Quantum statistics of a lossless beam splitter: SU(2) symmetry in phase space

Ulf Leonhardt

Arbeitsgruppe "Nichtklassische Strahlung" der Max-Planck-Gesellschaft an der Humboldt-Universität zu Berlin, Rudower Chaussee 5, 12484 Berlin, Germany

(Received 18 December 1992; revised manuscript received 6 May 1993)

A lossless beam splitter (a dielectric interface, a passive interferometer, or a linear coupler) changes the quantum state of two incident modes by an SU(2) transformation. Apart from phase shifting, the argument of the quadrature wave function of the system undergoes a rotation. Quasiprobabilities are changed by the inverse mode transformation. The use of balanced beam splitting allows the simultaneous measurement of conjugate quadrature components via homodyning the emerging beams with two strong coherent reference fields that differ in their phases by $\pi/2$. The measured probability distribution is given by a generalized Q function. It depends on the state of the field entering the second beam-splitter port. For a vacuum, the Q function will be obtained. The use of unbalanced beam splitting allows the measurement of a squeezed Q function without using squeezed states. Dissipation in Gaussian reservoirs corresponds exactly to a heuristic beam-splitter model. As a mathematical tool, the Fokker-Planck equation of damping in phase-sensitive reservoirs and the corresponding quantum master equation were solved. The dissipative decay of a Schrödinger-cat state was studied as an example. The sensitivity of quantum coherence with respect to damping can be interpreted geometrically.

PACS number(s): 42.50.Dv, 03.65.Bz, 42.50.Lc, 42.79.Fm

I. INTRODUCTION

A simple beam splitter is a nice device to demonstrate the quantum nature of light. In a number of subtle experiments [1,2] the wave-particle dualism was proved convincingly by counting photons of split [1] or optically mixed [2] light beams. A measurement of the phase-space distribution via homodyning the emerging beams with two strong coherent fields provides another example of simple yet fundamental quantum optical experiments. Especially attractive is the measurement of one quadrature on one beam and the canonically conjugate quadrature on the other [3]. In this way "position" and "momentum" are measured simultaneously, at the cost, however, of introducing additional (vacuum) noise via the second ("unused") port of the beam splitter. The recent experiment by Noh, Fougères, and Mandel [4] can be seen in this light. It was intended as an operational approach to the quantum phase of light and stimulated significantly the discussion about phase and phase-dependent properties [5]. Two recent Rapid Communications [6,7] showed that the simultaneously measured phase-space distribution is actually the Q function of the original light mode. One of the main goals of this paper is the analysis of a generalized Noh-Fougères-Mandel scheme in which a second field being in an arbitrary state enters the second port of the beam splitter and the splitting ratio is no longer restricted to 50%:50%. It turns out that the measured phase-space distribution is a generalized Q function [8,9]. Surprisingly, the unbalanced scheme allows a measurement of a squeezed Q function without using squeezed states.

Another subject of this paper is the phase-space analysis of nonclassical quantum states which are influenced by an external environment. It is well known that squeezed states [10] or Schrödinger-cat states [11] exhibit the lion's share of their typical quantum properties in phase space. The simplest model for the interaction of a light mode with an external environment is a beam splitter [12]. It models the attenuation of the field and the entering of external fluctuations via the second port of the beam splitter in accordance with the dissipation-fluctuation theorem. This paper shows that a heuristic beam-splitter model describes *exactly* the dissipation in Gaussian reservoirs. To demonstrate this the Fokker-Planck equation for damping in phase-sensitive reservoirs and the corresponding master equation are solved. To the author's knowledge only partial solutions of this problem were published [13,14] to date. An example will be the decay of quantum coherence of a Schrödinger-cat state due to thermal [15] or phase-sensitive reservoirs [16,17]. A surprisingly simple geometrical explanation of the extreme sensitivity of quantum coherence will be given.

The tailored formalism for the action of a lossless beam splitter in the phase space of the incident light modes is provided by transformation laws for wave functions and quasiprobability distributions. Although the quantum statistical theory of a lossless beam splitter has been already studied by a number of authors [18–24], a comprehensive phase-space description is still missing. The most general formalism was published by Campos, Saleh, and Teich [24]. In their paper the beam splitter is modeled by a SU(2) transformation of the bosonic operators of the incident light modes [19]. The same "black box" describes a passive interferometer [25], a linear coupler [26–28], or a dielectric interface [29]. Using the Jordan-Schwinger representation of an angular momentum system [30,31] they derived a general transformation law for the state vector and the density matrix of the in-

 ©1993 The American Physical Society

cident light modes. In this paper the basic model and the notations will be quoted from the work of Campos, Saleh, and Teich [24]. After a brief summary of their formalism in Sec. II the transformation law for wave functions will be derived in Sec. III. In Sec. IV generalized Noh-Fougères-Mandel schemes, i.e., simultaneous measurements of canonically conjugate variables are studied. In Sec. V the transformation law for quasiprobabilities will be derived. Reflection of light being in a Schrödinger-cat state off a semitransparent mirror will be studied as an example. In Sec. VI the equivalence of dissipation caused by Gaussian reservoirs and the corresponding beam-splitter model will be proved.

II. THE MODEL

The abstract model of a lossless beam splitter [22], a dielectric interface [29], an interferometer [25], or a linear coupler [26–28] is a linear four-port device (Fig. 1). Two radiation modes enter the instrument, interfere with each other, and leave it. Denoting the boson-annihilation operators of the input modes by $\hat{a}_1$ and $\hat{a}_2$ and the output-mode operators by $\hat{b}_1$ and $\hat{b}_2$ the input/output relation of the beam splitter is simply

$$\begin{bmatrix}\hat{b}_1\\ \hat{b}_2\end{bmatrix}=\underline{B}\begin{bmatrix}\hat{a}_1\\ \hat{a}_2\end{bmatrix}, \quad B=\begin{bmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{bmatrix}. \tag{1}$$

In order to conserve the bosonic commutation rules of the mode operators the transformation matrix $\underline{B}$ has to be unitary [24] and apart from an avoidable phase factor we find that

$$\underline{B}\in \mathrm{SU}(2). \tag{2}$$

Thus

$$\begin{bmatrix}\hat{a}_1\\ \hat{a}_2\end{bmatrix}=\underline{B}^{\dagger}\begin{bmatrix}\hat{b}_1\\ \hat{b}_2\end{bmatrix} \tag{3}$$

expresses the input-mode operators in terms of the output-mode operators. SU(2) matrices have a simple structure

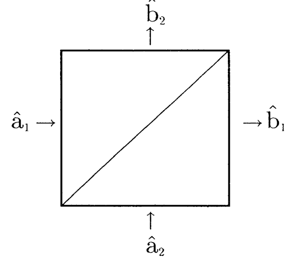

FIG. 1. Schematic diagram of a beam splitter. $\hat{a}_1$ and $\hat{a}_2$ denote the input modes and $\hat{b}_1$ and $\hat{b}_2$ the output modes.

$$\underline{B}=\begin{bmatrix}e^{i(\Psi/2)} & 0\\ 0 & e^{-i(\Psi/2)}\end{bmatrix}\times\begin{bmatrix}\cos(\Theta/2) & \sin(\Theta/2)\\ -\sin(\Theta/2) & \cos(\Theta/2)\end{bmatrix}\times\begin{bmatrix}e^{i(\Phi/2)} & 0\\ 0 & e^{-i(\Phi/2)}\end{bmatrix} \tag{4}$$

or explicitly written

$$\underline{B}=\begin{bmatrix}\cos(\Theta/2)e^{i(\Psi+\Phi)/2} & \sin(\Theta/2)e^{i(\Psi-\Phi)/2}\\ -\sin(\Theta/2)e^{i(-\Psi+\Phi)/2} & \cos(\Theta/2)e^{i(-\Psi-\Phi)/2}\end{bmatrix}. \tag{5}$$

The beam splitter acts in three steps: At first it shifts the phases of the input modes, then it mixes the modes via a rotation, and at last it shifts the phases again. The phase shifts could be eliminated by a proper phase redefinition of the incoming/outcoming modes. The rotation would remain. Mixing of modes is in fact the essential operation of a beam splitter. Note that the transmittance of the beam splitter is given by $\tau=\cos^2(\Theta/2)$ and the reflectance by $\rho=\sin^2(\Theta/2)$. The identity $\cos^2(\Theta/2)+\sin^2(\Theta/2)=1$ ensures energy conservation.

Equations (1) and (4) give the transformation law for mode operators, while the quantum state of the system remains unchanged. The situation is similar to quantum theory in the Heisenberg picture. In the Schrödinger picture the quantum state evolves while the mode operators remain fixed. In quantum theory the evolution operator of the system relates Heisenberg and Schrödinger picture. Thus, if the transformation law of the modes can be written as the action of an evolution operator $\hat{B}$,

$$\begin{bmatrix}\hat{b}_1\\ \hat{b}_2\end{bmatrix}=\hat{B}\begin{bmatrix}\hat{a}_1\\ \hat{a}_2\end{bmatrix}\hat{B}^{\dagger}, \quad \begin{bmatrix}\hat{a}_1\\ \hat{a}_2\end{bmatrix}=\hat{B}^{\dagger}\begin{bmatrix}\hat{b}_1\\ \hat{b}_2\end{bmatrix}\hat{B}, \tag{6}$$

the transformation law of quantum states in the Schrödinger picture is given by

$$|\psi_{\text{out}}\rangle=\hat{B}^{\dagger}|\psi_{\text{in}}\rangle \tag{7}$$

or for a statistical mixture

$$\hat{\rho}_{\text{out}}=\hat{B}^{\dagger}\hat{\rho}_{\text{in}}\hat{B}. \tag{8}$$

An explicit expression for $\hat{B}$ was obtained by Yurke, McCall, and Klauder [25] and Campos, Saleh, and Teich [24] using the Jordan-Schwinger representation of an angular momentum system in terms of bosonic operators [30,31]:

$$\hat{L}_1=\tfrac{1}{2}(\hat{a}_1^{\dagger}\hat{a}_2+\hat{a}_2^{\dagger}\hat{a}_1), \quad \hat{L}_2=\frac{1}{2i}(\hat{a}_1^{\dagger}\hat{a}_2-\hat{a}_2^{\dagger}\hat{a}_1),$$
$$\hat{L}_3=\tfrac{1}{2}(\hat{a}_1^{\dagger}\hat{a}_1-\hat{a}_2^{\dagger}\hat{a}_2). \tag{9}$$

Here $\hat{L}_1$, $\hat{L}_2$, and $\hat{L}_3$ obey the same commutation rules as angular momentum components. With the definition (9)

$\hat{B}$ can be written as [24]

$$\hat{B}=e^{-i\Phi\hat{L}_3}e^{-i\Theta\hat{L}_2}e^{-i\Psi\hat{L}_3} . \tag{10}$$

III. BEAM SPLITTER AND WAVE FUNCTIONS

A number of interesting states in quantum optics have simple wave functions but complicated photon-number distributions. Squeezed states are a good example [10]. In this case it is advantageous to use wave-function techniques. On the other hand (as it will be shown in this section), a beam splitter transforms wave functions in a rather simple and intuitive way. Let us introduce a quadrature decomposition of the mode operators

$$\hat{a}_\nu=\frac{1}{\sqrt{2}}(\hat{x}_\nu+i\hat{p}_\nu) . \tag{11}$$

The quadrature components $\hat{x}_\nu$ and $\hat{p}_\nu$ play the roles of position and momentum operators since

$$[\hat{x}_\nu,\hat{p}_\mu]=i\delta_{\nu\mu} . \tag{12}$$

Wave functions are defined as the scalar products of the position and momentum eigenstates and the quantum state $|\psi\rangle$:

$$\psi(x_1,x_2)=\langle x_1,x_2|\psi\rangle , \quad \varphi(p_1,p_2)=\langle p_1,p_2|\psi\rangle . \tag{13}$$

The state transformation $|\psi_{\text{out}}\rangle=\hat{B}^\dagger|\psi_{\text{in}}\rangle$ (7) and (10) leads to a transformation of the wave function

$$\psi'(x_1,x_2)=\langle x_1,x_2|e^{i\Psi\hat{L}_3}e^{i\Theta\hat{L}_2}e^{i\Phi\hat{L}_3}|\psi\rangle . \tag{14}$$

The elements of a wave function transformation are rotation and phase shifting.

A. Rotation

The basic operation of a beam splitter is the mixing of modes. The wave function will be transformed according to

$$\begin{aligned}\psi'(x_1,x_2)&=\langle x_1,x_2|e^{i\Theta\hat{L}_2}|\psi\rangle\\&=\langle x_1,x_2|\exp\left[\frac{\Theta}{2}(\hat{a}_1^\dagger\hat{a}_2-\hat{a}_2^\dagger\hat{a}_1)\right]|\psi\rangle\\&=\langle x_1,x_2|\exp\left[\frac{\Theta}{2}(i\hat{x}_1\hat{p}_2-i\hat{x}_2\hat{p}_1)\right]|\psi\rangle\\&=\exp\left[\frac{\Theta}{2}\left[x_1\frac{\partial}{\partial x_2}-x_2\frac{\partial}{\partial x_1}\right]\right]\psi(x_1,x_2) .\end{aligned} \tag{15}$$

This equation describes a rotation by $\Theta/2$ in the x_1,x_2 plane

$$\psi'(x_1,x_2)=\psi(\cos(\Theta/2)x_1-\sin(\Theta/2)x_2,\sin(\Theta/2)x_1+\cos(\Theta/2)x_2) . \tag{16}$$

B. Phase shifting

In the Heisenberg picture phase shifting is a transformation like

$$\hat{a}'=e^{-i\gamma}\hat{a}=e^{i\gamma\hat{a}^\dagger\hat{a}}\hat{a}e^{-i\gamma\hat{a}^\dagger\hat{a}} . \tag{17}$$

In the Schrödinger picture a wave function would be transformed as follows:

$$\psi'=\langle x|e^{-i\gamma\hat{a}^\dagger\hat{a}}|\psi\rangle . \tag{18}$$

A phase-shift transformation is equivalent to the harmonic evolution of a wave function. Phase is time. Using the Green's function of a harmonic oscillator the evolution of a wave packet can be described as

$$\psi'(x)=\int_{-\infty}^{+\infty}dx'G(x,x',\gamma)\psi(x') . \tag{19}$$

The Green's function G was calculated in Appendix A with the result

$$G(x,x',\gamma)=\frac{1}{\sqrt{2\pi\sin\gamma}}\exp\left[i\left[\frac{x^2\cos\gamma-2xx'+x'^2\cos\gamma}{2\sin\gamma}+\frac{\gamma}{2}+\frac{\pi}{4}\right]\right] . \tag{20}$$

It has the following properties: First,

$$\lim_{\gamma\to0}G(x,x',\gamma)=\lim_{\gamma\to0}\frac{1}{\sqrt{2\pi\gamma/i}}\exp\left[\frac{i}{2\gamma}(x-x')^2\right]=\delta(x-x') , \tag{21}$$

as it should be for a Green's function. Second,

$$G\left[x,x',\frac{\pi}{2}\right]=\frac{i}{\sqrt{2\pi}}e^{-ixx'} \tag{22}$$

or

$$\psi'\left[x,\frac{\pi}{2}\right]=i\int_{-\infty}^{+\infty}\frac{dx'}{\sqrt{2\pi}}e^{-ixx'}\psi(x')=i\varphi(p=x) . \tag{23}$$

A phase shift of $\pi/2$ means (apart from a factor of i) a Fourier transformation. It gives the momentum wave function. For example, phase squeezing is turned into amplitude squeezing [32]. Third, the Green's function is periodic with a periodicity of 2π.

C. General transformation

A general transformation of the input wave function is a combination of the basic operations rotation and phase shifting:

$$\psi'(x_1,x_2)=\int_{-\infty}^{+\infty}dx_1'\int_{-\infty}^{+\infty}dx_2'G(x_1,x_1'-\Psi/2)\times G(x_2,x_2',\Psi/2)\varphi(x_1',x_2') , \tag{24}$$

$$\varphi(x_1,x_2)=\chi(\cos(\Theta/2)x_1-\sin(\Theta/2)x_2,\sin(\Theta/2)x_1+\cos(\Theta/2)x_2) , \tag{25}$$

$$\chi(x_1,x_2)=\int_{-\infty}^{+\infty}dx_1'\int_{-\infty}^{+\infty}dx_2'G(x_1,x_1',-\Phi/2)\times G(x_2,x_2',\Phi/2)\psi(x_1',x_2') . \tag{26}$$

Note that a momentum wave function will be transformed in the same way. The essence of a beam-splitter operation is the rotation of the wave function. The squared cosine of the rotation angle is given by the transmittance, while the reflectance gives the squared sine. Phase shiftings are expressed by convolutions of the wave functions with the Green's function of a harmonic oscillator.

IV. SIMULTANEOUS MEASUREMENT OF CONJUGATE VARIABLES

In the preceding section wave functions of quadratures were introduced. A quadrature can be measured via homodyning [33] or balanced homodyning [34,35]. A measurement of a quadrature is similar to a measurement of position or momentum since the components $\hat{x}$ and $\hat{p}$ are canonically conjugate variables. According to quantum theory a simultaneous measurement of position *and* momentum is impossible. One has to divide a statistical ensemble of states and measure $\hat{x}$ on one half and $\hat{p}$ on the other. But if it were possible to make two copies of a state and measure $\hat{x}$ and $\hat{p}$ on each copy separately no objection from quantum theory had to be feared. A beam splitter is a device to make copies of an incident light beam. However, the additional noise emerging from the "unused" port of the beam splitter is the price to be paid for it. Two schemes will be studied: a balanced measurement by splitting with a 50%:50% beam splitter and a more general unbalanced measurement.

A. Balanced measurement

Here the incident light mode is divided by a perfect 50%:50% beam splitter to be characterized by the matrix

$$\underline{B}=\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} . \tag{27}$$

The state of the light beam will be denoted by $|\varphi\rangle$ and the state of the light (or darkness) at the unused port of the beam splitter by $|\chi^*\rangle$. The input wave function is written as

$$\psi(x_1,x_2)=\langle x_1,x_2|\varphi\rangle|\chi^*\rangle=\varphi(x_1)\chi^*(x_2) . \tag{28}$$

Now the emerging two light beams are homodyned with two strong coherent fields having a phase difference of $\pi/2$ in order to measure $\hat{x}_1$ on one beam and $\hat{p}_2$ on the other. Quantum theory predicts that the probability of finding a particular pair of values x_1 and p_2 is

$$|\psi'(x_1,p_2)|^2=\left|\int_{-\infty}^{+\infty}\frac{dx_2}{\sqrt{2\pi}}e^{ip_2x_2}\psi'(x_1,x_2)\right|^2 , \tag{29}$$

where

$$\psi'(x_1,x_2)=\varphi\left[\frac{x_1+x_2}{\sqrt{2}}\right]\chi^*\left[\frac{-x_1+x_2}{\sqrt{2}}\right] . \tag{30}$$

Setting $\xi=(-x_1+x_2)/\sqrt{2}$ one obtains

$$\begin{aligned}|\psi'(x_1,p_2)|^2 &=\frac{1}{\pi}\left|\int_{-\infty}^{+\infty}d\xi\,\chi^*(\xi)e^{-i\sqrt{2}p_2\xi}\varphi(\xi+\sqrt{2}x_1)\right|^2\\ &=\frac{1}{\pi}\left|\int_{-\infty}^{+\infty}d\xi\,\chi^*(\xi)\hat{D}(-x_1-ip_2)\varphi(\xi)\right|^2 ,\end{aligned} \tag{31}$$

where the position representation of the displacement operator has been used (see Appendix B). Finally, the probability of finding x_1 and p_2 is given by

$$|\psi'(x_1,p_2)|^2=\frac{1}{\pi}|\langle\chi;x_1+ip_2|\varphi\rangle|^2 , \tag{32}$$

where

$$|x_1+ip_2;\chi\rangle\equiv\hat{D}(x_1+ip_2)|\chi\rangle . \tag{33}$$

An expression of this kind was discussed by Aharonov, Albert, and Au [36] and O'Connell and Rajagopal [37]. They proposed a new interpretation of the scalar product in Hilbert space by relating it to a measurement of the observables $\hat{x}_1-\hat{x}_2$ and $\hat{p}_1+\hat{p}_2$ of a two-particle system. Having a look at the beam-splitter matrix $\underline{B}$ in Eq. (27) we see that our scheme is a physical realization of this procedure (apart from a factor of $\sqrt{2}$). In the words of Aharonow, Albert, and Au the probability $|\psi'(x_1,p_2)|^2$ is equivalent to the scalar product of a displaced "quantum-ruler" state and the "object" state. O'Connell and Rajagopal showed that $|\psi'(x_1,p_2)|^2$ can be expressed in terms of Wigner functions

$$\begin{aligned}|\psi'(x_1,p_2)|^2 &=2\int_{-\infty}^{+\infty}dq\int_{-\infty}^{+\infty}dp\,W_\varphi(q,p)\\ &\quad\times W_\chi(q-\sqrt{2}x_1,p-\sqrt{2}p_2) ,\end{aligned} \tag{34}$$

where W_φ denotes the Wigner function corresponding to $\varphi(x)$ and W_χ the Wigenr function corresponding to $\chi(x)$. It fits well into the concept of constructing generalized Husimi quasidistributions via smoothing the Wigner function [8,9]. Thus a scheme of simultaneous measurement of conjugate variables by splitting the "object" beam and homodyning the "copies" is an operational approach to generalized Husimi quasidistributions.

B. Unbalanced measurement

An example of a generalized Husimi function is a squeezed Q function

$$Q_s=\frac{1}{\pi}|\langle\varphi|\hat{D}(x_1+ip_2)\hat{S}|0\rangle|^2 . \tag{35}$$

It can be obtained when the second input of the beam splitter is in a squeezed vacuum state. Then the quantum noise of the second beam-splitter input is phase dependent and below the vacuum level for one quadrature and above vacuum noise for the conjugate one. But is it really necessary to recruit squeezed vacuum? Let the object light fall onto a more general beam splitter:

$$\underline{B}=\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} . \tag{36}$$

Then the probability of finding x_1 and p_2 is given by

$$|\psi'(x_1,p_2)|^2=\left|\int_{-\infty}^{+\infty}\frac{dx_2}{\sqrt{2\pi}}e^{-ip_2x_2}\psi'(x_1,x_2)\right|^2 . \qquad (37)$$

Here

$$\psi'(x_1,x_2)=\varphi(x_1\cos\alpha+x_2\sin\alpha)\chi^*(-x_1\sin\alpha+x_2\cos\alpha) . \qquad (38)$$

Substituting $\xi=-x_1\sin\alpha+x_2\cos\alpha$ one obtains

$$\begin{aligned}|\psi'(x_1,p_2)|^2&=\frac{1}{2\pi\cos^2\alpha}\left|\int_{-\infty}^{+\infty}d\xi\,\chi^*(\xi)\exp\left[-i\frac{p_2\xi}{\cos\alpha}\right]\varphi\left[\frac{\xi\sin\alpha+x_1}{\cos\alpha}\right]\right|^2\\&=\frac{1}{\pi\sin(2\alpha)}\left|\int_{-\infty}^{+\infty}d\xi\,\chi^*(\xi)\hat{S}(\ln(\tan\alpha))\exp\left[-i\frac{p_2\xi}{\sin\alpha}\right]\varphi\left[\xi+\frac{x_1}{\cos\alpha}\right]\right|^2 ,\end{aligned} \qquad (39)$$

where the position representation of the squeezing operator $\hat{S}$ was used (see Appendix C). The representation of the displacement operator (see Appendix B) enables us to write

$$|\psi'(x_1,p_2)|^2=\frac{1}{\pi\sin(2\alpha)}\left|\int_{-\infty}^{+\infty}d\xi\,\chi^*(\xi)\hat{S}(\ln(\tan\alpha))\hat{D}\left[-\frac{1}{\sqrt{2}}\left[\frac{x_1}{\cos\alpha}+i\frac{p_2}{\sin\alpha}\right]\right]\varphi(\xi)\right|^2 \qquad (40)$$

and since

$$[\hat{S}(\ln(\tan\alpha))]^\dagger=\hat{S}(-\ln(\tan\alpha))=\hat{S}(\ln(\cot\alpha)) \qquad (41)$$

we arrive at

$$|\psi'(x_1,p_2)|^2=\frac{1}{\pi\sin(2\alpha)}\left|\langle\varphi|\hat{D}\left[\frac{1}{\sqrt{2}}\left[\frac{x_1}{\cos\alpha}+i\frac{p_2}{\sin\alpha}\right]\right]\hat{S}(\ln(\cot\alpha))|\chi\rangle\right|^2 \qquad (42)$$

or using the notation of Sec. IV A

$$\begin{aligned}&|\psi'(x_1,p_2)|^2=\frac{1}{\pi\sin(2\alpha)}|\langle\chi;\beta,\zeta|\varphi\rangle|^2 ,\\&|\beta,\zeta;\chi\rangle=\hat{D}(\beta)\hat{S}(\zeta)|\chi\rangle ,\\&\beta=\frac{1}{\sqrt{2}}\left[\frac{x_1}{\cos\alpha}+i\frac{p_2}{\sin\alpha}\right] ,\\&\zeta=\ln(\cot\alpha) .\end{aligned} \qquad (43)$$

We see that when the unused port of the beam splitter is entered by the vacuum field (is in fact unused) a squeezed Q function will be measured. What does it mean? The noise from the second beam-splitter port is not equally distributed. It depends on the transmittance. When the transmittance is greater than 50% the measured quadrature on the transmitted beam contains less added vacuum fluctuations than using a balanced scheme whereby, of course, the reflected beam is influenced by respectively more fluctuations. For a transmittance lower than 50% we are in the reverse situation. Thus, in the unbalanced scheme for simultaneous measurements of conjugate variables fluctuations entering the second port of the beam splitter are suppressed on one quadrature and enhanced on the other. Note that this concept can be generalized to a complex squeezing parameter ζ by introducing an overall phase shift $\underline{B}\rightarrow\exp(-i\gamma)\underline{B}$.

C. Experiment

An experiment of the design in question has been carried out recently by Noh, Fougères, and Mandel [4]. It was intended as an operational approach to the quantum phase by photon counting. The phases of two coherent fields were compared in an interferometer depicted in Fig. 2. In case one coherent field is sufficiently strong to serve as a reference for balanced homodyning [35], phase from the Q function will be measured [6,7,38], whereby phase means the angle between position and momentum. The phase distribution is obtained by averaging over the "radius" in phase space. Having in mind the results obtained in Secs. IV A and IV B modifications of this experiment can be suggested.

First, it would be interesting to measure the Q function of squeezed states. Second, a squeezed Q function can be measured using the unbalanced scheme. Third, the Q function of displaced Kerr states produced in a nonlinear Mach-Zehnder interferometer [39] can be measured.

Especially attractive would be the measurement of the Q function or the squeezed Q function for a single photon produced in spontaneous parametric down-conversion [40].

D. Remarks

To the author's knowledge the balanced scheme of measurement was first analyzed by Lai and Haus [3]. They pointed out that the Q function or a squeezed Q

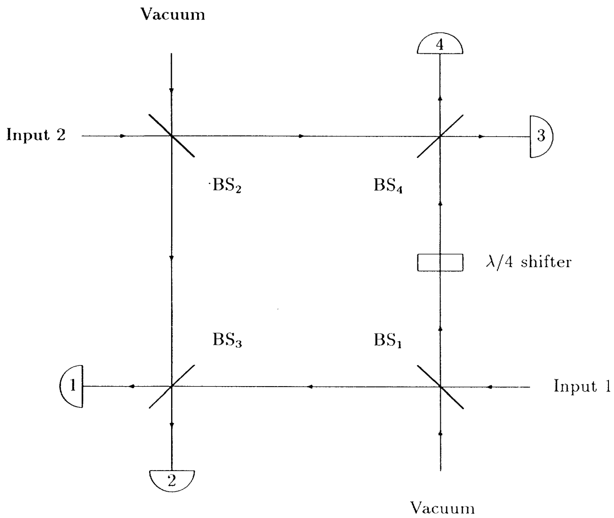

FIG. 2. Experimental scheme by Noh, Fougères, and Mandel intended as an operational approach to the quantum phase of light. The essence of the experiment is a simultaneous, however noisy, measurement of the two quadrature components of the field entering the interferometer at input 1. The field is split by the beam splitter BS$_1$, one beam is shifted in phase by $\pi/2$ and homodyned with two strong coherent reference beams. They are produced by splitting a laser beam entering at input 2 using the beam splitter BS$_2$.

function can be measured for vacuum or squeezed vacuum at the second input of the beam splitter. This paper shows that a squeezed Q function can be measured without using squeezed states. The experiment of Noh, Fougères, and Mandel [4] in the limit of a strong local oscillator was first successfully analyzed by Freyberger and Schleich [6]. They showed that for a coherent input, phase from the Q function is measured. Later Leonhardt and Paul [7] and Freyberger, Vogel, and Schleich [38] generalized this result with respect to an arbitrary state at the first beam-splitter port. In this paper arbitrary states at both the first and the second port of the beam splitter are taken into account.

Note that the detection of the Q function is the common feature of several different schemes of simultaneous measurements of conjugate variables. Phase measurements [41] via amplification [42,43] or heterodyning [44] as well as schemes based on the interaction of "meter systems" with the quantum object [45,46] are measurements of (generalized) Husimi quasidistributions.

It was pointed out by Wódkiewicz [47] and Arthurs and Goodman [48] that the price to be paid for measuring position and momentum *simultaneously* is an increasing of the uncertainty product

$$(\Delta x_1)(\Delta p_2)\geq 1 \tag{44}$$

instead of Heisenberg's relation

$$(\Delta x_1)(\Delta p_1)\geq \tfrac{1}{2} \ . \tag{45}$$

V. BEAM SPLITTER AND QUASIPROBABILITIES

The wave-function formalism of Sec. III works well if the system is in a pure state having a simple wave function. In many cases statistical mixtures are effectively described by quasiprobabilities. So let us introduce an operator [49]

$$\Delta(\alpha;s)\equiv\int\frac{d^2\beta}{\pi}\exp\left[\beta(\hat{a}^\dagger-\alpha^*)-\beta^*(\hat{a}-\alpha)+\frac{s}{2}\beta^*\beta\right] . \tag{46}$$

Quasiprobabilities can be extracted from the density operator $\hat{\rho}$ in the form

$$P(\alpha;s)\equiv\frac{1}{\pi}\mathrm{Tr}\{\hat{\rho}\Delta(\alpha;s)\} \tag{47}$$

and vice versa they allow a reconstruction of $\hat{\rho}$:

$$\hat{\rho}=\int d^2\alpha\, P(\alpha;s)\Delta(\alpha;-s) \ . \tag{48}$$

Here $P(\alpha;s)$ gives, in particular, (i) for $s=1$ the P function of Glauber [50] and Sudarshan [51], (ii) for $s=0$ (apart from a factor of 2) the Wigner function [52]

$$W(q,p)=\frac{1}{\pi}\int_{-\infty}^{+\infty}dx\, e^{2ipx}\langle q-x|\hat{\rho}|q+x\rangle \ ,$$

where $\alpha(q+ip)/\sqrt{2}$, and (iii) for $s=-1$ the Q function [53] $Q(\alpha)=\langle\alpha|\hat{\rho}|\alpha\rangle$. Now the beam-splitter transformation of quasiprobabilities will be derived. Let the input quasiprobability be denoted by

$$P(\alpha_1,\alpha_2;s)=\frac{1}{\pi^2}\mathrm{Tr}_{12}\{\hat{\rho}\Delta_1(\alpha_1;s)\Delta_2(\alpha_2;s)\} \ . \tag{49}$$

According to the density operator transformation (8) the output quasiprobability is given by

$$\begin{aligned}P'(\alpha_1,\alpha_2;s)&=\frac{1}{\pi^2}\mathrm{Tr}_{12}\{\hat{B}^\dagger\hat{\rho}\hat{B}\,\Delta_1(\alpha_1;s)\Delta_2(\alpha_2;s)\}\\&=\frac{1}{\pi^2}\mathrm{Tr}_{12}\{\hat{\rho}\hat{B}\,\Delta_1(\alpha_1;s)\Delta_2(\alpha_2;s)\hat{B}^\dagger\} \ .\end{aligned} \tag{50}$$

Then according to (6)

$$\begin{aligned}&\hat{B}\,\Delta_1(\alpha_1;s)\Delta_2(\alpha_2;s)\hat{B}^\dagger\\&\quad=\int\frac{d^2\beta_1}{\pi}\int\frac{d^2\beta_2}{\pi}\exp\Big[\beta_1(\hat{b}_1^\dagger-\alpha_1^*)-\beta_1^*(\hat{b}_1-\alpha_1)\\&\qquad+\beta_2(\hat{b}_2^\dagger-\alpha_2^*)-\beta_2^*(\hat{b}_2-\alpha_2)\\&\qquad+\frac{s}{2}(\beta_1^*\beta_1+\beta_2^*\beta_2)\Big] \ .\end{aligned} \tag{51}$$

Introducing

$$\begin{bmatrix}\alpha_1'\\ \alpha_2'\end{bmatrix}=\underline{B}^\dagger\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} , \quad \begin{bmatrix}\beta_1'\\ \beta_2'\end{bmatrix}=\underline{B}^\dagger\begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix} , \tag{52}$$

Eq. (51) can be written as

$$\hat{B}\Delta_1(\alpha_1;s)\Delta_2(\alpha_2;s)\hat{B}^\dagger = \int \frac{d^2\beta_1'}{\pi}\int \frac{d^2\beta_2'}{\pi}\exp\left[\beta_1'(\hat{a}_1^\dagger-\alpha_1'^*)-\beta_1'^*(\hat{a}_1-\alpha_1') + \beta_2'(\hat{a}_2^\dagger-\alpha_2'^*)-\beta_2'^*(\hat{a}_2-\alpha_2') + \frac{s}{2}(\beta_1'^*\beta_1'+\beta_2'^*\beta_2')\right] \quad (53)$$

and finally

$$\hat{B}\Delta_1(\alpha_1;s)\Delta_2(\alpha_2;s)\hat{B}^\dagger=\Delta_1(\alpha_1';s)\Delta_2(\alpha_2';s)\ . \quad (54)$$

Thus quasiprobabilites are transformed by the inverse beam-splitter transformation

$$P'(\alpha_1,\alpha_2;s)=P(\alpha_1',\alpha_2';s)\ ,\quad \begin{bmatrix}\alpha_1'\\ \alpha_2'\end{bmatrix}=\underline{B}^\dagger\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}\ . \quad (55)$$

Example: Schrödinger-cat state

A prominent candidate with simple but interesting quasiprobabilities is a Schrödinger-cat state [11]. The "cat" is a superposition of two (macroscopic distinct) coherent states

$$|\psi\rangle=\frac{1}{\sqrt{2}}(|\beta\rangle+|-\beta\rangle)\ . \quad (56)$$

Possibilities to generate a cat, its nonclassical properties, and its fragile behavior have attracted much interest [11,15–17,54–56]. Let us image a Schrödinger-cat state in front of a semitransparent mirror (an outcoupling mirror of a cavity with a cat inside) characterized by a rotation angle α. The Wigner function of a cat is (see Fig. 3)

$$W_1(q_1,p_1) = \frac{1}{2\pi(1+e^{-x_0^2})}e^{-p_1^2}\{e^{-(q_1-x_0)^2}+e^{-(q_1+x_0)^2} + e^{-q_1^2}[2\cos(2p_1x_0)]\}\ , \quad (57)$$

where $\mathrm{Re}\beta=x_0/\sqrt{2}$ and $\mathrm{Im}\beta=0$. The function consists of two terms localized at x_0 and $-x_0$ and one interference term. Without the latter the light amplitude of a cat state would be either x_0 or $-x_0$, but the oscillating term indicates quantum superposition, i.e., the amplitude is both x_0 and $-x_0$ in contradiction to macroscopic experience. Let the field beyond the mirror be in a thermal (or in a vacuum) state. The Wigner function of a thermal state is

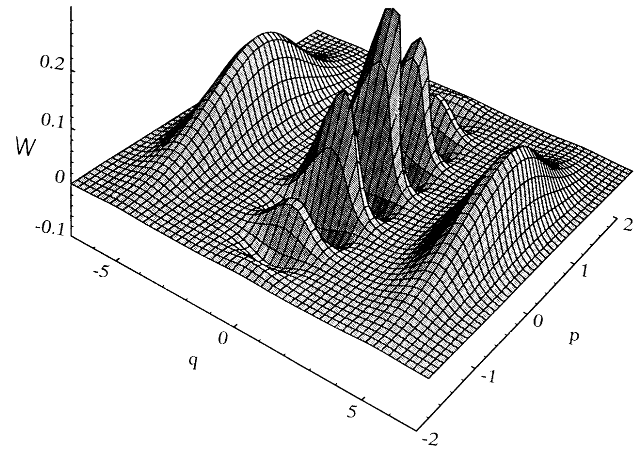

FIG. 3. Wigner function of a Schrödinger-cat state $|\psi\rangle=(1/\sqrt{2})(|\beta\rangle+|-\beta\rangle)$ with $\beta=5/\sqrt{2}$. The interference structure indicates quantum superposition.

$$W_2(q_2,p_2)=\frac{1}{\pi(2\bar{n}+1)}\exp\left[-\frac{q_2^2+p_2^2}{2\bar{n}+1}\right]\ . \quad (58)$$

In order to obtain the reflected state we rotate the total Wigner function by an angle of α and calculate the reduced one

$$W_1'(q_1,p_1)=\int_{-\infty}^{+\infty}dq_2\int_{-\infty}^{+\infty}dp_2\,W_1(q_1',p_1')W_2(q_2',p_2')\ . \quad (59)$$

The result is

$$W_1'(q_1,p_1)=\frac{1}{2\pi(1+e^{-x_0^2})2\delta_q\delta_p}\exp\left[-\frac{p_1^2}{2\delta_p^2}\right] \times\left\{\exp\left[-\frac{(q_1-x_0\cos\alpha)^2}{2\delta_q^2}\right]+\exp\left[-\frac{(q_1+x_0\cos\alpha)^2}{2\delta_q^2}\right] +\exp\left[-\frac{q_1^2}{2\delta_q^2}\right]\left[2\cos\left[\frac{2x_0p_1\cos\alpha}{2\delta_p^2}\right]\right]\exp\left[-\frac{x_0^2(1+2\bar{n})}{2\delta_p^2}\sin^2\alpha\right]\right\}\ , \quad (60)$$

where

$$2\delta_q^2=2\delta_p^2=\cos^2\alpha+(2\bar{n}+1)\sin^2\alpha\ . \quad (61)$$

A single nonperfect reflection "kills" a macroscopic cat (when x_0 is large) due to the quantum noise of the vacuum or the thermal reservoir beyond the mirror (at the unused input port of the beam splitter) since the superposition term in the Wigner function (60) decays at a rate of

$x_0^2(1+2\bar{n})$ for small α. The Schrödinger cat has to make a decision: It is localized either at x_0 or at $-x_0$. Mecozzi and Tombesi [56] suggested to introduce squeezed vacuum at the second input port of the beam splitter having the Wigner function

$$W_2(q_2,p_2)=\frac{1}{\pi}\exp\left[-2\delta^2 q_2^2-\frac{p_2^2}{2\delta^2}\right] . \quad (62)$$

Then the loss of quantum coherence will be reduced since in this case we get again the Wigner function (60) but now with

$$1+2\bar{n}=2\delta^2 ,$$
$$2\delta_q^2=\cos^2\alpha+\frac{1}{2\delta^2}\sin^2\alpha , \quad (63)$$
$$2\delta_p^2=\cos^2\alpha+2\delta^2\sin^2\alpha .$$

Now the destruction rate for small α is reduced to $2\delta^2 x_0^2$.

VI. BEAM SPLITTER AND DISSIPATION

A beam splitter provides a heuristic model of dissipation. When light is damped it can be imagined as being splitted in a transmitted and a removed part. On the other hand, an attenuated light mode needs an additional fluctuation mode in order to conserve the commutation rules. So intuition would suggest a model like [12]

$$\begin{bmatrix}\hat{a}(t)\\ \hat{b}(t)\end{bmatrix}=\begin{bmatrix}e^{-(\gamma/2)t} & \sqrt{1-e^{-\gamma t}}\\ -\sqrt{1-e^{-\gamma t}} & e^{-(\gamma/2)t}\end{bmatrix}\begin{bmatrix}\hat{a}(0)\\ \hat{b}(0)\end{bmatrix} , \quad (64)$$

where $\hat{a}(t)$ is the attenuated light mode, $\hat{b}$ is a fluctuation mode, and γ is a damping constant. One may argue that dissipation is an interaction with a reservoir of many degrees of freedom and not with one fluctuation mode only. However, it will be shown that the simple intuitive model is *exact* for dissipation in Gaussian reservoirs. An example of damping in a Gaussian reservoir is the evolution of a mode inside a lossy cavity [57]. Here the outcoupling mirror of the resonator couples the light inside the cavity to a reservoir of vacuum, thermal, or squeezed light outside.

Let us start from the master equation for dissipation in Gaussian reservoirs [58]:

$$\frac{d\hat{\rho}}{dt}=\frac{\gamma}{2}N(\hat{a}^\dagger\hat{\rho}\hat{a}-\hat{\rho}\hat{a}\hat{a}^\dagger)+\frac{\gamma}{2}(N+1)(\hat{a}\hat{\rho}\hat{a}^\dagger-\hat{a}^\dagger\hat{a}\hat{\rho})$$
$$+\frac{\gamma}{2}M(\hat{\rho}\hat{a}^{\dagger 2}+\hat{a}^{\dagger 2}\hat{\rho}-2\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger)+\text{H.c.} \quad (65)$$

Here N denotes the mean photon number of the reservoir and the complex parameter M indicates phase sensitivity. It is thermal if $M=0$ and ideally squeezed if $|M|^2=N(N+1)$, while for the general case we have $|M|^2\leq N(N+1)$. The master equation (65) can be translated into a Fokker-Planck equation for quasiprobabilities $P(\alpha;s)$ [59]

$$\frac{\partial P}{\partial t}=\frac{\gamma}{2}\left[\frac{\partial(\alpha P)}{\partial\alpha}+\frac{\partial(\alpha^* P)}{\partial\alpha^*}+M\frac{\partial^2 P}{\partial\alpha^2}+2D\frac{\partial^2 P}{\partial\alpha^*\partial\alpha}+M^*\frac{\partial^2 P}{\partial\alpha^{*2}}\right] \quad (66)$$

with the s-dependent diffusion constant

$$D=N+\tfrac{1}{2}-s . \quad (67)$$

The general solution of the Fokker-Planck equation (66) is given by

$$P(\alpha,t;s)=\int d^2\beta\, P_a(e^{-(\gamma/2)t}\alpha-\sqrt{1-e^{-\gamma t}}\beta;s)\times P_0(\sqrt{1-e^{-\gamma t}}\alpha+e^{-(\gamma/2)t}\beta;s) \quad (68)$$

(see Appendix D), where

$$P_0(\beta;s)=\frac{1}{\pi\sqrt{D^2-|M|^2}}\times\exp\left[\frac{M^*\beta^2-2D\beta^*\beta+M\beta^{*2}}{2(D^2-|M|^2)}\right] \quad (69)$$

denotes the stationary solution of the Fokker-Planck equation (66). Already here we see that dissipation mixes the initial and the final quasidistribution. Let us return to the density operator. Using Eq. (47) and $\text{Tr}_B\{\Delta(\beta;-s)\}=1$ [49] we write

$$\hat{\rho}=\text{Tr}_B\left\{\int d^2\alpha'\int d^2\beta' P_a(\alpha';s)P_0(\beta';s)\times\Delta_A(\alpha;-s)\Delta_B(\beta;-s)\right\} , \quad (70)$$

where

$$\begin{bmatrix}\alpha'\\ \beta'\end{bmatrix}=\begin{bmatrix}e^{-(\gamma/2)t} & -\sqrt{1-e^{-\gamma t}}\\ \sqrt{1-e^{-\gamma t}} & e^{-(\gamma/2)t}\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} . \quad (71)$$

Substituting α' and β' for α and β we obtain

$$\hat{\rho}=\text{Tr}_B\left\{\int d^2\alpha\int d^2\beta P_a(\alpha';s)P_0(\beta';s)\times\Delta'_A(\alpha;-s)\Delta'_B(\beta;-s)\right\} , \quad (72)$$

where

$$\Delta'_A(\alpha;-s)\Delta_B(\beta;-s)=\int d^2\tilde{\alpha}\int d^2\tilde{\beta}\exp\left[\tilde{\alpha}(\hat{a}'^\dagger-\alpha'^*)-\tilde{\alpha}^*(\hat{a}'-\alpha')+\tilde{\beta}(\hat{b}'^\dagger-\beta'^*)-\tilde{\beta}^*(\hat{b}'-\beta')-\frac{s}{2}(\tilde{\alpha}^*\tilde{\alpha}+\tilde{\beta}^*\tilde{\beta})\right] \quad (73)$$

and

$$\begin{pmatrix} \hat{a}' \\ \hat{b}' \end{pmatrix} = \begin{pmatrix} e^{-(\gamma/2)t} & -\sqrt{1-e^{-\gamma t}} \\ \sqrt{1-e^{-\gamma t}} & e^{-(\gamma/2)t} \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} \tag{74}$$

has been introduced. In Sec. II beam-splitter transformations were expressed by means of the Jordan-Schwinger representation of angular momenta. Transformation formula (6) gives us

$$\Delta'_A(\alpha;-s)\Delta_B(\beta;-s) = e^{i\Theta\hat{L}_2}\Delta_A(\alpha';-s)\Delta_B(\beta;-s)e^{-i\Theta\hat{L}_2} \tag{75}$$

with

$$\cos(\Theta/2)=\exp[-(\gamma/2)t] \ . \tag{76}$$

Thus the density operator $\hat{\rho}$ can be expressed as

$$\rho=\mathrm{Tr}_B\{e^{i\Theta\hat{L}_2}\int d^2\alpha' P_a(\alpha';s)\Delta_A(\alpha';-s) \times \int d^2\beta' P_0(\beta';s)\Delta_B(\beta';-s)e^{-i\Theta\hat{L}_2}\} \ . \tag{77}$$

Here $\int d^2\alpha' P_a(\alpha';s)\Delta_A(\alpha';s)$ gives the initial density operator $\hat{\rho}_a$ [see Eq. (48)] and $\int d^2\beta' P_0(\beta';s)\Delta_B(\beta';s)$ the final one $\hat{\rho}_0$—a Gaussian density operator. Using the relations between Gaussian quasidistributions and Gaussian density operators [60] we finally arive at

$$\begin{aligned} &\hat{\rho}=\mathrm{Tr}_B\{e^{i\Theta\hat{L}_2}\hat{\rho}_a\hat{\rho}_0 e^{-i\Theta\hat{L}_2}\} \ , \\ &\cos(\Theta/2)=\exp[-(\gamma/2)t] \ , \\ &\hat{\rho}_0=\mathcal{N}\exp[-\tfrac{1}{2}(m^*\hat{b}^2+2n\hat{b}^\dagger\hat{b}+m\hat{b}^{\dagger 2})] \ , \\ &n=\frac{2(N+\frac{1}{2})}{\sqrt{(N+\frac{1}{2})^2-|M|^2}} \times \mathrm{arccoth}[2\sqrt{(N+\tfrac{1}{2})^2-|M|^2}] \ , \\ &m=\frac{2M}{\sqrt{(N+\frac{1}{2})^2-|M|^2}} \times \mathrm{arccoth}[2\sqrt{(N+\tfrac{1}{2})^2-|M|^2}] \ . \end{aligned} \tag{78}$$

($\mathcal{N}$ denotes a normalization constant.) Equation (78) gives the general solution of the master equation (65) for dissipation in Gaussian reservoirs. It describes the mixing of an initial density operator $\hat{\rho}_a$ with the final Gaussian state $\hat{\rho}_0$. In the Heisenberg picture we get the intuitive model (64). Dissipation is interference with a fluctuation mode. One has to translate only $\cos(\Theta/2) = \exp[-(\gamma/2)t]$.

Returning to the Schrödinger-cat example we see that our beam-splitter model reproduces in a simple way the known results about the dissipative decay of quantum coherence in vacuum [11], thermal [15], or phase-sensitive [17] reservoirs. Moreover, the sensitivity of macroscopic quantum superposition can be interpreted geometrically: Let us consider the oscillating term in the Wigner function of the cat at $q_1=0$ (where its maximum is situated) together with the Wigner function of the Gaussian state at $q_2=0$ in a common two-dimensional momentum space (see Fig. 4). We see a rapid oscillating function in p_1 direction. Dissipation acts like a beam splitter. It rotates this function (see Fig. 5) and projects it onto the p_1 axis. Now the oscillations will cancel each other. The efficiency of this cancellation is determined by the broadening in p_2 direction. Thermal reservoirs have a broad distribution, squeezed vacuum a narrow one. Thus thermal dissipation leads to a rapid decay of the quantum superposition but squeezing may save the life of the cat.

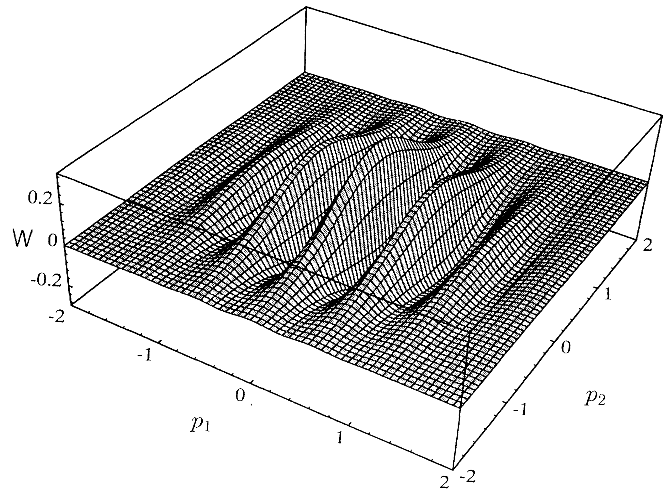

FIG. 4. Plot of the oscillating term in the Wigner function of the Schrodinger-cat state of input 1 together with the Wigner function of the Gaussian state of input 2 in a common two-dimensional momentum space at $q_1=q_2=0$.

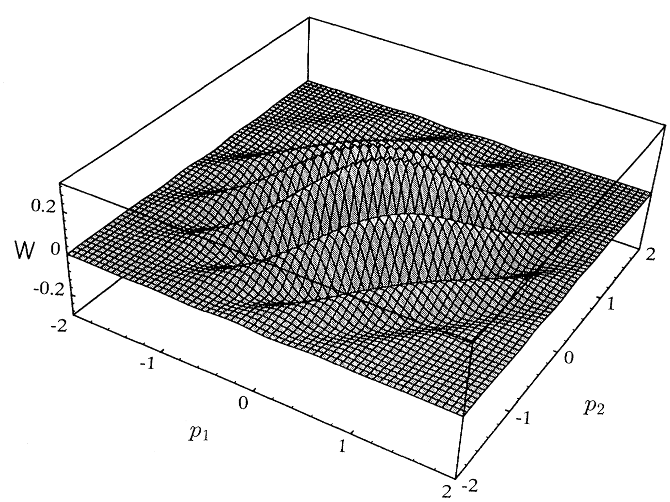

FIG. 5. Dissipation "rotates" the input Wigner function depicted in Fig. 4 and projects it onto the p_1 axis. The oscillations of the quantum interference structure cancel each other.

VII. CONCLUDING REMARKS

For theoretical simplicity the description of interference in a beam splitter, an interferometer or a linear coupler should be adapted to the measured observables. When photons are counted directly then a Fock basis formalism works well [24]. When quadrature components are measured then the natural theoretical tool is a wave-function formalism. Apart from phase shiftings the beam splitter rotates the total wave function of the incident light modes. The Green's function for phase shifting was derived. Quasiprobabilities such as the P function, the Wigner function, or the Q function undergo a unitary transformation of their arguments in the process of beam splitting.

A measurement of a quadrature component on one beam and the canonically conjugated variable on the other beam emerging from a balanced 50%:50% beam splitter is a measurement of a generalized Husimi function (a smoothed Wigner function). When one input port of the beam splitter is unused (vacuum is entering) the detected phase-space probability distribution is given by the Q function. Using an unbalanced beam splitter a squeezed Q function is measured without using squeezed states. An experiment in this spirit is the operational approach to the quantum phase by Noh, Fougères, and Mandel [4]. Experimental improvements are suggested.

A beam splitter provides a heuristic model for damping. Dissipation corresponds to a finite reflectivity of the beam splitter and fluctuation to the contact with fluctuations of the second input state. It was shown that this relationship is *exact* for damping in Gaussian reservoirs. As the mathematical tool to prove it the Fokker-Planck equation for damping in phase-sensitive reservoirs and the corresponding quantum master equation were solved. The decay of a Schrödinger-cat state in Gaussian reservoirs was studied as an example. A geometrical interpretation of the extreme sensitivity of macroscopic quantum coherence with respect to damping was found.

Note added in proof. The author recently learned from P. L. Knight that the eight-port homodyne detector sketched in Fig. 2 has already been extensively analyzed by N. G. Walker, J. Mod. Opt. **34**, 15 (1987).

ACKNOWLEDGMENTS

First of all I would like to thank H. Paul. I gained from him the greatest part of the physical insight into the subject of this paper. It is a pleasure to thank my colleagues A. Bandilla, V. Bužek, M. Freyberger, U. Herzog, J. Jansky, L. Knöll, A. Orlowski, Th. Richter, H.-H. Ritze, S. Stenholm, and A. Wünsche for stimulating discussions and valuable hints.

APPENDIX A: PHASE SHIFTING OF WAVE FUNCTIONS

Let a phase shift (time evolution) operator $\hat{U}(\gamma)=\exp(-i\gamma\hat{a}^\dagger\hat{a})$ act on a wave function

$$\hat{U}(\gamma)\psi(x)=\langle x|\exp(-i\gamma\hat{a}^\dagger\hat{a})|\psi\rangle = \int \frac{d^2\alpha}{\pi}\langle x|e^{-i\gamma}\alpha\rangle\langle\alpha|\psi\rangle ,$$

where a coherent state basis was introduced. The wave function of a coherent state $|\alpha\rangle$ is

$$\langle x|\alpha\rangle=\pi^{-1/4}\exp\left[-\frac{(x-x_0)^2}{2}+ip_0x-i\frac{p_0x_0}{2}\right] ,$$

where $\alpha=(x_0+ip_0)/\sqrt{2}$. So the wave function of a coherent state $|\alpha e^{-\gamma}\rangle$ can be expressed as

$$\langle x|\alpha e^{-i\gamma}\rangle=\pi^{-1/4}\exp\left[-\tfrac{1}{2}[x-(x_0\cos\gamma+p_0\sin\gamma)]^2 + i(-x_0\sin\gamma+p_0\cos\gamma)x-\frac{i}{2}(x_0\cos\gamma+p_0\sin\gamma)(-x_0\sin\gamma+p_0\cos\gamma)\right] ,$$

$$\langle x|\alpha e^{-i\gamma}\rangle=\pi^{-1/4}\exp\{-\tfrac{1}{2}[x^2-2e^{-i\gamma}(x_0+ip_0)x+(x_0\cos\gamma+p_0\sin\gamma)^2+i(x_0\cos\gamma+p_0\sin\gamma)(-x_0\sin\gamma+p_0\cos\gamma)]\}$$
$$\times\frac{e^{i(\gamma/2+\pi/4)}}{\sqrt{2\pi\sin\gamma}}\int_{-\infty}^{+\infty}dx'\exp\left[\frac{ie^{i\gamma}}{2\sin\gamma}\{x'-e^{-i\gamma}[x+i(x_0+ip_0)\sin\gamma]\}^2\right] ,$$

$$\langle x|\alpha e^{-i\gamma}\rangle=\frac{e^{i(\gamma/2+\pi/4)}}{\sqrt{2\pi\sin\gamma}}\int_{-\infty}^{+\infty}dx'\exp\left[i\frac{\cos\gamma x^2-2xx'+\cos\gamma x'^2}{2\sin\gamma}\right]\pi^{-1/4}\exp\left[-\frac{(x'-x_0)^2}{2}+ip_0x'-i\frac{p_0x_0}{2}\right]$$

after some algebra. Defining a function G

$$G(x,x',\gamma)\equiv\frac{1}{\sqrt{2\pi\sin\gamma}}\exp\left[i\frac{\cos\gamma x^2-2xx'+\cos\gamma x'^2}{2\sin\gamma}+i\frac{\gamma}{2}+i\frac{\pi}{4}\right] \tag{A1}$$

we get $\langle x|\alpha e^{-i\gamma}\rangle=\int_{-\infty}^{+\infty}dx'G(x,x',\gamma)\langle x'|\alpha\rangle$ and finally

$$\hat{U}(\gamma)\psi(x)=\int_{-\infty}^{+\infty}dx'G(x,x',\gamma)\psi(x') . \tag{A2}$$

G means the Green's function for phase shifting (time evolution).

APPENDIX B: DISPLACEMENT OF WAVE FUNCTIONS

Let a displacement operator [10] $\hat{D}(\alpha)=\exp(\alpha\hat{a}^{\dagger}-\alpha^*\hat{a})$ act on a wave function, where $\alpha=(q+ip)/\sqrt{2}$ and $\hat{a}=(\hat{x}+i\hat{p})/\sqrt{2}$:

$$\hat{D}(\alpha)\psi(x)=\exp(ip\hat{x}-iq\hat{p})\psi(x)$$
$$=\exp\left[-\frac{ipq}{2}\right]\exp(ip\hat{x})\exp(-iq\hat{p})\psi(x)$$
$$=\exp\left[-\frac{ipq}{2}+ipx\right]\exp\left[-q\frac{\partial}{\partial x}\right]\psi(x)\ ,$$

where the Baker-Hausdorff formula was used. The final result is

$$\hat{D}\left[\frac{q+ip}{\sqrt{2}}\right]\psi(x)=\exp\left[-\frac{ipq}{2}+ipx\right]\psi(x-q)\ . \tag{B1}$$

APPENDIX C: SQUEEZING OF WAVE FUNCTIONS

Let a (real) squeezing operator [10] $\hat{S}(s)=\exp[(s/2)(\hat{a}^2-\hat{a}^{\dagger 2})]$ act on a wave function, where $\hat{a}=(\hat{x}+i\hat{p})/\sqrt{2}$:

$$\hat{S}(s)\psi(x)=\exp\left[\frac{s}{2}(i\hat{x}\hat{p}-i\hat{p}\hat{x})\right]\psi(x)$$
$$=\exp\left[\frac{s}{2}\left[x\frac{\partial}{\partial x}+\frac{\partial}{\partial x}x\right]\right]\psi(x)\ .$$

Since the scaling transformation

$$\psi'(x;s)=e^{s/2}\psi(e^s x)$$

has the generator

$$\left.\frac{\partial\psi(x;s)}{\partial s}\right|_{s=0}=\frac{1}{2}\left[x\frac{\partial}{\partial x}+\frac{\partial}{\partial x}x\right]\psi(x)$$

squeezing mean scaling:

$$\hat{S}(s)\psi(x)=e^{s/2}\psi(e^s x)\ , \tag{C1}$$

where the normalization $\int_{-\infty}^{+\infty}\psi^*\psi\,dx=1$ is conserved.

APPENDIX D: GENERAL SOLUTION OF THE FOKKER-PLANCK EQUATION

In this appendix the general solution of the Fokker-Planck equation for dissipation in phase-sensitive reservoirs is given

$$\frac{\partial P}{\partial t}=\mathcal{L}P=\frac{\gamma}{2}\left[\frac{\partial(\alpha P)}{\partial\alpha}+\frac{\partial(\alpha^* P)}{\partial\alpha^*}+M\frac{\partial^2 P}{\partial\alpha^2}+2D\frac{\partial^2 P}{\partial\alpha\partial\alpha^*}+M^*\frac{\partial^2 P}{\partial\alpha^{*2}}\right]\ . \tag{D1}$$

It can be easily proved that the stationary solution ($\mathcal{L}P_0=0$) reads

$$P_0=\frac{1}{\pi\sqrt{D^2-|M|^2}}\exp\left[\frac{M^*\alpha^2-2D\alpha^*\alpha+M\alpha^{*2}}{2(D^2-|M|^2)}\right]\ . \tag{D2}$$

Note that P_0 is normalized to unity. We will prove that the general solution of (D1) is given by

$$P=\int d^2\beta\, P_a(e^{-(\gamma/2)t}\alpha-\sqrt{1-e^{-\gamma t}}\beta)\times P_0(\sqrt{1-e^{-\gamma t}}\alpha+e^{-(\gamma/2)t}\beta)\ . \tag{D3}$$

By means of the transformation $\alpha_0\equiv e^{-(\gamma/2)t}\alpha-\sqrt{1-e^{-\gamma t}}\beta$ we obtain

$$P=\int d^2\alpha_0 G(\alpha,\alpha_0,t)P_a(\alpha_0)\ ,$$

where

$$G(\alpha,\alpha_0,t)=\frac{1}{1-e^{-\gamma t}}P_0(\tilde{\alpha})\ ,\quad \tilde{\alpha}=\frac{\alpha-\alpha_0 e^{-(\gamma/2)t}}{\sqrt{1-e^{-\gamma t}}}\ . \tag{D4}$$

First we prove that G solves the Fokker-Planck equation (D1). Let the differential operator $\mathcal{L}$ act on G:

$$\mathcal{L}G=\frac{1}{1-e^{-\gamma t}}\mathcal{L}P_0(\tilde{\alpha})=\frac{\gamma/2}{1-e^{-\gamma t}}\left\{1+\alpha\frac{\partial}{\partial\alpha}+M\frac{\partial^2}{\partial\alpha^2}+D\frac{\partial^2}{\partial\alpha\partial\alpha^*}+\text{c.c.}\right\}P_0$$
$$=\frac{\gamma/2}{1-e^{-\gamma t}}\left\{1+\frac{\alpha}{\sqrt{1-e^{-\gamma t}}}\frac{\partial}{\partial\tilde{\alpha}}+\frac{1}{1-e^{-\gamma t}}\left[M\frac{\partial^2}{\partial\tilde{\alpha}^2}+D\frac{\partial^2}{\partial\tilde{\alpha}^*\partial\tilde{\alpha}}\right]+\text{c.c.}\right\}P_0\ .$$

Using the stationarity of P_0

$$\mathcal{L}G=\frac{\gamma/2}{1-e^{-\gamma t}}\left\{1+\frac{\alpha}{\sqrt{1-e^{-\gamma t}}}\frac{\partial}{\partial\tilde{\alpha}}+\frac{1}{1-e^{-\gamma t}}\left[-1-\tilde{\alpha}\frac{\partial}{\partial\tilde{\alpha}}\right]+\text{c.c.}\right\}P_0$$

and inserting the definition (D4) of $\tilde{\alpha}$ we find

$$\mathcal{L}G = \frac{\gamma/2}{1-e^{-\gamma t}} \left\{ \frac{-e^{-\gamma t}}{1-e^{-\gamma t}} + \frac{1}{\sqrt{1-e^{-\gamma t}}} \left[\alpha - \frac{\alpha - \alpha_0 e^{-(\gamma/2)t}}{1-e^{-\gamma t}} \right] \frac{\partial}{\partial \tilde{\alpha}} + \text{c.c.} \right\} P_0$$

$$= \frac{1}{1-e^{-\gamma t}} \left\{ \frac{-\gamma e^{-\gamma t}}{2(1-e^{-\gamma t})} + \frac{\partial \tilde{\alpha}}{\partial t} \frac{\partial}{\partial \tilde{\alpha}} + \text{c.c.} \right\} P_0 = \frac{\partial G}{\partial t} .$$

Since G solves the Fokker-Planck equation (D1) P is a solution, too. It remains to check that it is the general solution. For $t=0$ we get

$$P(\alpha, t=0) = \int d^2\beta \, P_a(\alpha) P_0(\beta) = P_a(\alpha) ,$$

since P_0 is normalized. P_a means an arbitrary initial quasiprobability and so P is in fact the general solution of the Fokker-Planck equation (D1). G gives the Green's function.

---

[1] J. Brendel, S. Schütrumpf, R. Lange, W. Martienssen, and M. O. Scully, Europhys. Lett. **5**, 223 (1988); R. Lange, J. Brendel, E. Mohler, and W. Martienssen, *ibid.* **5**, 619 (1988).

[2] C. K. Hong, Z. Y. Ou, and L. Mandel, Phys. Rev. Lett. **59**, 2044 (1987); Z. Y. Ou and L. Mandel, *ibid.* **61**, 54 (1988); J. G. Rarity and P. R. Tapster, J. Opt. Soc. Am. B **6**, 1221 (1989).

[3] Y. Lai and H. A. Haus, Quantum Opt. **1**, 99 (1989).

[4] J. W. Noh, A. Fougères, and L. Mandel, Phys. Rev. Lett. **67**, 1426 (1991); Phys. Rev. A **45**, 424 (1992); **46**, 2840 (1992).

[5] Phys. Scr. **T48**, special issue on quantum phase and phase dependent measurements (to be published).

[6] M. Freyberger and W. Schleich, Phys. Rev. A **47**, R30 (1993).

[7] U. Leonhardt and H. Paul, Phys. Rev. A **47**, R2460 (1993).

[8] R. F. O'Connell and E. P. Wigner, Phys. Lett. **85A**, 121 (1981).

[9] D. Lalović, D. M. Davidović, and N. Bijedić, Phys. Rev. A **46**, 1206 (1992).

[10] R. Loudon and P. L. Knight, J. Mod. Opt. **34**, 709 (1987).

[11] W. Schleich, M. Pernigo, and F. Le Kien, Phys. Rev. A **44**, 2172 (1991), and references cited therein.

[12] H. Fearn, Quantum Opt. **2**, 103 (1990).

[13] V. Peřinova, A. Lukš, and P. Szlachetka, J. Mod. Opt. **36**, 1435 (1989).

[14] A. K. Ekert and P. L. Knight, Phys. Rev. A **42**, 487 (1990).

[15] M. S. Kim and V. Bužek, J. Mod. Opt. **39**, 1609 (1992); Phys. Rev. A **46**, 4239 (1992).

[16] T. A. B. Kennedy and D. F. Walls, Phys. Rev. A **37**, 152 (1988).

[17] M. S. Kim and V. Bužek, Phys. Rev. A **47**, 610 (1993).

[18] W. H. Brunner, H. Paul, and G. Richter, Ann. Phys. (Leipzig) **15**, 17 (1965); H. Paul, W. H. Brunner, and G. Richter, *ibid.* **17**, 262 (1966); for a review see H. Paul, Rev. Mod. Phys. **54**, 1061 (1982).

[19] Y. Aharonov, D. Falkoff, E. Lerner, and H. Pendleton, Ann. Phys. (N.Y.) **39**, 498 (1966).

[20] S. Prasad, M. O. Scully, and W. Martienssen, Opt. Commun. **62**, 139 (1987).

[21] Z. Y. Ou, C. K. Hong, and L. Mandel, Opt. Commun. **63**, 118 (1987).

[22] H. Fearn and R. Loudon, Opt. Commun. **64**, 485 (1987).

[23] B. Huttner and Y. Ben-Aryeh, Phys. Rev. A **38**, 204 (1988).

[24] R. A. Campos, B. E. A. Saleh, and M. C. Teich, Phys. Rev. A **40**, 1371 (1989).

[25] B. Yurke, S. L. McCall, and J. R. Klauder, Phys. Rev. A **33**, 4033 (1986).

[26] J. Janszky, C. Sibilia, M. Bertolotti, and Y. Yushin, J. Mod. Opt. **35**, 1757 (1988); J. Janszky, C. Sibilia, and M. Bertolotti, *ibid.* **38**, 2467 (1991); J. Janszky, P. Adam, M. Bertolotti, and C. Sibilia, Quantum Opt. **4**, 163 (1992).

[27] V. Peřinova, A. Lukš, J. Křepelka, C. Sibilia, and M. Bertolotti, J. Mod. Opt. **38**, 2429 (1991).

[28] W. K. Lai, V. Bužek, and P. L. Knight, Phys. Rev. A **43**, 6323 (1991).

[29] L. Allen and S. Stenholm, Opt. Commun. **93**, 253 (1992).

[30] P. Jordan, Z. Phys. **94**, 531 (1935).

[31] J. Schwinger, U.S. Atomic Energy Commission Report No. NYO-3071 (U.S. GPO, Washington, DC, 1952), reprinted in *Quantum Theory of Angular Momentum,* edited by L. C. Biedenharn and H. van Dam (Academic, New York, 1965).

[32] M. D. Levenson and R. M. Shelby, J. Mod. Opt. **34**, 775 (1987).

[33] H. P. Yuen and J. H. Shapiro, *Quantum Statistics of Homodyne and Heterodyne Detection in Coherence and Quantum Optics IV,* edited by L. Mandel and E. Wolf (Plenum, New York, 1978).

[34] H. P. Yuen and V. W. S. Chan, Opt. Lett. **8**, 177 (1983).

[35] S. L. Braunstein, Phys. Rev. A **42**, 474 (1990).

[36] Y. Aharonov, D. Z. Albert, and C. K. Au, Phys. Rev. Lett. **47**, 1029 (1981).

[37] R. F. O'Connell and A. K. Rajagopal, Phys. Rev. Lett. **48**, 525 (1982).

[38] M. Freyberger, K. Vogel, and W. Schleich, Quantum Opt. **5**, 65 (1993); Phys. Lett. A **176**, 41 (1993).

[39] H.-H. Ritze and A. Bandilla, Opt. Commun. **29**, 126 (1979); M. Kitagawa and Y. Yamamoto, Phys. Rev. A **34**, 3974 (1986).

[40] C. K. Hong and L. Mandel, Phys. Rev. Lett. **56**, 58 (1986).

[41] An overview is given in U. Leonhardt and H. Paul, in Ref. [5] (to be published).

[42] W. Schleich, A. Bandilla, and H. Paul, Phys. Rev. A **45**, 6652 (1992); H. Paul, Fortschr. Phys. **22**, 657 (1974). An experiment was reported in H. Gerhardt, U. Buchler, and G. Litfin, Phys. Lett. **49A**, 119 (1974).

[43] G. S. Agarwal and K. Tara, Phys. Rev. A **47**, 3160 (1993).

[44] J. H. Shapiro and S. S. Wagner, IEEE J. Quantum Electron. **QE-20**, 803 (1984).

[45] E. Arthurs and J. L. Kelly, Jr., Bell. Syst. Tech. J. **44**, 725 (1965); S. L. Braunstein, C. M. Caves, and G. J. Milburn, Phys. Rev. A **43**, 1153 (1991); or see the excellent review of S. Stenholm, Ann. Phys. (N.Y.) **218**, 233 (1992).
[46] K. Wódkiewicz, Phys. Rev. Lett. **52**, 1064 (1984); Phys. Lett. A **115**, 304 (1986); for the propensity concept as an approach to quantum phase see D. Burak and K. Wódkiewicz, Phys. Rev. A **46**, 2744 (1992).
[47] K. Wódkiewicz, Phys. Lett. A **124**, 207 (1987).
[48] E. Arthurs and M. S. Goodman, Phys. Rev. Lett. **60**, 2447 (1988).
[49] J. Peřina, *Coherence of Light* (Van Nostrand Reinhold, London, 1971), Chap. 16. Note that in this paper $P(\alpha;s)=\Phi(\alpha,s)/\pi$.
[50] R. Glauber, Phys. Rev. Lett. **10**, 84 (1963).
[51] E. C. G. Sudarshan, Phys. Rev. Lett. **10**, 277 (1963).
[52] E. P. Wigner, Phys. Rev. **40**, 749 (1932).
[53] K. Husimi, Proc. Phys. Math. Soc. Jpn. **22**, 264 (1940).
[54] M. Brune, S. Haroche, J. M. Raimond, L. Davidovich, and N. Zagury, Phys. Rev. A **45**, 5193 (1992).
[55] V. Bužek, A. Vidiella-Barranco, and P. L. Knight, Phys. Rev. A **45**, 6570 (1992).
[56] A. Mecozzi and P. Tombesi, Phys. Lett. A **121**, 101 (1987); Phys. Rev. Lett. **58**, 1055 (1987); J. Opt. Soc. Am. B **4**, 1700 (1987).
[57] For a first-principles approach to resonators in quantum optics see L. Knöll, W. Vogel, and D.-G. Welsch, Phys. Rev. A **43**, 543 (1991); or L. Knöll and D.-G. Welsch, Prog. Quantum Electron. **16**, 135 (1992).
[58] C. W. Gardiner and M. J. Collett, Phys. Rev. A **31**, 3761 (1985).
[59] N. Lu, S.-Y. Zhu, and G. S. Agarwal, Phys. Rev. A **40**, 258 (1989).
[60] C. W. Gardiner, *Quantum Noise* (Springer, Berlin, 1991), Chap. 4.4.5.