{n}$ is the boson distribution function.

A quantum statistical-mechanical system is described by a density matrix $\rho(t)$ in the Hilbert space $\mathcal{H}$. The time evolution of the system is governed by the Liouville–von Neumann equation,

$$\frac{\partial}{\partial t}\rho(t)=-i[H,\rho(t)]=-iL\rho(t)\ , \tag{5.14}$$

where H is the Hamiltonian of the system and L is the Liouvillian superoperator. In the Liouville space $\mathcal{L}$, the system is described by the state vector $|\rho(t)\rangle$ corresponding to $\rho(t)$ and the time evolution of the state vector $|\rho(t)\rangle$ is determined by

$$\frac{\partial}{\partial t}|\rho(t)\rangle=-i\hat{H}|\rho(t)\rangle,\quad \hat{H}=H-\tilde{H}\ , \tag{5.15}$$

where $\tilde{H}$ is the tilde conjugate of H. Equation (5.15) is derived from (5.14) using the correspondence rules (5.8)–(5.9). In the Liouville space $\mathcal{L}$ the Liouville–von Neumann equation takes the same form as the Schrödinger equation. This is true for dissipative systems [45–47]. In the presence of dissipation, $\hat{H}=H-\tilde{H}$ is modified as follows:

$$\hat{H}=H-\tilde{H}+i\hat{\Pi}\ , \tag{5.16}$$

where $\hat{\Pi}$ is a damping operator satisfying $(\hat{\Pi})^{\sim}=\tilde{\Pi}$. In general, $\hat{\Pi}$ includes the product of tilde and nontilde operators such as $a\tilde{a}$ and $a^{\dagger}\tilde{a}^{\dagger}$. On the other hand, the time evolution of any operator $\hat{A}$ in the Liouville space $\mathcal{L}$ is determined by

$$\frac{d}{dt}\hat{A}(t)=i[\hat{H},\hat{A}(t)]\ . \tag{5.17}$$

This is the same form as the conventional Heisenberg equation except that $\hat{H}=H-\tilde{H}$ or $\hat{H}=H-\tilde{H}+i\hat{\Pi}$ is used instead of H. The details are given in Refs. [41–48].

B. Phase representation

In this section, we introduce a phase representation in the Liouville space $\mathcal{L}$ in terms of the relative-number

states. It will be found that a phase representation in $\mathcal{L}$ is convenient and suitable for investigating the time evolution of quantum coherence or the phase properties of physical systems.

In the Liouville space $\mathcal{L}$, the relative-number state is defined by

$$|n,m\rangle\rangle=\theta(n)|m+n\rangle\otimes|\tilde{m}\rangle + \theta(-n-1)|m\rangle\otimes|m-n\rangle^{\sim} . \tag{5.18}$$

It is easily seen that $|n,m\rangle\rangle$ is an eigenstate of $\hat{N}=a^{\dagger}a-\tilde{a}^{\dagger}\tilde{a}$ with eigenvalue n. In this case, the relative number represents the boson-number difference between the physical system and its tilde conjugate system. The set of relative-number states, $S_R=\{|n,m\rangle\rangle|m\in\mathbb{Z}_+,n\in\mathbb{Z}\}$, becomes a complete orthonormal basis in the Liouville space $\mathcal{L}$. By using the relative-number states $\{|n,m\rangle\rangle\}$, the RNS phase operator $\hat{D}$ in $\mathcal{L}$ is defined by $\hat{D}=\sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty}|n,m\rangle\rangle\langle\langle m,n+1|$, and the eigenstate of $\hat{D}$ is given by

$$|\phi,m\rangle\rangle=\frac{1}{(2\pi)^{1/2}}\sum_{n=-\infty}^{\infty}|n,m\rangle\rangle e^{-in\phi} . \tag{5.19}$$

A set given by $S_{\phi}=\{|\phi,m\rangle\rangle|\phi\in[-\pi,\pi)\subset\mathbb{R},m\in\mathbb{Z}_+\}$ is also a complete orthonormal basis in $\mathcal{L}$. Here, we have assumed that the domain of phase variables is $[-\pi,\pi)$, for simplicity.

Consider a system described by the state vector $|\rho(t)\rangle$ in the Liouville space $\mathcal{L}$. Since $\{|n,m\rangle\rangle\}$ and $\{|\phi,m\rangle\rangle\}$ are complete orthonormal sets in $\mathcal{L}$, $|\rho(t)\rangle$ is expanded as follows:

$$|\rho(t)\rangle=\sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty}f(t;n,m)|n,m\rangle\rangle , \tag{5.20}$$

$$|\rho(t)\rangle=\sum_{m=0}^{\infty}\int_{-\pi}^{\pi}d\phi g(t;\phi,m)|\phi,m\rangle\rangle , \tag{5.21}$$

where $f(t;n,m)=\langle\langle m,n|\rho(t)\rangle$ and $g(t;\phi,m)=\langle\langle m,\phi|\rho(t)\rangle$. It is easily found from (5.19) that the relation between $f(t;n,m)$ and $g(t;\phi,m)$ is given by the Fourier transformation,

$$g(t;\phi,m)=\frac{1}{(2\pi)^{1/2}}\sum_{n=-\infty}^{\infty}e^{in\phi}f(t;n,m) , \tag{5.22}$$

$$f(t,n,m)=\frac{1}{(2\pi)^{1/2}}\int_{-\pi}^{\pi}d\phi\, e^{-in\phi}g(t;\phi,m) . \tag{5.23}$$

The function $g(t;\phi,m)$ is called the phase representation of the state vector $|\rho(t)\rangle$. Since $|\rho(t)\rangle$ is tilde invariant, the expansion coefficients $f(t;n,m)$ and $g(t;\phi,m)$ should satisfy

$$f^*(t;n,m)=f(t;-n,m) , \quad g^*(t;\phi,m)=g(t;\phi,m) , \tag{5.24}$$

where we have used the relations, $(|n,m\rangle\rangle)^{\sim}=|-n,m\rangle\rangle$ and $(|\phi,m\rangle\rangle)^{\sim}=|\phi,m\rangle\rangle$. It should be noted that the phase variable ϕ in $g(t;\phi,m)$ characterizes the quantum coherence or the off-diagonal matrix elements of the statistical operator $\rho(t)$.

Now, consider a simple harmonic oscillator, for example, whose Hamiltonian is given by $H=\omega a^{\dagger}a$ ($\hbar=1$). In such a case, the time-evolution generator in the Liouville space $\mathcal{L}$ becomes $\hat{H}=\omega a^{\dagger}a-\omega\tilde{a}^{\dagger}=\omega\hat{N}$. From (5.15), (5.20), and (5.21), the expansion coefficients $f(t;n,m)$ and $g(t;\phi,m)$ are determined by the following equations:

$$\frac{\partial}{\partial t}f(t;n,m)=-i\omega nf(t;n,m) , \quad \frac{\partial}{\partial t}g(t;\phi,m)=-\omega\frac{\partial}{\partial\phi}g(t;\phi,m) . \tag{5.25}$$

The solutions of these equations are given, respectively, by

$$f(t,n,m)=e^{-i\omega nt}F(n,m) , \quad g(t;\phi,m)=G(\phi-\omega t,m) , \tag{5.26}$$

where F and G are determined by the initial condition. For an arbitrary stationary state, we have $f(t;n,m)=\delta_{n0}F_0(m)$ and $g(t;\phi,m)=G_0(m)$, where $F_0(m)$ and $G_0(m)$ depend only on m. These results are the well-known fact that in the stationary state the density matrix in the boson-number basis becomes diagonal and coherence is lost completely.

Next, we consider a damped harmonic oscillator in the phase representation. Under the Markovian approximation, the statistical operator $\rho(t)$ of the damped harmonic oscillator in the Hilbert space $\mathcal{H}$ satisfies the Liouville–von Neumann equation as follows:

$$\frac{\partial}{\partial t}\rho(t)=-i\omega[a^{\dagger}a,\rho(t)]-\kappa\{(\bar{n}+1)[a\rho(t),a^{\dagger}]+\bar{n}[a^{\dagger},\rho(t)a]+(\bar{n}+1)[a,\rho(t)a^{\dagger}]+\bar{n}[a^{\dagger}\rho(t),a]\} , \tag{5.27}$$

where κ is a damping constant determined by the reservoir correlation function and $\bar{n}$ is the equilibrium boson distribution function. According to the correspondence rules (5.8)–(5.10), in the Liouville space $\mathcal{L}$ the time evolution of the state vector $|\rho(t)\rangle$ of the damped harmonic oscillator is determined by

$$\frac{\partial}{\partial t}|\rho(t)\rangle=-i\hat{H}|\rho(t)\rangle , \tag{5.28}$$

with

$$\hat{H}=\omega(a^{\dagger}a-\tilde{a}^{\dagger}\tilde{a})-i\kappa[(2\bar{n}+1)(a^{\dagger}a+\tilde{a}^{\dagger}\tilde{a})-2(\bar{n}+1)a\tilde{a}-2\bar{n}a^{\dagger}\tilde{a}+2\bar{n}] . \tag{5.29}$$

When we define a phase probability distribution $P(t,\phi)$ by

$$P(t,\phi)=\sum_{m=0}^{\infty} g(t;\phi,m) , \tag{5.30}$$

which is normalized as $\int_{-\pi}^{\pi} d\phi\, P(t,\phi)=1$, where $g(t;\phi,m)$ is the phase representation of $|\rho(t)\rangle$, then $P(t,\phi)$ satisfies the following equation:

$$\frac{\partial}{\partial t}P(t,\phi)=-\omega\frac{\partial}{\partial\phi}P(t,\phi)+\sum_{m=0}^{\infty} F\left[-i\frac{\partial}{\partial\phi},m\right]g(t;\phi,m) , \tag{5.31}$$

where the coefficient $F(-i\partial/\partial\phi,m)$ which describes the dissipative effect is given by

$$F\left[-i\frac{\partial}{\partial\phi},m\right]e^{in\phi}=F(n,m)e^{in\phi} , \tag{5.32}$$

$$F(n,m)=2\kappa(\bar{n}+1)\sqrt{m(m+|n|)}+2\kappa\bar{n}\sqrt{(m+1)(m+|n|+1)}-2\kappa\bar{n}-\kappa(2\bar{n}+1)(2m+|n|) . \tag{5.33}$$

Note that the relation $F(0,m)=0$ ensures the normalization of $P(t,\phi)$. When the averaged boson number of the system is extremely large, this equation is simplified as follows:

$$\frac{\partial}{\partial t}P(t,\phi)=-\omega\frac{\partial}{\partial\phi}P(t,\phi)+2\kappa\frac{\partial^2}{\partial\phi^2}P(t,\phi) . \tag{5.34}$$

This is the Fokker-Planck equation describing the phase relaxation process in the semiclassical regime. Equations (5.31) and (5.34) describe the decay of phase information or quantum coherence in the relaxation process from an arbitrary initial state to the thermal equilibrium state.

In this section, we have presented the phase representation in the Liouville space $\mathcal{L}$ based on the relative-number state $\{|n,m\rangle\rangle\}$. Stenholm [59] gave another phase representation for the density matrix in Hilbert space $\mathcal{H}$, which is defined by

$$\rho(t;\phi,\mu=m+n-2\bar{n})=\sum_{\nu=n-m=-\infty}^{\infty} e^{i\nu\phi}\langle n|\rho(t)|m\rangle , \tag{5.35}$$

where $\bar{n}$ is the average boson number. As pointed out by Stenholm, since m and n are non-negative integers and so the range of ν depends on μ, (5.35) is not rigorously correct, and is only approximately correct when $\bar{n}\gg 1$. Thus, (5.35) can be used only to investigate the properties of physical systems in the semiclassical regime. On the other hand, the phase representation in terms of the relative-number states is rigorously correct in any case and can be used to investigate both quantum and semiclassical systems. Furthermore, thermofield dynamical method [43,45–47] can be used in the RNS phase representation. Bialynicki-Birula and Bialynicki-Birula [60] also introduced the phase representation of wave functions and used it to investigate the semiclassical properties of photons. However, this phase representation is difficult to apply to dissipative systems interacting with thermal reservoir.

C. Time operator

In this section, we will construct a time operator in the Liouville space $\mathcal{L}$ for a system whose Hamiltonian H has a continuous spectrum. We assume that the lowest energy is zero. In the Hilbert space $\mathcal{H}$, the system can be described by a complete orthonormal set, $S_E=\{|E\rangle|H|E\rangle=E|E\rangle, E\in\mathbb{R}_+\}$. The tilde conjunction derives a complete orthonormal set in the tilde conjugated space $\tilde{\mathcal{H}}$, which is given by $S_{\tilde{E}}=\{|\tilde{E}\rangle|\tilde{H}|\tilde{E}\rangle=E|\tilde{E}\rangle, E\in\mathbb{R}_+\}$, where $\tilde{H}$ is the tilde conjugate of H. Thus the system in $\mathcal{L}$ is described by the complete orthonormal set,

$$S_{E+\tilde{E}}=\{|E_1,E_2\rangle=|E_1\rangle\otimes|\tilde{E}_2\rangle\,||E\rangle\in\mathcal{H},|\tilde{E}\rangle\in\tilde{\mathcal{H}},E_1,E_2\in\mathbb{R}_+\} , \tag{5.36}$$

where $|E_1,E_2\rangle$ satisfies the following relations:

$$\begin{aligned}&\langle E_2,E_1|E_1',E_2'\rangle=\delta(E_1-E_1')\delta(E_2-E_2') ,\\ &\int_0^{\infty}dE_1\int_0^{\infty}dE_2|E_1,E_2\rangle\langle E_2,E_1|=1 .\end{aligned} \tag{5.37}$$

when $|\rho(t)\rangle$ is a state vector of the system in $\mathcal{L}$, the time evolution of $|\rho(t)\rangle$ is governed by (5.15). It is important to remember that in $\mathcal{L}$ the time-evolution generator $\hat{H}$ is different from the Hamiltonian energy operator H. The Hamiltonian energy operator has a lower bounded spectrum while the spectrum of the time-evolution generator extends over all real values. It should be noted that this fact makes it possible to define a time operator in $\mathcal{L}$.

The relative-energy state in the Liouville space $\mathcal{L}$ is defined by

$$|\mathcal{E},E\rangle\rangle=\Theta(\mathcal{E})|E+\mathcal{E},E\rangle+\Theta(-\mathcal{E})|E,E-\mathcal{E}\rangle . \tag{5.38}$$

The relative-energy state is the eigenstate of the time-evolution generator $\hat{H}$ with eigenvalue $\mathcal{E}$ which can take any real value. A set defined by $S=\{|\mathcal{E},E\rangle\rangle|\mathcal{E}\in\mathbb{R},E\in\mathbb{R}_+\}$ becomes a complete orthonormal basis in the Liouville space $\mathcal{L}$, satisfying

$$\langle\langle E,\mathcal{E}|\mathcal{E}',E'\rangle\rangle=\delta(\mathcal{E}-\mathcal{E}')\delta(E-E')\ ,$$
$$\int_0^\infty dE\int_{-\infty}^{\infty}d\mathcal{E}|\mathcal{E},E\rangle\rangle\langle\langle E,\mathcal{E}|=1\ . \tag{5.39}$$

Using the relative-energy states, we define a unitary operator

$$D(\xi)=\int_0^\infty dE\int_{-\infty}^{\infty}d\mathcal{E}|\mathcal{E}-\xi,E\rangle\rangle\langle\langle E,\mathcal{E}|\ , \tag{5.40}$$

which is a displacement operator with relative energy $\mathcal{E}$, and which satisfies the following commutation relation:

$$[D(\xi)\hat{H}]=\xi D(\xi)\ . \tag{5.41}$$

It is easily seen that $\{D(\xi)|\xi\in\mathbb{R}\}$ becomes a one-parameter unitary group and $D(\xi)$ is strongly continuous. Thus, from the Stone theorem [52], a Hermitian operator $\hat{T}$ exists such that

$$D(\xi)=\exp[-i\xi\hat{T}]\ . \tag{5.42}$$

In the limit as $\xi\to 0$ in (5.41), we can obtain the commutation relation,

$$[\hat{T},\hat{H}]=i\ . \tag{5.43}$$

Thus, $\hat{T}$ is a canonical conjugate of the time-evolution generator $\hat{H}$, but not the Hamiltonian energy operator H. It is found that the eigenstates of $\hat{T}$ are given by

$$|\tau,E\rangle\rangle=\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}d\mathcal{E}|\mathcal{E},E\rangle\rangle e^{i\mathcal{E}\tau}\ , \tag{5.44}$$

and a set given by $S_T=\{|\tau,E\rangle\rangle|E\geq 0,\tau\in\mathbb{R},E\in\mathbb{R}_+\}$ becomes a complete orthonormal basis in $\mathcal{L}$. Furthermore, from (5.42) and (5.43) we can obtain the following relations:

$$\hat{H}\|\tau,E\rangle\rangle=i\frac{\partial}{\partial\tau}|\tau,E\rangle\rangle,\quad \langle\langle E,\tau|\hat{H}=-i\frac{\partial}{\partial\tau}\langle\langle E,\tau|\ , \tag{5.45}$$

$$\hat{T}|\mathcal{E},E\rangle\rangle=-i\frac{\partial}{\partial\mathcal{E}}|\mathcal{E},E\rangle\rangle,\quad \langle\langle E,\mathcal{E}|\hat{T}=i\frac{\partial}{\partial\mathcal{E}}\langle\langle E,\mathcal{E}|\ . \tag{5.46}$$

Since $\{|\tau,E\rangle\rangle\}$ is a complete orthonormal basis, we can expand an arbitrary state $|\Psi(t)\rangle$ at time t as follows:

$$|\Psi(t)\rangle=\int_0^\infty dE\int_{-\infty}^{\infty}d\tau\, g(t;\tau,E)|\tau,E\rangle\rangle\ . \tag{5.47}$$

Then, using (5.45), the time-evolution equation (5.15) gives $g(t;\tau,E)=g(t-\tau,E)$. Furthermore, the tilde invariance of $|\Psi(t)\rangle$ requires $g(t-\tau,E)$ to be real. For any stationary state, $g(t;\tau,E)$ depends on neither t nor τ.

Now, let us consider the properties of $\hat{T}$. In the Liouville space $\mathcal{L}$, the time evolution of an arbitrary operator A is determined by a Heisenberg-like equation (5.17). Thus, from the commutation relation (5.43), we obtain,

$$\frac{d}{dt}\hat{T}(t)=1,\quad \hat{T}(t)=\hat{T}(0)+t\hat{1}\ . \tag{5.48}$$

Taking the average values of these equations, we obtain the following relations:

$$d\tau(t)=dt,\quad \tau(t)=\tau(0)+t\ , \tag{5.49}$$

where $\tau(t)=\langle\hat{T}(t)\rangle$ and $\langle\ \rangle$ means a certain expectation value. These relations show that an increment of the expectation value of $\hat{T}(t)$ during time dt is equal to an increment of time t, that is dt. It is found that the difference between $\tau(t)$ and $t,\tau(0)$, can be attributed to the initial preparation of the system. Thus, $\tau(t)$ is equivalent to t with regard to dynamics, and $\hat{T}$ can be regarded a time operator in the Liouville space $\mathcal{L}$.

It is important to notice that $\hat{T}$ and $\hat{D}(\xi)$ do not preserve the factorizability of state. In the Liouville space $\mathcal{L}$, when a physical state $|\Psi\rangle$ is factorizable, we have

$$|\Psi\rangle=|\psi\rangle\otimes|\tilde{\psi}\rangle\ , \tag{5.50}$$

since $|\Psi\rangle$ has to be invariant under tilde conjugation. It is easily seen that (5.50) corresponds to $|\psi\rangle\langle\psi|$ in the Hilbert space $\mathcal{H}$, and so (5.50) represents a pure state of the system. Thus, $\hat{T}$ and $\hat{D}(\xi)$ change a pure state of the system into a mixed state. This fact was first pointed out by Prigogine and Misra [49,50], who also showed that for a system and with time and entropy operators, the distinction between pure and mixed states becomes meaningless. The underlying idea for constructing a time operator as discussed in this section is the same as that given by Prigogine and Misra [49,50] who pointed out that the lack of a time operator in quantum mechanics stems from the fact that the time-evolution generator is identical with the Hamiltonian energy operator of the system. The Hamiltonian energy operator must have a lower bounded spectrum in a stable of world, and thus so does the time-evolution generator. This lower bound of the spectrum prevents the existence of a time operator. Thus, they have proposed that in order to introduce a time operator into quantum mechanics the physical system should be described in the Liouville space since the time-evolution generator is different from the Hamiltonian energy operator in the Liouville space. In this section, using the relative-energy states in the Liouville space $\mathcal{L}$, we have realized this idea of Prigogine and Misra and have constructed the time operator systematically.

VI. SUMMARY

In this paper, we introduced relative-coordinate states, relative-number states, and relative-energy states, and we presented a relative-state formulation for quantum systems. We considered two cases; physical systems consisting of two subsystems in the Hilbert space $\mathcal{H}$, and physical systems in the Liouville space $\mathcal{L}$.

In Sec. II, using the relative-coordinate states, we gave a description of quantum systems consisting of relevant and reference subsystems in position and momentum representations. When the reference system satisfies a certain condition, the relative-state formulation reduces to the usual description of a relevant system. We also showed that the probability distribution calculated by the probability operator measure, which is constructed in terms of the relative-coordinate states, is equivalent to the functional definition of quantum probability distribution or the propensity in phase space considered by Aharonov, Albert, and Au [21], O'Connell and Rajagopal

[22], Prugoveçki [23], and Wódkiewicz [24,25]. In Sec. III, we considered a boson-number measurement and defined a quantum-mechanical phase operator in terms of the relative-number states. We showed that by using the relative number states it is possible to remove the well-known difficulties in defining the phase operator, which stem from the fact that the number operator has a lower bounded spectrum. We have shown that if we assume that the reference system is in the vacuum state, the average values of physical quantities thus calculated are equal to those given by the Pegg-Barnett phase-operator method [27–30]. Furthermore, we discussed the relation to other phase-operator methods presented by Newton [31], Shapiro *et al.* [32–34], and Hradil [35,36]. In Sec. IV, we have introduced the relative-energy states and investigated the energy measurement in terms of them.

In Sec. V, we developed the relative-state formulation in the Liouville space. In this case, we did not assume that the system consists of two subsystems. Using the relative-number states, we introduced the phase representation in the Liouville space. It was shown that the phase representation is useful for investigating the time evolution of quantum coherence or off-diagonal matrix elements of the statistical operator. Furthermore, we showed that when the system has a continuous spectrum, a time operator can be defined systematically in terms of the relative-energy states. The time operator is a canonical conjugate of the time-evolution generator, but not the Hamiltonian energy operator. Finally, we would like to point that, although we considered the various problems, all the problems can be treated systematically by the relative-state method.

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