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Micromaser with Kerr nonlinearity

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A micromaser containing a Kerr medium in its cavity has been investigated. We have presented an exact solution to the model Hamiltonian, which, apart from the usual Jaynes-Cummings terms, contains a Kerr term ($a^{\dagger 2}a^2$). We have shown that the Kerr cavity nonlinearity significantly modifies the photon statistics of the micromaser field. Particularly, it is interesting to note that with the introduction of appropriate nonlinearity a micromaser can be made to approach towards a number state with a low number of photons.

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I. INTRODUCTION

Experimental studies [1–3] with micromasers that use highly excited Rydberg atoms in high-Q superconducting microcavities have demonstrated various nonclassical features of electromagnetic fields, such as collapse and revivals, modification of spontaneous emission, sub-Poissonian statistics, etc. In fact, resonator quantum electrodynamics has become a good testing ground for fundamental models of quantum optics. Quantum theory of a micromaser device, proposed by two groups [4,5] independently, predicted several novel features, such as sub-Poissonian photon statistics for the cavity field above threshold and the abrupt transition of field states, which are usually averaged out in ordinary lasers and masers due to large macroscopic fluctuations of the number of interacting atoms with the field. Since then a number of theoretical investigations [6–14] have been carried out in connection with atomic-beam noise suppression [6], number-state generation [7,8], relation between microscopic and macroscopic maser theory [9], two-photon micromaser theory [10], trapping states in micromaser [11], phase transition in micromaser field state [12], micromaser spectrum [13], cavity field noise reduction by regulating pumping statistics [14], etc.

In this paper, we have studied theoretically a microscopic maser where the high-Q resonator cavity through which a monoenergetic beam of excited two-level atoms is passed contains a Kerr nonlinear medium [15–17]. The model pertaining to such a situation is a simple modified Jaynes-Cummings model [Eq. (2.1)] which additionally contains a Kerr term. We show that the model is exactly solvable and can be adopted within the framework of micromaser theory. Furthermore, due to the introduction of an intracavity nonlinear element, photon statistics gets significantly modified in a number of ways. First, sub-Poissonian character of the field is more enhanced due to the presence of an intracavity Kerr medium. Second, the photon distribution gets sharper with the increase of nonlinearity and the peak shifts towards the lower photon numbers. This implies that by introducing a Kerr nonlinear element the micromaser can be made to approach towards a number state with a relatively low number of photons. This is opposite to the situation considered by Krause, Scully, and Walther [8], who showed that by increasing the number of injected atoms in the cavity one can approximately realize a number state with a relatively high number of photons.

Before closing this section, two pertinent points are to be noted. First, the model considered here is a variant of a coupled-oscillator model considered by Agarwal and Puri [18] in a different context, where instead of the two-level atoms a harmonic-oscillator field mode was considered. While in Ref. [18] the coupled oscillator model was solved approximately, we have shown that the present model is exactly solvable. Second, recently, it has been noted [19] that amplitude noise reduction in lasers can be achieved by introducing intracavity nonlinear elements, such as a two-photon absorber. Also intracavity second-harmonic generation is known to produce squeezed light with reduced amplitude fluctuations [20]. These suggest that an intracavity Kerr nonlinear element may also be effective in sharpening a photon-number distribution in micromaser and subsequently facilitating the realization of photon number state.

The rest of the paper is organized as follows. In Sec. II we introduce the modified Jaynes-Cummings model and its solutions. In Sec. III the standard quantum theory of the micromaser has been adopted to this model. The modifications of the photon statistics due to intracavity Kerr nonlinearity are discussed in Sec. IV. The paper is concluded in Sec. V.

II. THE MODEL AND THE DYNAMICS OF THE SYSTEM

The total Hamiltonian of the system is given by

$$H=\hbar\omega_0 a^\dagger a+(\tfrac{1}{2})\hbar\omega\sigma_z+\hbar q a^{\dagger 2}a^2+\hbar g(a^\dagger\sigma_-+a\sigma_+)\ , \tag{2.1}$$

where a ($a^\dagger$) represents the field annihilation (creation) operator, ω_0 and ω are the field and atomic transition frequencies, respectively, g is the atom-field coupling constant, and q indicates the nonlinear parameter [18] of the Kerr medium.

We can separate the total Hamiltonian into two parts H_{I} and H_{II} as

$$H = H_{\mathrm{I}} + H_{\mathrm{II}} ,$$

$$H_{\mathrm{I}} = \hbar\omega_0 a^{\dagger}a + (\tfrac{1}{2})\hbar\omega_0\sigma_z , \tag{2.2}$$

$$H_{\mathrm{II}} = (\tfrac{1}{2})\hbar\Delta\sigma_z + \hbar q a^{\dagger 2}a^2 + \hbar g(a^{\dagger}\sigma_- + a\sigma_+) , \tag{2.3}$$

where $\Delta = \omega - \omega_0$ is the detuning of the system. It can be easily verified that

$$[H_{\mathrm{I}}, H_{\mathrm{II}}] = 0 . \tag{2.4}$$

Since H_{I} is diagonal in the number-state representation and commutes with H_{II}, it is always possible to find a representation where both H_{I} and H_{II} are diagonal. This implies that the eigenstates of H_{II} must be a linear combination of eigenstates of H_{I}.

Let $|a\rangle$ ($|b\rangle$) represent the upper (lower) level atomic state and $|an\rangle$ and $|b\,n+1\rangle$ the basis kets for the atom-field system with the atom in the upper and lower levels, respectively. For obtaining a diagonal energy representation for H_{II} we consider the following eigenkets and eigenvalue equations:

$$|\psi_{n1}\rangle = (\cos\theta_n)|b\,n+1\rangle + (\sin\theta_n)|an\rangle , \tag{2.5a}$$

$$|\psi_{n2}\rangle = -(\sin\theta_n)|b\,n+1\rangle + (\cos\theta_n)|an\rangle , \tag{2.5b}$$

$$H_{\mathrm{II}}|\psi_{n1}\rangle = \lambda_n|\psi_{n1}\rangle , \tag{2.6a}$$

$$H_{\mathrm{II}}|\psi_{n2}\rangle = \lambda_n'|\psi_{n2}\rangle . \tag{2.6b}$$

Here θ_n is a parameter at our disposal to make $|\psi_{n1}\rangle$ and $|\psi_{n2}\rangle$ the eigenkets of H_{II}. λ_n, λ_n' are the eigenvalues of H_{II}. Following the standard procedure [21] we get

$$\lambda_n = \hbar q n^2 + \hbar[g^2(n+1)^2 + \{(\tfrac{1}{2})\Delta - qn\}^2]^{1/2} , \tag{2.7a}$$

$$\lambda_n' = \hbar q n^2 - \hbar[g^2(n+1)^2 + \{(\tfrac{1}{2})\Delta - qn\}^2]^{1/2} , \tag{2.7b}$$

$$\tan\theta_n = \frac{g(n+1)^{1/2}}{qn^2 + [g^2(n+1) + \{(\tfrac{1}{2})\Delta - qn\}^2 - (\tfrac{1}{2})\Delta - q(n^2-n)]^{1/2}} . \tag{2.8}$$

The eigenstates of Eq. (2.5) define a transformation matrix

$$T = \begin{bmatrix} \cos\theta_n & -\sin\theta_n \\ \sin\theta_n & \cos\theta_n \end{bmatrix} \tag{2.9}$$

that transforms the bare-atom probability to "dressed"-atom probability.

The Schrödinger state vector $|\psi(\tau)\rangle$, where τ is the interaction time, can be expanded in terms of bare states as

$$|\psi(\tau)\rangle = \sum_{n=0}^{\infty} [C_{an}(\tau)|an\rangle + C_{b\,n+1}(\tau)|b\,n+1\rangle] . \tag{2.10}$$

In an interaction picture the amplitude coefficients are given by

$$\begin{bmatrix} C_{an}(\tau) \\ C_{b\,n+1}(\tau) \end{bmatrix} = T^{-1} \begin{bmatrix} \exp(-i\lambda_n\tau/\hbar) & 0 \\ 0 & \exp(i\lambda_n'\tau/\hbar) \end{bmatrix} T . \tag{2.11}$$

Assuming the atoms initially in the upper level, i.e., $C_{an}(0) = 1$ and $C_{b\,n+1}(0) = 0$, we have

$$|C_{an}(\tau)|^2 = 1 - \sin^2 2\theta_n \sin^2\{(\lambda_n - \lambda_n')\tau/2\hbar\} , \tag{2.12}$$

$$|C_{b\,n+1}(\tau)|^2 = \sin^2 2\theta_n \sin^2\{(\lambda_n - \lambda_n')\tau/2\hbar\} . \tag{2.13}$$

For convenience, we introduce A_n such that

$$|C_{bn}(\tau)|^2 = A_n(\tau) , \quad |C_{an}(\tau)|^2 = 1 - A_{n+1}(\tau) . \tag{2.14}$$

By adopting quantum theory of micromaser into this model system, in the two subsequent sections III and IV, we investigate how the cavity-field Kerr nonlinearity comes into play to modify the well-known features of micromaser photon statistics.

III. MICROMASER WITH KERR CAVITY NONLINEARITY

We consider that the atoms in the upper level without initial coherence are injected into the cavity containing a Kerr medium. We assume, as in the standard micromaser theory, that the rate of injection of atoms is low enough so that almost one atom at a time is inside the cavity and the interaction time τ is much smaller than the cavity relaxation time γ^{-1}. We also assume that the field is initially diagonal in the number-state representation. Therefore the initial atom-field density operator is given by

$$\rho_{a\text{-}f}(t_i) = |a\rangle\langle a| \otimes \sum_n P_n(t_i)|n\rangle\langle n| , \tag{3.1}$$

where t_i is the time at which the ith atom enters the cavity. The atom interacts with the field for a time τ. During interaction the Hamiltonian [Eq. (2.3)] couples $|an\rangle$ and $|b\,n+1\rangle$ Tracing over the atomic states we obtain the field-reduced density operator at the time $t_i + \tau$,

$$\rho_f(t_i+\tau) = \sum_{n=0}^{\infty} P_n(t_i)[|C_{an}(\tau)|^2|n\rangle\langle n| + |C_{b\,n+1}(\tau)|^2|n+1\rangle\langle n+1|] . \tag{3.2}$$

Identifying the diagonal elements of the above equation we obtain

$$P_n(t_i+\tau)=[1-A_{n+1}(\tau)]P_n(t_i)+A_n(\tau)P_{n-1}(t_i)\ , \tag{3.3}$$

where

$$A_n(\tau)=\frac{n}{\left[\dfrac{q(n-1)-(\frac{1}{2})\Delta}{g}\right]^2+n}\times\sin^2\left\{\left[\left[\frac{q(n-1)-(\frac{1}{2})\Delta}{g}\right]^2+n\right]^{1/2}g\tau\right\}. \tag{3.4}$$

To introduce the decay of the cavity field we further assume that the field is coupled to a continuum of thermal modes. Then in the interval between $t_i+\tau$ and t_{i+1} (i.e., the time at which the next atom is injected) the decay of the field is governed by the standard master equation of the form [22]

$$\dot{\rho}=L\rho=-(\tfrac{1}{2})\gamma(\bar{n}+1)[a^\dagger a\rho(t)-a\rho(t)a^\dagger] -(\tfrac{1}{2})\gamma\bar{n}[\rho(t)aa^\dagger-a^\dagger\rho(t)a]+\text{adj}\ , \tag{3.5}$$

where γ is the rate of decay of the field and $\bar{n}$ is the average number of thermal photons. Here the damping operator L is the usual Liouvillian operator. At the time t_{i+1} when the $(i+1)$th atom enters the cavity, the field density matrix is given by

$$\rho_f(t_{i+1})=\exp(Lt_P)\rho_f(t_i+\tau)\ , \tag{3.6}$$

where $t_P=t_{i+1}-t_i-\tau\cong t_{i+1}-t_i$.

We assume that the atoms enter into the cavity according to a Poisson process with mean spacing $1/R$ between the successive entering events. Then after appropriate averaging the "steady-state" [4] solution of Eq. (3.6) with $\rho_f(t_{i+1})=\rho_f(t_i)$ yields the photon distribution of the form

$$P_n=P_0\{\bar{n}/(1+\bar{n})\}^n\prod_{k=1}^{n}[1+(N/\bar{n})(A_k/k)]\ , \tag{3.7}$$

where $N\ (=R/\gamma)$ is the average number of atoms that is injected into the cavity during the lifetime of the field. Here P_0 is the normalization constant. The derivation of Eq. (3.7) follows the similar procedure as in Ref. [4]. Since the expression (3.7) and the values of averages and variances do not form summable analytical series one resorts to numerical analysis. Instead of dealing with the absolute magnitudes of nonlinear parameter q and the interaction time τ, it is helpful to work with dimensionless parameters, such as q/g, $g\tau$, etc. This is particularly important to avoid the problem of numerical overflow. In the next section we study numerically the features of steady-state photon statistics under exact resonance conditions, i.e., $\Delta=0$.

IV. THE PHOTON STATISTICS

In Fig. 1 we show the plots of photon distribution for different values of dimensionless nonlinear parameter (q/g) with dimensionless interaction time $g\tau=0.4$. A single peak structure appears for $q/g=0$ [Fig. 1(a)]. As q/g increases from 0.1 to 1 the distribution becomes sharper and sharper and shifts towards the lower number of photons. This implies that the field state approaches towards a pure number state with a low number of photons. It is interesting to recall at this point that Krause, Scully, and Walther [8] observed that by increasing average number of injected atoms into the cavity the micromaser-field state can be made to approach towards a number state with a relatively high number of photons. Shown in Figs. 2 and 3 is a photon-number distribution for relatively higher values of interaction time $g\tau$ ($=2.0$, 3.0, respectively). The symmetrical nature of the multipeak photon distribution for $q/g=0$ turns asymmetrical due to nonlinearity ($q/g=0.5$) indicating that the character of multistability of the photon distribution for longer interaction times is modified.

The normalized average number of photons is given by

$$\langle n\rangle/N=\sum_n nP_n/N \tag{4.1}$$

and is plotted in Figs. 4–6 as a function of pump parameter, $\Theta\ (=0.5\sqrt{N}g\tau)$ for different values of nonlinear parameter q, with $N=150$ and $\bar{n}=0.1$. The field starts growing from almost zero intensity in all cases and as usual after crossing the threshold one can observe the oc-

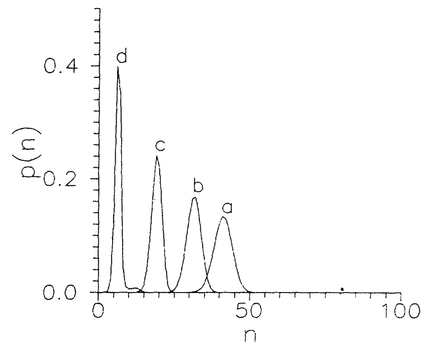

FIG. 1. Normalized photon distribution function $[p(n)]$ is plotted against the integer photon number (n) for different values of the dimensionless nonlinear parameter (a) $q/g=0$, (b) $q/g=0.1$, (c) $q/g=0.5$, and (d) $q/g=1$, with fixed $\bar{n}=0.1$, $N=150$, and $g\tau=0.4$.

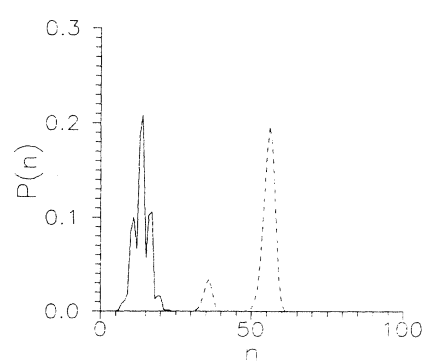

FIG. 2. Normalized photon distribution function $[P(n)]$ vs the integer photon number (n) for $q/g=0$ (dashed lines) and $q/g=0.5$ (solid line) with $\bar{n}=0.1$, $N=150$, and $g\tau=2.0$.

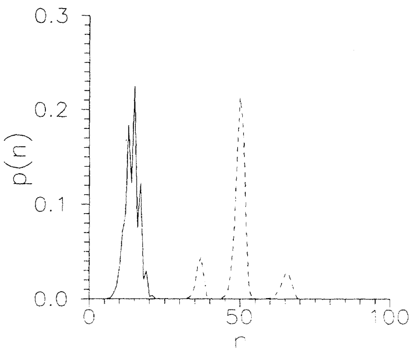

FIG. 3. Same as in Fig. 2 but for $q/g=0$ (dashed lines) and $q/g=0.5$ (solid line) with $g\tau=3.0$.

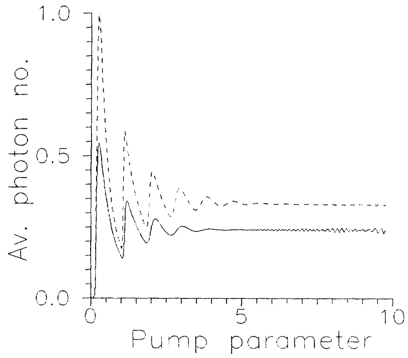

FIG. 4. Average photon number ($\langle n \rangle/N$) is plotted as a function of pump parameter (Θ in units of π^{-1}) for $q/g=0$ (dashed line) and $q/g=0.1$ (solid line) with $\bar{n}=0.1$ and $N=150$.

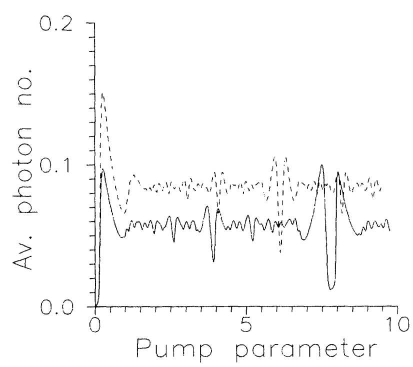

FIG. 5. Same as in Fig. 4 but for $q/g=0.5$ (dashed line) and $q/g=1$ (solid line).

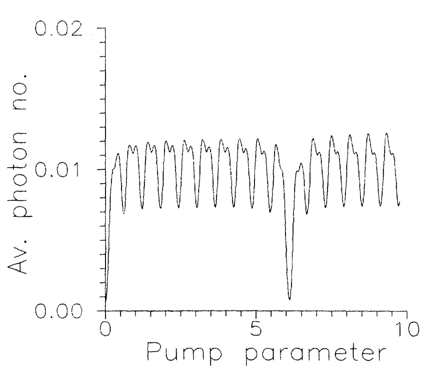

FIG. 6. Same as in Fig. 4 but for $q/g=10$.

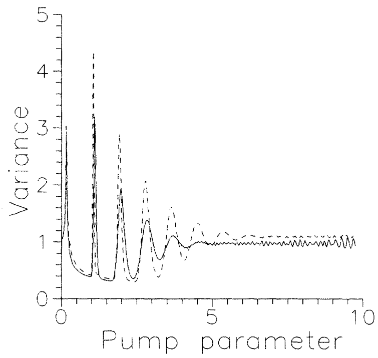

FIG. 7. Normalized variance (σ) in the photon-number distribution is plotted against the pump parameter (Θ in units of π^{-1}) for $q/g=0$ (dashed line) and $q/g=0.1$ (solid line) with $\bar{n}=0.1$ and $N=150$.

currence of an abrupt transition of the field states at regular intervals. As the nonlinearity is increased the stationary regime (over which $\langle n \rangle/N$ is nearly independent of Θ) is reached at a much lower value of θ. Also the average number of photons in the cavity is decreased drastically. Beyond the scale of the graph there are additional features which are reminiscent of Jaynes-Cummings revivals [23].

It is also interesting to see how the trapping condition [24] which results from coherent atom-field interaction is modified by nonlinearity. On resonance the zeros of Eq. (3.4) with $n \to n+1$ implies that there exist number states $|n_j\rangle$ such that

$$g\tau\sqrt{(qn_j/g)^2+(n_j+1)}=\pi j \ , \tag{4.2}$$

where j is an integer during the interaction time τ. In the limit $q \to 0$, the normal trapping condition is recovered.

The normalized variance in the photon-number distribution is given by

$$\sigma=\left[\frac{\langle n^2\rangle-\langle n\rangle^2}{\langle n\rangle}\right]^{1/2} . \tag{4.3}$$

Figures 7–9 show the plots of σ as a function of θ for different values of dimensionless nonlinear parameter q/g. Without field nonlinearity ($q/g=0$) and above threshold ($\theta>1$) the field exhibits sub-Poissonian statis-

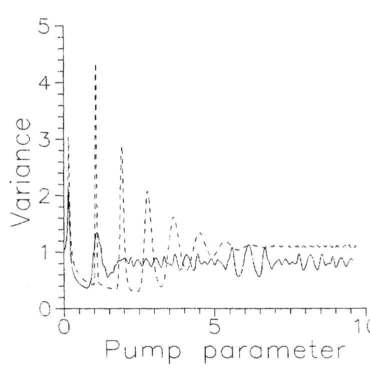

FIG. 8. Same as in Fig. 7 but for $q/g=0$ (dashed line) and $q/g=0.5$ (solid line).

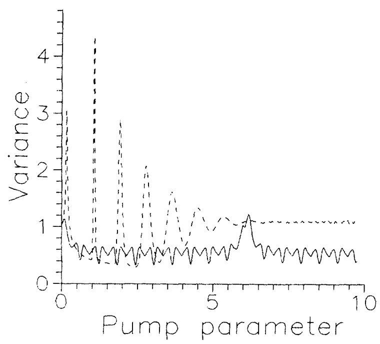

FIG. 9. Same as in Fig. 7 but for $q/g=0$ (dashed line) and $q/g=10$ (solid line).

tics ($\sigma<1$) with periodic sharp transitions to the super-Poissonian region ($\sigma>1$) for lower values of θ. For $q/g=0.1$ (Fig. 7, solid curve), though $\sigma(\theta)$ retains the nature of the periodic transition from the sub- to the super-Poissonian region, the period of transition is less than what it is for $q/g=0$ and finally $\sigma(\theta)$ remains confined approximately along the line of Poissonian distribution ($\sigma=1$). When $q/g=0.5$ (Fig. 8, solid curve) the period during which the field remains sub-Poissonian is prolonged, though it is less sub-Poissonian than what it is for $q/g=0$. At $q/g=10$ (Fig. 9, solid curve), for a long range of pump parameter the field remains sub-Poissonian. For $q/g=10$ the sub-Poissonian character of the field is strongly enhanced and furthermore some periodic dips appear in the $\sigma(\theta)$ curve indicating that the field has a tendency to go into a still deeper sub-Poissonian region.

V. CONCLUSIONS

We have presented a theory of micromaser in which a monoenergetic beam of excited two-level atoms is injected into a high-Q resonator filled with a Kerr medium. We have shown that due to the introduction of nonlinearity photon statistics of the cavity field gets profoundly modified in a number of ways. Particularly interesting is the narrowing of the photon distribution profile and its shift towards the lower photon number, since the realization of the number state with the low photon number is an important issue from the point of view of generation of nonclassical states. Another important aspect is the generation of squeezed states, the possibility of which can be explored by injecting the atoms in a coherent superposition of states. We hope to address this and other related issues in a future paper.

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