Quantum violation of stochastic noncontextual hidden-variable theories

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We formulate stochastic noncontextual hidden-variable theories. In such theories the hidden state λ specifies not definite values, but expectation values $\overline{A}(\lambda), \ldots$ of observables $A, \ldots$ and noncontextuality means that $\overline{A}(\lambda)$ is independent of which other commuting observables commuting with A are measured together with A. We show via Bell inequalities that such theories conflict with quantum theory and propose a two-photon experimental test. We also show that for a single particle of spin $(2^n-1)/2$, quantum violation of classical noncontextuality grows exponentially with n.

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In quantum theory, the expectation value of an observable A is unaffected by the previous (or simultaneous) measurement of any set of mutually commuting observables commuting with A. This "statistical noncontextuality" provides the inspiration for the "noncontextuality" hypothesis in hidden-variable or deeper-level theories. In a deterministic noncontextual hidden-variable theory, the value of an observable A is independent of which particular set of mutually commuting observables commuting with A are measured together with A. The Gleason and Kochen-Specker theorems [1] prove the impossibility of such a hidden-variable theory for quantum mechanics. A particularly interesting case of noncontextuality arises when observables A and B commute due to spacelike separation, and the noncontextuality hypothesis becomes the Einstein locality hypothesis [2]. In this case Bell's theorem [3] proves a stronger result, viz. the impossibility, not only of deterministic but also of stochastic local hidden-variable theories for quantum mechanics.

Further Mermin [4] and Roy and Singh [5] (MRS) have shown that for n spin-$\frac{1}{2}$ particles quantum theory violates the Einstein-Bell locality by an exponentially large factor $2^{(n-1)/2}$.

The purpose of the present work is threefold. (1) We extend the Gleason-Kochen-Specker theorem to stochastic noncontextual hidden-variable theories. (2) We propose a two-photon experiment to test inequalities on photon polarization correlations implied by such theories and violated by quantum theory. (3) For a single particle of spin S, with $2S+1=2^n$, n integer, we show that quantum violation of noncontextuality grows exponentially with n. Previously [6] only violations (of Einstein locality) for two spinning particles by a factor up to $\sqrt{2}$ have been known. The violations we find are illustrations of the quantum theory conflicting with the classical idea of noncontextual realism. Evidently, noncontextuality is more stringent than locality because it applies to a greater variety of physical situations. For example, for a *single* high-spin particle there are no locality inequalities. Implications for measurement theory will be discussed.

Stochastic noncontextual hidden variable theories. Let $A_1(a_1), \ldots, A_n(a_n)$ be dynamical variables corresponding to different degrees of freedom of a given physical system, the settings of the apparatus measuring $A_j(a_j)$ being denoted by a_j. In quantum theory the dynamical variables are represented by observables $A_i(a_i)$, with

$$[A_i(a_i), A_j(a_j)]=0 . \tag{1}$$

In a hidden-variable theory, the state of the system may be characterized by variables λ (which may include the quantum-state vector as well), with $\rho(\lambda)$ being their probability distribution obeying

$$\rho(\lambda)\geq 0, \quad \int d\lambda\, \rho(\lambda)=1 . \tag{2}$$

For given λ, the dynamical variables have expectation values $\overline{A}_1(\lambda,a_1), \overline{A}_2(\lambda,a_2), \ldots, \overline{A}_n(\lambda,a_n)$, noncontextuality implying that $\overline{A}_i$ depends only on λ and a_i, but not on a_j with $j\neq i$. We assume that by the very definition of the A_i, $|A_i|\leq 1$ (e.g., $A_i=+1$ for transmission through a polarizer and -1 for nontransmission), and hence that

$$|\overline{A}_i(\lambda,a_i)|\leq 1, \quad \forall i . \tag{3}$$

Further, in complete analogy to Bell's argument for local stochastic theories [3] we define noncontextual stochastic theories to be those in which $\overline{A_1A_2}(\lambda,a_1,a_2) =\overline{A}_1(\lambda,a_1)\overline{A}_2(\lambda,a_2)$. To motivate this, suppose that $\overline{A}_i(\lambda,a_i)$ is the average over hidden variables λ_i of the apparatus which measures A_i to be $A_i(\lambda,\lambda_i,a_i)$. Noncontextuality requires that $A_1(\lambda,\lambda_1,a_1)$ and the probability distribution $\rho_1(\lambda_1)$ of λ_1 must be independent of which commuting variables are measured together with A_1 and in particular of λ_2,a_2. Hence

$$\begin{aligned}\overline{A_1A_2}(\lambda,a_1,a_2)&=\int d\lambda_1 d\lambda_2 \rho_1(\lambda_1)\rho_2(\lambda_2)\\ &\quad\times A_1(\lambda,\lambda_1,a_1)A_2(\lambda,\lambda_2,a_2)\\ &=\overline{A}_1(\lambda,a_1)\overline{A}_2(\lambda,a_2) .\end{aligned}$$

The n-variable correlation function thus has the representation

$$P(a_1,a_2,\ldots,a_n)=\int d\lambda\,\rho(\lambda)\prod_{i=1}^{n}\overline{A}_i(\lambda,a_i) . \tag{4}$$

The corresponding quantum correlation function in a state ψ is

$$[P(a_1,a_2,\ldots,a_n)]_\psi=\left\langle\psi\left|\prod_{i=1}^{n}A_i(a_i)\right|\psi\right\rangle . \quad (5)$$

As we show below, Bell's work applied to the above formulation entails the following theorem.

Theorem. There exist quantum systems for which some experimentally verifiable quantum predictions cannot be reproduced by any noncontextual stochastic hidden-variable model; i.e., there exist quantum states ψ such that

$$P(a_1,a_2,\ldots,a_n)\neq[P(a_1,a_2,\ldots,a_n)]_\psi . \quad (6)$$

On the other hand it is obvious that the representation for n-variable correlation functions assumed for stochastic noncontextual hidden-variable theories necessarily holds in any classical theory. Hence the noncontextuality inequalities derived below yield a quantitative measure of the amount by which quantum theory must violate any classical theory.

Proof of the theorem for two spin-$\frac{1}{2}$ particles and an experimental test of noncontextuality. Let $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ be the Pauli spin operators for two particles. Then

$$[\boldsymbol{\sigma}_1\cdot\mathbf{a}_1\boldsymbol{\sigma}_2\cdot\mathbf{a}_2,\boldsymbol{\sigma}_1\cdot\mathbf{b}_1\boldsymbol{\sigma}_2\cdot\mathbf{b}_2]=0 , \quad (7)$$

if $\mathbf{a}_1,\mathbf{a}_2,\mathbf{b}_1,\mathbf{b}_2$ are unit vectors obeying $\mathbf{a}_i\perp\mathbf{b}_i$, i.e., $\mathbf{a}_1\cdot\mathbf{b}_1=\mathbf{a}_2\cdot\mathbf{b}_2=0$. Notice that Eq. (7) is not related to locality. Let us denote $a=\{\mathbf{a}_1,\mathbf{a}_2\}$, $b=\{\mathbf{b}_1,\mathbf{b}_2\}$, and $A(a)=\boldsymbol{\sigma}_1\cdot\mathbf{a}_1\boldsymbol{\sigma}_2\cdot\mathbf{a}_2$; then $A(a)$ has eigenvalues ±1, and $[A(a),A(b)]=0$. Hence, in a stochastic noncontextual hidden-variable theory we must have the correlation function $P(a,b)$ obeying (4) with $[A(\lambda,a)]^2\leq1$. This leads to Bell's inequalities,

$$|P(a,b)-P(a,b')|+|P(a',b)+P(a',b')|\leq2 , \quad (8)$$

provided that $\mathbf{a}_i\perp\mathbf{b}_i$, $\mathbf{a}_i\perp\mathbf{b}'_i$, $\mathbf{a}'_i\perp\mathbf{b}_i$, $\mathbf{a}'_i\perp\mathbf{b}'_i$ for $i=1$ and 2. On the other hand quantum mechanics gives in the singlet state ψ

$$[P(a,b)]_{\rm QM}=\langle\psi|A(a)A(b)|\psi\rangle=(\mathbf{a}_1\times\mathbf{b}_1)\cdot(\mathbf{a}_2\times\mathbf{b}_2) . \quad (9)$$

The orthogonality conditions on a,b are obeyed if we choose $\mathbf{b}_1$, $\mathbf{b}'_1$, $\mathbf{a}_2$, and $\mathbf{a}'_2$ along the negative z axis, $\mathbf{b}_1=\mathbf{b}'_1=\mathbf{a}_2=\mathbf{a}'_2=(0,0,-1)$, and the remaining vectors in the x-y plane. In particular, the choice $\mathbf{a}_1=(1,0,0)$, $\mathbf{a}'_1=(0,-1,0)$, $\mathbf{b}_2=(-1/\sqrt{2},1/\sqrt{2},0)$, $\mathbf{b}'_2=(1/\sqrt{2},1/\sqrt{2},0)$ leads to

$$|\{P(a,b)-P(a,b')\}_{\rm QM}|+|\{P(a',b)+P(a',b')\}_{\rm QM}| =2\sqrt{2} , \quad (10)$$

which violates the noncontextuality inequality by a factor $\sqrt{2}$. This proves the announced theorem. For experimental purposes, a two-photon version of the violation of noncontextuality by quantum mechanics might be more practical to test for the two-photon state $\psi=(|x\rangle|y\rangle-|y\rangle|x\rangle)/\sqrt{2}$, where $|x\rangle$ and $|y\rangle$ denote photon states plane polarized along the x and y axes, respectively. Replacing $\boldsymbol{\sigma}$ by the 3×3 photon spin operator $\boldsymbol{\Sigma}$ which equals $\boldsymbol{\sigma}$ for x and y components of photon spin wave function and has zeros on the third row and third column, we obtain as before Eq. (9), which violates the Bell inequalities following from noncontextuality by a factor $\sqrt{2}$, for the choice of a,b,a',b' given already.

The experiment (Fig. 1) with a two-photon source of the kind used in Refs. 7 or 8 will provide a test of the quantum superposition principle against the classical idea of noncontextual realism in a situation where locality is not the issue [9].

Power-law violation of classical behavior for a particle of high spin. A striking consequence of the above formulation of stochastic noncontextual theories (entirely outside the scope of Bell's locality theorem) is the following result. For a single particle of spin $S=(2^n-1)/2$, where $n=1,2,3,\ldots$, quantum theory violates noncontextual realism by a factor $(S+\frac{1}{2})^{1/2}$ if n is odd, and $(S+\frac{1}{2})^{1/2}/\sqrt{2}$ if n is even.

A particle with spin S may be described quantum mechanically by means of a $(2S+1)$-component wave function ψ in a suitable orthonormal basis $|\alpha\rangle$. When $2S+1=2^n$, we may choose the labels α to be n-tuples: $\alpha\equiv m_1m_2\cdots m_n$, where $m_i=\pm1$ (or simply $m_i=\pm$). Thus

$$|\psi\rangle=\sum_\alpha\psi_\alpha|\alpha\rangle,\quad \alpha\equiv m_1m_2\cdots m_n,\ m_i=\pm .$$

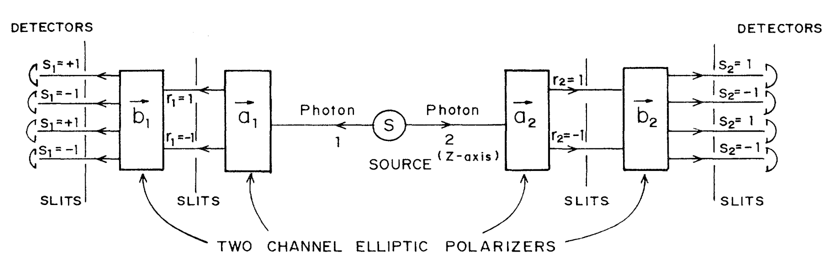

FIG. 1. Schematic diagram of experimental apparatus to test the noncontextuality inequality. The source S emits two photons in the polarization state $(|x\rangle|y\rangle-|y\rangle|x\rangle)/\sqrt{2}$, along $-z$ and $+z$ axes. The left photon goes through an elliptic polarizer measuring $\boldsymbol{\Sigma}_1\cdot\mathbf{a}_1=r_1=\pm1$, and then through another elliptic polarizer measuring $\boldsymbol{\Sigma}_1\cdot\mathbf{b}_1=s_1=\pm1$. The right photon similarly goes through two channel elliptic polarizers measuring $\boldsymbol{\Sigma}_2\cdot\mathbf{a}_2=r_2=\pm1$ and $\boldsymbol{\Sigma}_2\cdot\mathbf{b}_2=s_2=\pm1$. Here, $\mathbf{a}_1\perp\mathbf{b}_1$ and $\mathbf{a}_2\perp\mathbf{b}_2$. Detectors on the left and right are wired to measure $2^4=16$ coincidence counts $N_{r_1s_1,r_2s_2}$, and hence

$$P(a,b)=\sum_{r_1,s_1,r_2,s_2}r_1r_2s_1s_2N_{r_1s_1r_2s_2}\Big/\sum_{r_1,s_1,r_2,s_2}N_{r_1s_1,r_2s_2} .$$

Consider the Hermitian operators $A_i(a_i)$ with matrix elements

$$\langle \alpha' | A_i(a_i) | \alpha \rangle = (\boldsymbol{\sigma}\cdot\mathbf{a}_i)_{m_i' m_i} \prod_{j\neq i} \delta_{m_j' m_j} , \quad i=1,\ldots,n , \tag{11}$$

where $\alpha' \equiv m_1' \cdots m_n'$ and the $\boldsymbol{\sigma}$ are Pauli matrices, i.e.,

$$\boldsymbol{\sigma}\cdot\mathbf{a} = \begin{bmatrix} a_z & a_x - i a_y \\ a_x + i a_y & -a_z \end{bmatrix} .$$

Then $A_i^2(a_i)$ equals the unit matrix and hence each $A_i(a_i)$ has eigenvalues ± 1. Further, for $i\neq j$,

$$\langle \alpha' | A_i(a_i) A_j(a_j) | \alpha \rangle = (\boldsymbol{\sigma}\cdot\mathbf{a}_i)_{m_i' m_i} (\boldsymbol{\sigma}\cdot\mathbf{a}_j)_{m_j' m_j} \prod_{k(\neq i,j)} \delta_{m_k' m_k} ,$$

which shows that the $A_1(a_1),\ldots,A_n(a_n)$ are mutually commuting operators. Hence we obtain the representation (4) for their correlation function in stochastic noncontextual theories.

We now derive inequalities on linear combinations of the correlation functions $P(a_1,\ldots,a_n)$ using the methods of MRS [4,5]. Consider

$$F^{(n)} = \prod_{i=1}^{n} [A_i(a_i) + i\eta_i A_i(a_i')] ,$$
$$A^{(n)} = (F^{(n)} + F^{(n)\dagger})/2 , \tag{12}$$
$$B^{(n)} = (F^{(n)} - F^{(n)\dagger})/(2i) ,$$

where $\eta_i = \pm 1$. Then, $A_\psi^{(n)} = \langle \psi | A^{(n)} | \psi \rangle$, $B_\psi^{(n)} = \langle \psi | B^{(n)} | \psi \rangle$ involve only linear combinations of quantum correlation functions in the state ψ. Their hidden-variable analogs are

$$A_{\mathrm{HV}}^{(n)} = \int d\lambda\, \rho(\lambda) A^{(n)}(\lambda), \quad B_{\mathrm{HV}}^{(n)} = \int d\lambda\, \rho(\lambda) B^{(n)}(\lambda) , \tag{13}$$

where $A^{(n)}(\lambda) = \mathrm{Re} F^{(n)}(\lambda)$, $B^{(n)}(\lambda) = \mathrm{Im} F^{(n)}(\lambda)$, and

$$F^{(n)}(\lambda) = \prod_{i=1}^{n} [\overline{A}_i(\lambda, a_i) + i\eta_i \overline{A}_i(\lambda, a_i')] . \tag{14}$$

Since the $A^{(n)}(\lambda)$ and $B^{(n)}(\lambda)$ are linear functions of each of the arguments $\overline{A}_i(\lambda,a_i)$ and $\overline{A}_i(\lambda,a_i')$ their maxima and minima on varying the arguments between -1 and $+1$ must be reached on the boundary, i.e., for each argument $=\pm 1$. It follows that

$$|A_{\mathrm{HV}}^{(n)}| \le p_n , \quad |B_{\mathrm{HV}}^{(n)}| \le p_n , \tag{15}$$

where

$$p_n = 2^{(n-1)/2} \text{ for } n \text{ odd}, \quad p_n = 2^{n/2} \text{ for } n \text{ even} . \tag{16}$$

We now show that these noncontextuality inequalities can be violated by quantum correlations. Choose all $\eta_i = +1$, all $\mathbf{a}_i = \hat{\mathbf{x}}$, all $\mathbf{a}_i' = \hat{\mathbf{y}}$, and for $|\psi\rangle$ consider the choices

$$|\phi_\pm\rangle = (|+ + \cdots +\rangle \pm i |- - \cdots -\rangle)/\sqrt{2} ,$$
$$|\chi_\pm\rangle = (|+ \cdots +\rangle \pm |- \cdots -\rangle)/\sqrt{2} .$$

Then,

$$(A^{(n)} \mp 2^{n-1})|\chi_\pm\rangle = 0, \quad (B^{(n)} \mp 2^{n-1})|\phi_\pm\rangle = 0 ; \tag{17}$$

hence the quantum-mechanical expectation values $A_\psi^{(n)}$ and $B_\psi^{(n)}$ equal $\pm 2^{n-1}$ for $\psi = \chi_\pm$ and $\phi_\pm$, respectively, violating the noncontextuality bounds (15) by a factor $2^{(N-1)/2}$, with $N=n$ for n odd, and $N=n-1$ for n even, as announced. We expect that the result for even n can be improved following the methods of Ref. [5] and that similar violations could be proved also for $S\neq(2^n-1)/2$.

Consequences for measurement theory. The above power-law violation of classical behavior for a single particle is qualitatively new with respect to the MRS violations for n spin-$\frac{1}{2}$ particles. The MRS violations arise from quantum states which are superpositions of distinct states of a macroscopic number of particles. Since such states have not been observed in nature, one possibility is that such states undergo spontaneous localization jumps as in certain measurement theories [10] thus removing the exponential departure from classical behavior. The jumps proposed in Ref. [10] have significant probability only for a large number of particles. In contrast the growing power-law violations here reported are for a single particle of high spin; modifications of quantum theory more general than Ref. [10] are needed to avert these large violations.

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