Superstructures, fractional revivals, and optical Schrödinger-cat states in the Jaynes-Cummings model

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It is shown that if the initial photon distribution is narrow (sub-Poissonian), long-time behavior of the atomic inversion is dominated by very spectacular and almost-periodical superstructures. These superstructures generalize phenomena described previously (fractional revivals). They result from beating of not-nearest-neighbor eigenstates of the system, and we show which beats lead to particular types of revivals. In addition, the cavity field develops macroscopically distinguishable states (optical Schrödinger-cat states). These states emerge when the fractional revivals with a frequency of $2/t_R$, $3/t_R$, and $4/t_R$, where t_R is the single-revival period, appear for the second, fourth, sixth, etc., time in the superstructure pattern.

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I. INTRODUCTION

The Jaynes-Cummings model [1] of a two-level atom interacting with a single mode of electromagnetic radiation has been attracting interest as one of the few models that can be solved exactly and give nontrivial results, and this interest was further enhanced when progress in experimental techniques involving Rydberg atoms and high-Q cavities made it possible to observe "practically two-level" atoms in the laboratory [2]. Using these techniques it is in principle possible to build a single-atom maser [3], and the interaction between a single atom and its own radiation field has been observed for the first time by Rempe, Walther, and Klein [4]. Periodical collapses and revivals of the initial atomic population [5] form the most spectacular feature of the Jaynes-Cummings (JC) model. These quantum revivals were first described for the case of an initially coherent state of the field. In that case the wave function of the system is not an eigenstate of the total Hamiltonian, and during the time evolution, beats between eigenstates that oscillate with their respective Rabi frequencies occur. These beats lead to revivals of the atomic population. However, as an overall phase difference between the beating eigenstates accumulates, revivals start to overlap and then they vanish altogether.

Many authors have also been investigating the Jaynes-Cummings model with a different initial state of the field and/or the atom. In particular, some unexpected long-time behavior of the atomic inversion has been reported for a case of an initially sub-Poissonian statistics of the field [6]. It has been shown that during the time evolution of the system, revivals of the quantum inversion, following each other three times faster than the usual ones, appear, and that they are associated with macroscopically distinguishable (optical Schrödinger-cat) states of the field. This phenomenon has been explained in terms of expanding the Rabi frequencies in powers of $(n-\bar{n})$ ($\bar{n}$ is the initial average number of the photons) under the assumption that the initial distribution is narrow. We show in the present paper that the reported fractional revivals are a part of a more complicated superstructure that dominates the very-long-time behavior of the system. This superstructure results from beating of not-nearest-neighbor eigenstates of the system. In particular, apart from "standard" revivals with ever-decreasing amplitudes, "super-revivals" (or revivals of the revivals) with amplitudes very close to the original one appear.

Results of the present paper provide a full quantum-mechanical interpretation of earlier considerations of wave packets emitted by Rydberg atoms, performed in the classical limit [7,8]. In particular, the "fractional-revival scenario" has been first proposed in Ref. [8], also in the classical limit. It should be stressed at this point that the results of Ref. [6], though fully quantum mechanical, focus on formalism rather than on explaining the physical mechanism underlying the observed phenomena.

It should also be mentioned that similar superstructures have been also observed in a nonlinear generalization of the JC model [9] and in the linear JC model with the field initially coherent but with a very small initial average number of photons and the field and atom slightly detuned [10].

This paper is organized as follows: In Sec. II we briefly describe the JC model and introduce a measure of the overall phase difference between the beating eigenstates. In Sec. III we discuss the long-time behavior of the atomic inversion for the present case, explain the physical origin of the superstructure, and show which eigenstates lead to particular types of fractional revivals. In Sec. IV we present optical Schrödinger-cat states of the field corresponding to various types of fractional revivals, and we present some concluding remarks in Sec. V.

II. JAYNES-CUMMINGS MODEL

A. Quantum inversion

The Hamiltonian of the Jaynes-Cummings model in the rotating-wave approximation (RWA) reads

$$H = \tfrac{1}{2}\omega_0\sigma_3 + \omega a^\dagger a + g\left(a^\dagger\sigma^- + a\sigma^+\right) , \tag{2.1}$$

where $a^\dagger$ and a are standard creation and annihilation operators for the harmonic oscillator, σ^+ and σ^- are spin raising and lowering operators, and g is the coupling constant. The model can be solved exactly. If we assume for simplicity that the atom is initially in its lower state, we obtain for the expectation value of σ_3, or the quantum inversion,

$$W(t) = \sum_{n=0}^{\infty} P_n^2 B_n \cos(2\Lambda_n t) + C , \tag{2.2}$$

where P_n^2 stands for the initial photon distribution,

$$B_{n+1} = \frac{ng^2}{\Lambda_n^2} , \tag{2.3}$$

and the Rabi frequencies Λ_n are defined as

$$\Lambda_n = \sqrt{\tfrac{1}{4}\Delta^2 + ng^2} \tag{2.4}$$

with the detuning parameter $\Delta = \omega_0 - \omega$. The constant C is such that $\langle\sigma_3(0)\rangle = -1$. If $\Delta \neq 0$, this constant, or simply, the initial condition, makes the system "remember" its initial state; $\langle\sigma_3(t)\rangle$ does not oscillate around zero, but rather around a nonzero value. If the atom and the field are in resonance ($\Delta = 0$), (2.2) simplifies to

$$W(t) = \sum_{n=0}^{\infty} P_n^2 \cos\left(2gt\sqrt{n}\right) . \tag{2.5}$$

This formula is very well known. In the case of an initially coherent field with a large initial average number of photons ($\bar{n} \sim 10$), it leads to periodical collapses and revivals of the atomic population. However, for longer times, the revivals start to overlap and eventually vanish altogether, and an irregular pattern of quantum beats emerges [5].

Any two terms, or components, that appear in (2.5) acquire a common phase at some times. Note that in the interaction picture also the corresponding eigenstates acquire a common phase at the same times. The revival period T_R can be estimated [5] as a time when the $\bar{n}$th and $(\bar{n}+1)$-th components acquire a common phase. Thus

$$2\Lambda_{\bar{n}+1}T_R - 2\Lambda_{\bar{n}}T_R = 2\pi , \tag{2.6}$$

from which for large $\bar{n}$ one obtains

$$T_R = \frac{2\pi\sqrt{\bar{n}}}{g} . \tag{2.7}$$

B. A measure of the overall phase difference

It is generally agreed that overlapping of the revivals results from the accumulation of an overall phase difference between the beating eigenstates. We propose the following procedure to associate numbers with this notion of accumulating phase "disorder." First, we want this quantity to be a continuous function of time. Therefore for each pair of beating eigenstates we take their relative phase difference modulo 2π, and, if the resulting number is less that π, we take that number or 2π, minus that number otherwise (note that phase differences equal 0 or 2π both lead to a fully constructive interference). It seems also natural that eigenstates that do not bring an important contribution to the quantum inversion should not bring one to the "overall phase difference" either. We therefore multiply the contribution from each pair of beating components by the product of total weights with which the corresponding terms enter the sum in (2.5). We thus arrive at (we assume that the atom and field are in resonance for simplicity)

$$\varphi_{mn}(t) = \begin{cases} c\,P_m^2 P_n^2 x_{mn} , & x_{mn} < \pi , \\ c\,P_m^2 P_n^2 (2\pi - x_{mn}) , & x_{mn} > \pi , \end{cases}$$

$$x_{mn} = (2\Lambda_m t - 2\Lambda_n t) \bmod 2\pi , \tag{2.8}$$

$$\varphi(t) = \sum_{m=0}^{\infty}\sum_{n=0}^{m} \varphi_{mn}(t) , \tag{2.9}$$

where $\varphi(t)$ is our new measure for the "overall phase difference," φ_{mn} is the contribution from the mth and nth components, and c is a normalization constant. We choose c such that if all $\varphi_{mn} = \pi$ (fully destructive interference between each pair of components), also $\varphi(t) = \pi$. Therefore

$$c = \left(\sum_{m=0}^{\infty}\sum_{n=0}^{m} P_m^2 P_n^2\right)^{-1} . \tag{2.10}$$

In practice, $\varphi(t)$ never reaches π; except for the $t = 0$ there are always some components that are in phase and some that are out of phase. For instance, when after the initial collapse $\bar{n}$th and $(\bar{n}+1)$-th components have a phase difference of π, $(\bar{n}+1)$-th and $(\bar{n}-1)$-th components have, to a good approximation, a phase difference of 2π, etc.

To illustrate a possible usefulness of the function (2.9), we have computed it along with the inversion for the well known large-$\bar{n}$ case (Fig. 1). Note that each revival of the quantum inversion is associated with a minimum of the function $\varphi(t)$, and that the latter takes values around $\pi/2$ in the regions where the inversion is practically zero. As we will see, the function (2.9) works well also in the case of a sub-Poissonian initial photon distribution.

It should be perhaps noted that other quantities have been proposed to measure the accumulating "disorder" of the system. In particular, let us mention works of Phoenix and Knight [11], who used entropy of the cavity field

$$S = -\mathrm{Tr}_F\left[\varrho_F \ln \varrho_F\right] , \tag{2.11}$$

where ϱ_F is the reduced density operator of the field, and Tr_F means tracing over the field variables. The quantity (2.11) has a direct physical interpretation, but our phase function (2.9) is perhaps simpler to calculate and behaves less irregularly around the revivals than (2.11). Moreover, as we will see, it provides a very convincing explanation of the fractional revivals.

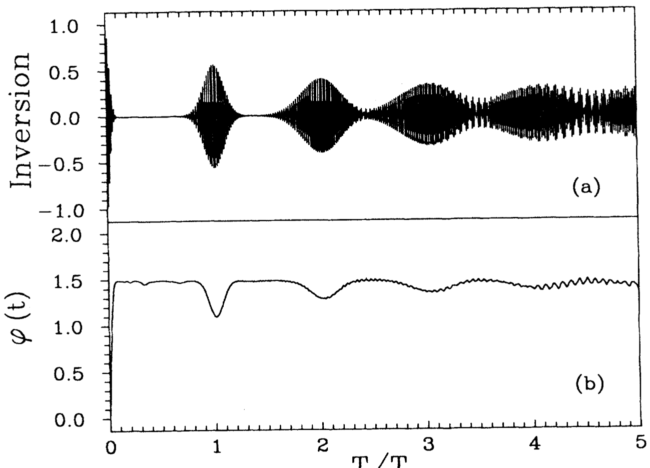

FIG. 1. Revivals of the quantum inversion (a) and the phase function $\varphi(t)$ (2.9) (b) for the case of an initially coherent field with an average number of photons $\bar{n} = 30$.

C. Cavity-field distribution

Distribution of the cavity field is another quantity of interest, apart from the quantum inversion and a measure of the "disorder" in the system. Under present initial conditions the total wave function of the system reads

$$|\Psi(t)\rangle=\sum_{n=0}^{\infty} P_n e^{-i(n-\frac{1}{2})\omega t}[\cos\Lambda_n t|n,\downarrow\rangle - i\sin\Lambda_n t|n-1,\uparrow\rangle] , \tag{2.12}$$

where it is to be understood that the rightmost term vanishes for $n = 0$. $|n\rangle$ is the harmonic-oscillator number state, and $|\downarrow\rangle$, $|\uparrow\rangle$ are the atom ground and excited states, respectively. For the reduced density operator of the field we get

$$\varrho_F(t) = \mathrm{Tr}_A|\Psi(t)\rangle\langle\Psi(t)| . \tag{2.13}$$

Tr_A stands for tracing over atomic variables. Note that even though the complete system is in a pure state (2.12), the reduced density operator (2.13) describes in general a mixed state.

It is convenient to analyze (2.13) in terms of the so-called Husimi representation

$$Q(\alpha,\alpha^\star) = \langle\alpha|\varrho_F|\alpha\rangle , \tag{2.14}$$

where $|\alpha\rangle$ is a coherent state of the harmonic oscillator and $\alpha^\star$ denotes the complex conjugate of α. After simple algebra, from (2.12) we get

$$Q(\alpha,\alpha^\star) = \left|\sum_{n=0}^{\infty} P_n \frac{(\alpha^\star)^n}{\sqrt{n!}} e^{-in\omega t}\cos\sqrt{n}gt\right|^2 + \left|\sum_{n=0}^{\infty} P_{n+1} \frac{(\alpha^\star)^n}{\sqrt{n!}} e^{-in\omega t}\sin\sqrt{n+1}gt\right|^2 . \tag{2.15}$$

Since the Hamiltonian (2.1) conserves the number of excitations and the atom has only two states, the average number of photons in the cavity can vary only between $(\bar{n}-1)$ and $\bar{n}$. Therefore, if the initial distribution of the field is narrow, we expect the function Q to be concentrated around a ring with a radius of $\sqrt{\bar{n}}$ in the complex α plane. However, the phase properties of the field can vary considerably as the system evolves. Note also that for a fixed time ω causes only a global rotation of the coordination frame in the complex α plane: $\alpha = |\alpha|e^{i\phi} \to \alpha' = |\alpha|e^{i(\phi+\omega t)}$. Therefore, while computing Q for various times we will always put $\omega = 0$.

III. SUPERSTRUCTURES IN THE LONG-TIME BEHAVIOR OF THE INVERSION

As we have mentioned in the preceding section, when the field is initially coherent and $\bar{n}$ is sufficiently large, distinct revivals appear. For longer times these revivals overlap. The situation changes dramatically if the field has initially a sub-Poissonian statistics. Following Averbukh [6], we choose for our initial state of the field a squeezed state with a Gaussian distribution of P_n:

$$P_n^2 = \frac{1}{\sqrt{2\pi}\Delta n}\exp\left[-\frac{(n-\bar{n})^2}{2\Delta n^2}\right] , \tag{3.1}$$

and we put $\bar{n} = 50$, $\Delta n = 2$. In this case, for short times distinct Gaussian revivals appear [(Fig. 2(a)], but instead of overlapping for longer times, they become non-Gaussian and develop side peaks. The revivals' amplitude decreases with time. Around $t = 25T_R$, revivals follow each other four times faster than for small times [Fig. 2(b)]. The revivals are now distinct, although not very well separated. For longer times the revivals partially overlap, but around $t = 33T_R$ distinct revivals reappear, and this time they follow each other three times faster than the original one [Fig. 2(c)]. They again partially overlap and reappear around $t = 50T_R$, this time following each other two times faster than for initial times [Fig. 2(d)]. Then the scenario repeats itself in the reversed order: distinct revivals appear around $t = 67T_R$ [see Fig. 1(d) of Ref. [6]] and around $t = 75T_R$ (not plotted). The revivals' amplitudes start to increase, and finally around $t = 100T_R$ distinct revivals with amplitudes very close to the original one appear. They follow each other with the same period as for initial times [Fig. 2(e)]. We call these revivals with amplitudes close to the original one super-revivals, or revivals of the revivals. Then the whole pattern starts again, but the revivals are now less distinct that for shorter times. Overall, a very spectacular superstructure emerges [Fig. 3(a)]. Each individual revival is associated with a minimum of the phase function $\varphi(t)$ (2.9), which also shows a distinct long-time superstructure [Fig. 3(b)]. The revivals are very persistent: we have observed distinct individual revivals and clear, though distorted, superstructure for times as long as $t \sim 10^4 T_R$.

Following Averbukh [6,8] we call revivals whose periods are simple fractions of the original revival period T_R "fractional revivals." Super-revivals may be thought of as fractional revivals with period T_R, but we reserve this special name for them. It should be also noted that the super-revivals are not Gaussian; instead, they are accompanied by sets of "ringing revivals" [12].

A. Physical origin of the superstructure

The superstructure finds its origin in the quantum nature of the process under consideration. Much as in the coherent case, revivals result from the beating of all terms in (2.5). In the coherent case, for very short times, all components are approximately in phase and the inversion displays oscillations with an amplitude close to unity. For longer times, the components become out of phase and destructive interference prevails. For $t = T_R$ the $\bar{n}$th and $(\bar{n}+1)$-th components are in phase, and the phase difference between the $\bar{n}$th and $(\bar{n}-1)$-th components approximately equals $\pi/\bar{n}$, which is small if $\bar{n}$ is large. Similar small-phase differences occur between other pairs of components and the first revival appears. However, as time increases, these small-phase differences grow larger, destructive interference appears, and consecutive revivals have smaller amplitudes and start to overlap.

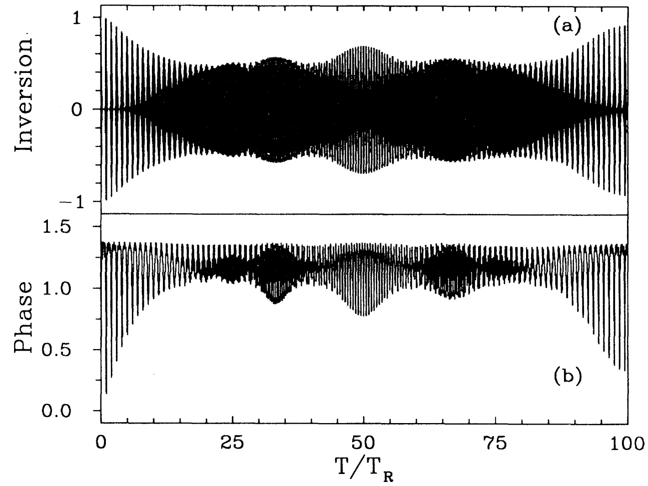

FIG. 3. Superstructures in the long-time behavior of the JC model with a sub-Poissonian statistics of the field. (a) The quantum inversion; (b) the phase function $\varphi(t)$ (2.9). Parameters as in Fig. 2.

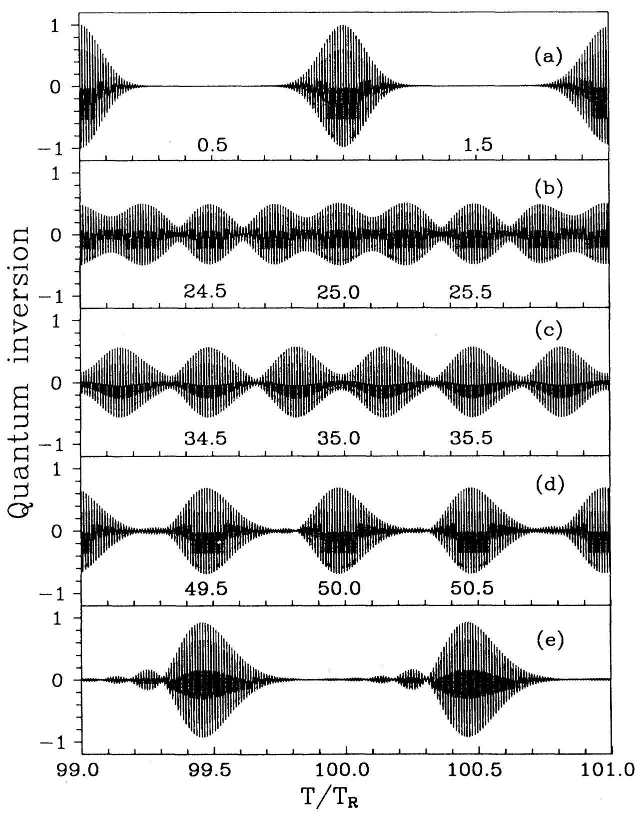

FIG. 2. Various types of revivals during the first superstructure period for the initial sub-Poissonian field. Each panel shows a different range of time T, but the time scale is kept the same. Plotted are (a) initial revivals, $0 < T < 2T_R$; (b) period $T_R/4$ revivals, $24T_R < T < 26T_R$; (c) period $T_R/3$ revivals, $32T_R < T < 34T_R$; (d) period $T_R/2$ revivals, $49T_R < T < 51T_R$; (e) super-revivals, $99T_R < T < 101T_R$. Parameters are $\bar{n} = 50$, $\Delta n = 2$.

If the photon distribution is narrow, i.e., if there are only a few eigenstates that bring non-negligible contributions to the sum in (2.5), the situation changes: if a pair of components related to strongly populated eigenmodes is in phase, some pairs of components have intermediate phases and there might be too few components that are out of phase for the destructive interference to prevail. Consider now the two pairs of components related to the most populated eigenstates, namely the $(\bar{n}+1)$-th and $\bar{n}$th, and $\bar{n}$th and $(\bar{n}-1)$-th components. If $t > 0$, when one pair is in phase, there is always a nonzero phase difference between the other pair. However, for some long times the phase difference between the $(\bar{n}+1)$-th and $\bar{n}$th components approaches a value of $2k\pi$ for some integer k, while the phase difference between the $\bar{n}$th and $(\bar{n}-1)$-th components approaches a value of $2(k+1)\pi$. Since there are too few components out of phase and populated enough to matter, constructive interference prevails once more and revivals regain their original amplitudes. As a consequence, the super-revival appears.

We can estimate the "super-revival period" T_S as time for which

$$2\Lambda_{\bar{n}+1}T_S - 2\Lambda_{\bar{n}}T_S = 2k\pi\,,$$
$$2\Lambda_{\bar{n}}T_S - 2\Lambda_{\bar{n}-1}T_S = 2(k+1)\pi\,, \tag{3.2}$$

for some integer k. Solving (3.3) for T_S, we obtain

$$T_S = \frac{\pi}{2\Lambda_{\bar{n}} - \Lambda_{\bar{n}+1} - \Lambda_{\bar{n}-1}}\,, \tag{3.3}$$

which for large $\bar{n}$ gives

$$T_S \approx 2\bar{n}T_R\,. \tag{3.4}$$

To find components responsible for fractional revivals and to estimate times for which various types of fractional revivals occur, one should consider such pairs related to the most populated eigenstates whose relative frequencies correspond to that of fractional revivals. In the following, we will restrict our analysis to the seven components related to the most populated eigenstates: $(\bar{n}-3)$, . . . , $(\bar{n}+3)$; the other components are so sparsely populated that they can bring only small corrections to the total inversion. Table I lists 20 pairs from this set with their relative frequencies f and their "weight factors" β: the (n,m)-th pair gives a contribution to the phase function (2.9) that is proportional to $\exp(-\beta)$ with $\beta = [(n-\bar{n})^2+(m-\bar{n})^2]/2(\Delta n)^2$. The pair $(\bar{n}+3)\leftrightarrow(\bar{n}-3)$ has been excluded because in order to admit it we should also admit pairs $(\bar{n}\pm 4)\leftrightarrow\bar{n}$ that have smaller weight factors.

The pair $(\bar{n}+1)\leftrightarrow(\bar{n}-1)$ is the most populated pair with a relative frequency $2/T_R$. A period $T_R/2$ fractional revival occurs when this pair and any of the $(\bar{n}\pm 1)\leftrightarrow\bar{n}$ pairs are simultaneously in phase. Accordingly (l is integer),

$$2\Lambda_{\bar{n}+1}T_2 - 2\Lambda_{\bar{n}-1}T_2 = 2l\pi\,, \qquad 2\Lambda_{\bar{n}}T_2 - 2\Lambda_{\bar{n}-2}T_2 = 2(l+1)\pi\,, \tag{3.5}$$

from which for large $\bar{n}$ we get

$$T_2 \approx \bar{n}T_R\,. \tag{3.6}$$

Similarly, a period $T_R/3$ fractional revival occurs when the two most populated pairs with relative frequency $3/T_R$ are simultaneously in phase:

$$2\Lambda_{\bar{n}+2}T_3 - 2\Lambda_{\bar{n}-1}T_3 = 2r\pi\,, \qquad 2\Lambda_{\bar{n}+1}T_3 - 2\Lambda_{\bar{n}-2}T_3 = 2(r+1)\pi\,, \tag{3.7}$$

$$T_3 \approx \frac{2\bar{n}}{3}T_R\,. \tag{3.8}$$

Finally, the time when period $T_R/4$ fractional revivals occur can be estimated from

$$2\Lambda_{\bar{n}+2}T_4 - 2\Lambda_{\bar{n}-2}T_4 = 2s\pi\,, \qquad 2\Lambda_{\bar{n}+1}T_4 - 2\Lambda_{\bar{n}-3}T_4 = 2(s+1)\pi\,, \tag{3.9}$$

$$T_4 \approx \frac{\bar{n}}{2}T_R\,. \tag{3.10}$$

All the above estimates of times when various fractional revivals occur are very accurate.

In principle, one could also estimate times for which revivals with periods $T_R/5, T_R/6, \ldots$ could occur. However, as the corresponding components are very scarcely populated, these revivals do not show up in the present case. It can even be argued that increasing the population of the corresponding components would mean introducing many odd phases into the system, which would blur the whole pattern altogether. Such high-order fractional revivals can show up if the initial photon distribution is multipeaked with very narrow peaks. This observation provides a possible explanation of the phenomenon of "ringing revivals" reported in Ref. [12] for such a multipeaked distribution of the field.

To end this subsection, let us make two comments. First, Eqs. (3.3), (3.5), (3.8), and (3.10) cannot be solved for the *integers* k, l, r, s, respectively. Rabi frequencies are in general irrational and incommensurate, and there is no exact periodicity in the superstructure. These deviations from periodicity blur the superstructure for longer times, and eventually make it disappear for extremely long times. Second, as will become clear in the next subsection, even when pairs of components with relative frequency n/T_R (n=1,2,3,4) dominate the system, the influence of other components cannot be neglected, and we did not take it into account while deriving Eqs. (3.4), (3.6), (3.8), and (3.10). Still, our estimates of fractional revival times work quite well.

TABLE I. Pairs of components that bring most important contributions to (2.5) and (2.9), along with their relative frequencies f and "weight factors" β (see text). The first pair excluded has $\beta = 16/8$.

Components	f	β
$\bar{n}\leftrightarrow\bar{n}-1$	$1/T_R$	1/8
$\bar{n}+1\leftrightarrow\bar{n}$		
$\bar{n}+1\leftrightarrow\bar{n}-1$	$2/T_R$	2/8
$\bar{n}\leftrightarrow\bar{n}-2$	$2/T_R$	4/8
$\bar{n}+2\leftrightarrow\bar{n}$		
$\bar{n}-1\leftrightarrow\bar{n}-2$	$1/T_R$	5/8
$\bar{n}+2\leftrightarrow\bar{n}+1$		
$\bar{n}+1\leftrightarrow\bar{n}-2$	$3/T_R$	
$\bar{n}+2\leftrightarrow\bar{n}-1$		
$\bar{n}+2\leftrightarrow\bar{n}-2$	$4/T_R$	8/8
$\bar{n}\leftrightarrow\bar{n}-3$	$3/T_R$	9/8
$\bar{n}+3\leftrightarrow\bar{n}$		
$\bar{n}-1\leftrightarrow\bar{n}-3$	$2/T_R$	10/8
$\bar{n}+3\leftrightarrow\bar{n}+1$		
$\bar{n}+1\leftrightarrow\bar{n}-3$	$4/T_R$	
$\bar{n}+3\leftrightarrow\bar{n}-1$		
$\bar{n}-2\leftrightarrow\bar{n}-3$	$1/T_R$	13/8
$\bar{n}+3\leftrightarrow\bar{n}+2$		
$\bar{n}+2\leftrightarrow\bar{n}-3$	$5/T_R$	
$\bar{n}+3\leftrightarrow\bar{n}-2$		

B. Details of the fractional revivals

We will now discuss the structure of fractional revivals in more detail. We start with period $T_R/2$ revivals. As we have argued, they occur when pairs of components related to the most populated eigenmodes and with relative frequency $2/T_R$ all interfere constructively. In Fig. 4 are plotted the revivals [panel (a), solid line] and the total phase function [panel (a), dashed line], and contributions from various pairs of components [panel (b), components of number difference 1; panel (c), components of number difference 1 and 4; panel (d), components of number difference 2; panel (e), components of number difference 3]; only ten of the pairs listed in Table I are plotted—otherwise the whole plot would be too complicated. As one can see, in this time domain, relative phases of the three pairs of components with frequency $2/T_R$ follow each other almost exactly [the difference of "amplitudes" in Fig. 4(d) results from different weights of these pairs, which in turn reflect different populations of the components; c.f. (2.8)]. The constructive interference between these components is further enhanced by the constructive interference of the $(\bar{n}+2)\leftrightarrow(\bar{n}-2)$ pair [panel (c), frequency $4/T_R$] and other even-frequency pairs (not plotted). Odd-frequency components take intermediate values and, as a result, distinct revivals appear and reach the maxima of their amplitudes when the even-frequency components are in phase. The revivals are associated with pronounced minima of the phase function $\varphi(t)$. Between the revivals, the phase function is fairly flat and takes larger values. Tiny "ringing revivals" between the main revivals can be associated with times when some of the odd-frequency pairs are simultaneously in phase. However, at these times all even-frequency components and some other odd-frequency are out of phase and the amplitude of these ringing revivals is very small.

The situation is slightly more complicated for the period $T_R/3$ revivals. In Fig. 5 we have plotted revivals occurring around $t = 67T_R$, along with the total phase function and contributions to this function from pairs of the most populated components. Note that this is the *second* occurrence of period $T_R/3$ revivals; we have chosen to plot this one and not the first one around $t = 33T_R$, because it was precisely this time domain that had been originally discussed in Ref. [6]. As one can see, relative

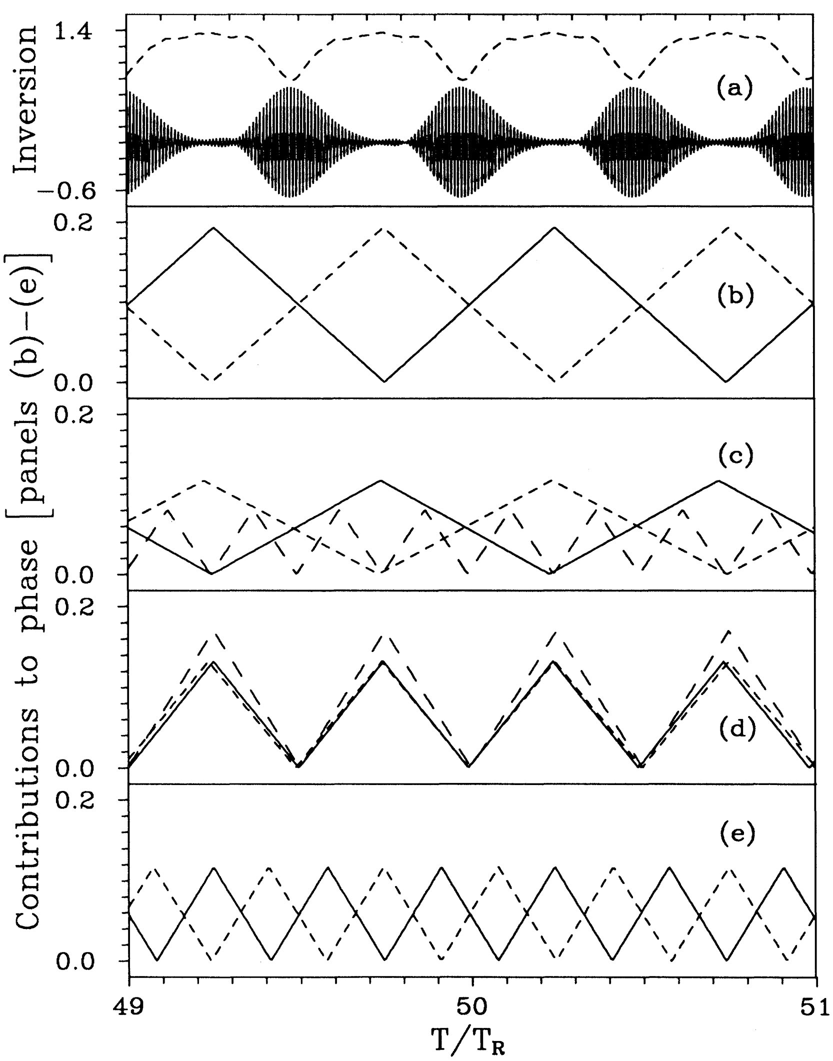

FIG. 4. Quantum inversion [panel (a), solid line] and the total phase function [panel (a), dashed line] in the region of period $T_R/2$ fractional revivals. Other panels show contributions to the phase function from pairs of the most populated components: (b) solid line, 50$\leftrightarrow$49 pair; dashed line, 51$\leftrightarrow$50 pair; (c) solid line, 49$\leftrightarrow$48 pair; short dashed line, 52$\leftrightarrow$51 pair; long dashed line, 52$\leftrightarrow$48 pair; (d) solid line, 50$\leftrightarrow$48 pair; short dashed line, 52$\leftrightarrow$50 pair; long dashed line, 51$\leftrightarrow$49 pair; (e) solid line, 51$\leftrightarrow$48 pair; dashed line, 52$\leftrightarrow$49 pair. Parameters as in Fig. 2.

phases of the components with relative frequency $3/T_R$ follow each other almost completely, and the revivals occur when both pairs interfere constructively. However, the revivals are also associated with constructive interference between the $1/T_R$ pairs. For instance, the pair $\bar{n}\leftrightarrow(\bar{n}-1)$ brings a contribution to the revival at $(66+2/3)T_R$, the pairs $(\bar{n}-1)\leftrightarrow(\bar{n}-2)$ and $(\bar{n}+2)\leftrightarrow(\bar{n}+1)$ to the revival at $67T_R$, and the pair $(\bar{n}+1)\leftrightarrow\bar{n}$ to the one at $(67+1/3)T_R$. Hence we have an admixture of two frequency $1/T_R$ pairs to the central revival, and admixtures of one pair to either of its nearest neighbors. However, the two pairs that contribute to the central revival are less populated than the other two pairs and, as a result, the revival at $67T_R$ dominates only slightly—note that $T_3 = (66+2/3)T_R$ and we could have expected this revival to dominate. There are also contributions from the $2/T_R$ frequency pairs to the revivals.

Since in this time domain there are no clear-cut regions of prevailing destructive interference, the revivals are not well separated and the phase function does not flatten in between revivals. It is interesting to note that all 20 pairs of Table I are needed to obtain even this degree of separation. In Fig. 6(a) we have plotted the exact inversion, and on the other two panels approximate inversions obtained by replacing the infinite (or, in practice, very large) sum in (2.5) by a finite one: $\bar{n}-3 \leq n \leq \bar{n}+3$ in Fig. 6(b) and $\bar{n}-2 \leq n \leq \bar{n}+2$ in Fig. 6(c). The former corresponds to keeping all pairs listed in Table I, while the latter to keeping only the terms that are plotted in Figs. 4 and 5. As one can see, in the latter approximation the positions of individual revivals are correct, but the amplitudes both at maxima and at minima are not given correctly: the revivals are poorly separated and the whole picture flattens. On the other hand, the approximation that led to Fig. 6(b) gives satisfactory results. The ten pairs are enough to explain the positions of the revivals, but all 20 contribute to the destructive interference. It supports the point that the more populated

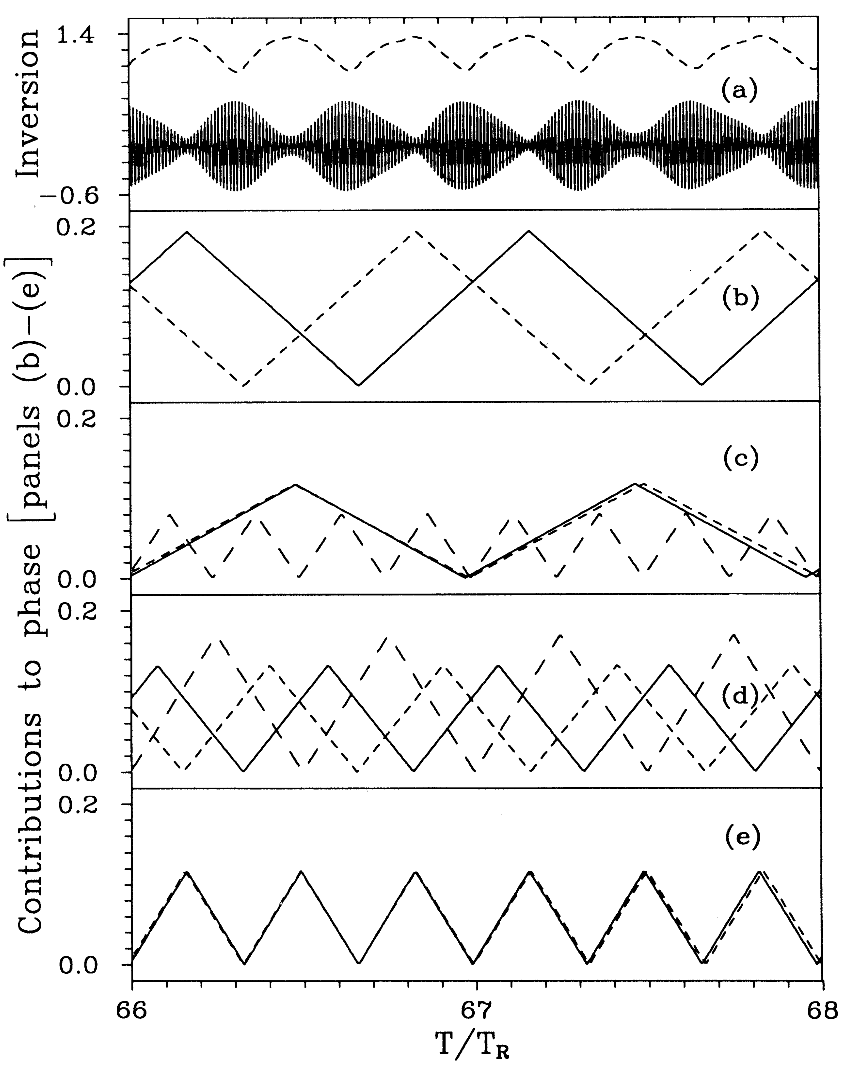

FIG. 5. Same as Fig. 4 for the region of period $T_R/3$ fractional revivals.

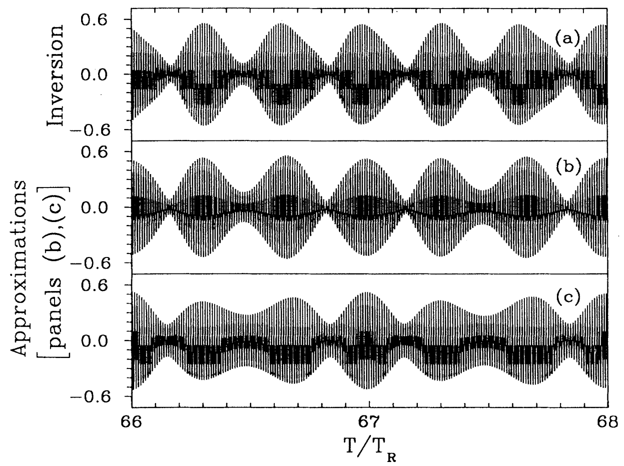

FIG. 6. Exact (a) and approximate (b),(c) inversion in the period $T_R/3$ fractional revivals region. The approximation in (b) has been obtained by replacing the infinite sum in (2.5) by a sum of seven terms $\bar{n}-3 \leq n \leq \bar{n}+3$, while in (c) the sum has been replaced by a sum of five terms $\bar{n}-2 \leq n \leq \bar{n}+2$.

the eigenstates in the system, the more dominant the destructive interference is.

Details of the period $T_R/4$ fractional revivals, as well as the super-revivals, can be explained in a similar manner. For instance, the most pronounced of the "ringing revivals" that precede the super-revivals can be attributed to the constructive interference of the $3/T_R$ pairs. The other "ringing revivals" come from even higher order corrections.

IV. EVOLUTION OF THE FIELD DISTRIBUTION

As pointed out in Ref. [6], fractional revivals are associated with macroscopically distinguishable states of the cavity field, or optical Schrödinger cat states. However, these states emerge only when each type of fractional revival occurs for the second, fourth, sixth,..., during the time evolution of the system: there are no cat states for $t = \bar{n}T_R/2$, and for $t = \bar{n}T_R$ we observe a four-peaked Q function, and not a two-peaked one, as one might have expected. Note that at $t = T_2 = \bar{n}T_R$ the period $T_R/4$ fractional revivals should occur for the second time. They do not show up in the quantum inversion, though, as they are screened off by much stronger period $T_R/2$ fractional revivals, which just appear for the first time. They do show up in the field distribution. Similarly, at $t = 4\bar{n}T_R/3$ or when the period $T_R/3$ appear for the second time, the Q function develops three distinct peaks. Super-revivals at $t = 2\bar{n}T_R$ occur when period $T_R/2$ fractional revivals should appear for the second time, and accordingly the Q function develops two distinct peaks for these times. Finally, at $t = 4\bar{n}T_R = 2T_S$ the field also completes its evolution and the Q function is single peaked. Because of the accumulated phase differences this peak is broader and lower than the original one, and there are also some structures that do no belong to this peak.

To understand this apparent paradox it is enough to remember that the cavity field is a superposition of the initial distribution and the oscillating atomic dipole field, and it is very well known that the latter evolves with half the atomic evolution frequency. Note that in the formula (2.5) for the quantum inversion all Rabi frequencies are multiplied by 2, but they are not multiplied in the formula (2.15) for the Q function.

In Fig. 7 the Q function for various times is plotted: $t = 0$ in (a), $t = 50T_R$ in (b), $t = 67T_R$ in (c), $t = 100T_R$ in (d), $t = 150T_R$ (e), and $t = 200T_R$ in (f). Note that Fig. 7(e) corresponds to the sixth occurrence of period $T_R/4$ fractional revivals of the inversion, and because of accumulated phase difference (deviations from ideal periodicity) the cat states are now distorted. All distributions from Fig. 7 have been calculated with the exact function (2.15) but with $\omega = 0$.

V. SUMMARY AND CONCLUSIONS

Fractional revivals and super-revivals, which may be thought of as a particular type of fractional revival, are parts of a very spectacular superstructure that dominates the long-time behavior of the JC model for the initially sub-Poissonian field. They result from the beating of not nearest neighbors of the eigenstates of the system. We have shown which pairs of components contribute to various parts of the superstructure. Note that the previous attempt to explain some of these phenomena [6] concentrated on expanding the Rabi frequencies in powers of $(n-\bar{n})$, and gave no physical interpretation of the whole process.

We have also constructed the phase function $\varphi(t)$ (2.9), which measures the phase disorder of the system. Revivals are associated with minima of this function, and $\varphi(t)$ helped us to single out pairs of components responsible for various types of fractional revivals.

If the initial photon distribution is narrow, or in other words, if there are only few eigenstates that bring non-negligible contributions to the sums in (2.5) and (2.15), there is a series of "magic times" when various pairs of components become in-phase and interfere constructively, while other pairs of components have intermediate phase differences. This phenomenon does not occur if the distribution is wide and there are many components that are approximately in phase and many others that are out of phase—in this situation one observes irregular quantum beats, like those in the long-time behavior of the coherent JC model.

It is interesting to observe that the superstructure described in the present paper reveals the quantum nature, or "granularity," of radiation even stronger than ordinary revivals do: It is enough to consider only the beats between the nearest-neighbor components to explain the ordinary revivals, while fractional revivals occur only if a couple of pairs of components (at least three pairs of components related to eigenstates with similar and large populations, as in the cases discussed in detail above) simultaneously interfere constructively. Thus, fractional

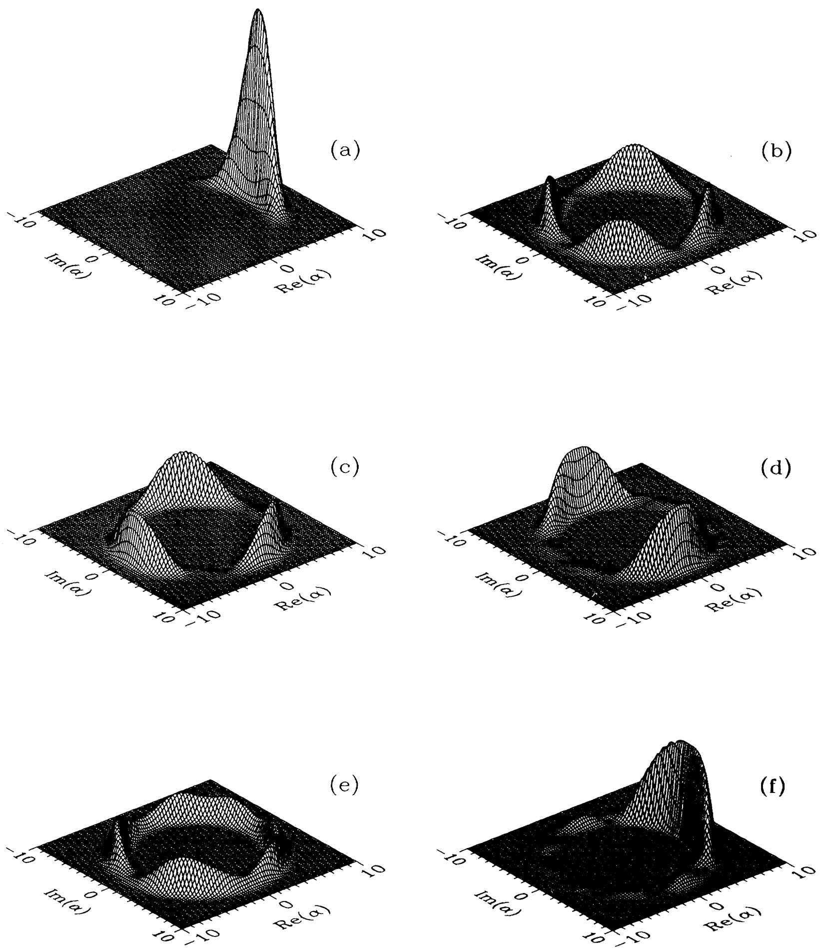

FIG. 7. Husimi distributions Q of the cavity field for various times. (a) $t = 0$, (b) $t = 50T_R$, (c) $t = 67T_R$, (d) $t = 100T_R$, (e) $t = 150T_R$, and (f) $t = 200T_R$.

revivals may be called a cooperative phenomenon. From this point of view, fractional revivals can appear only if the Rabi frequency spectrum can be locally approximated by a linear function in the vicinity of the most populated eigenstates. If the Rabi frequency spectrum were linear, one would observe an infinite series of ideal Gaussian revivals, and it is precisely the deviations from this linearity that cause small phase differences between various pairs to appear. As these deviations accumulate and eventually amount to 2π, a fractional revival emerges. However, if the Rabi frequency spectrum is far from being linear, there is no regularity in these small deviations, and when two pairs of components [say, $(\bar{n}+1)\leftrightarrow(\bar{n}-1)$ and $(\bar{n}+2)\leftrightarrow\bar{n}$] happen to interfere constructively, there is no chance of a simultaneous constructive interference with other pairs [say, $\bar{n}\leftrightarrow(\bar{n}-2)$], which under the present conditions do contribute to the fractional revival (period $T_R/2$ in this example). The coherent JC model with a small value of $\bar{n}$ is a good example of this situation: if $\bar{n}$ is small, there is only a very limited number of eigenstates that effectively contribute to the sum in (2.5), but if there is no detuning, no fractional revivals, or ordinary revivals for that matter, occur precisely because the square root function is strongly nonlinear for small values of its argument. This problem will be discussed in more detail elsewhere [10].

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