Quantum-noise limits to matter-wave interferometry

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We derive the quantum limits for an atomic interferometer from a second-quantized theory in which the atoms obey either Bose-Einstein or Fermi-Dirac statistics. It is found that the limiting quantum noise is due to the uncertainty associated with the particle sorting between the two branches of the interferometer, and that this noise can be reduced in a sufficiently dense atomic beam by using fermions as opposed to bosons. As an example, the quantum-limited sensitivity of a generic matter-wave gyroscope is calculated and compared with that of a laser gyroscope.

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Matter-wave interferometry dates from the inception of quantum mechanics, i.e., the early electron-diffraction experiments [1]. More recent neutron-interferometry experiments have yielded insight into many fundamental aspects of quantum mechanics [2]. Presently, atom interferometry has been demonstrated and holds promise as a new field of optics—matter-wave optics [3]. This field is particularly interesting since the potential sensitivity of matter-wave interferometers [4] far exceeds that of their light-wave or "photon" antecedents [5].

However, as was emphasized at the recent Solvay conference on quantum optics, there is at present no paradigm available for calculating or estimating the quantum-noise limits to matter-wave interferometers, and therefore we have no basis for estimating the potential sensitivity of devices based on matter-wave interferometry (e.g., gyroscopes) [6].

In order to motivate the analysis and derive the quantum limits, we proceed as follows: First, we "set the stage" by considering a simple gyroscope and deriving the rotation-induced signal in matter-wave optics. Next, we proceed to develop the theory for atomic interferometers, cast in an operator formalism that is well suited to a quantum-noise analysis, and then we obtain the quantum-noise limits for matter-wave interferometry. Finally, we compare current laser gyroscope sensitivity to that of a near-term, matter-wave device.

We begin by considering an idealized atom interferometer used as a rotation detector or gyroscope, as shown in Fig. 1. From this diagram it is easy to see that the atomic path difference between the upper branch α and the lower branch β is given by $\delta l = 2r\Omega t$, where Ω is the angular velocity of the interferometer, r is the radius of the circle, v the particle velocity, and $t = \pi r/v$ is the particle transit time through the interferometer. This readily translates into a Sagnac phase difference of $\delta\varphi_{\alpha\beta} = k(l_\alpha - l_\beta) = 2\pi r^2\Omega/\lambda\!\!\!^{-} v = 2A\Omega/\lambda\!\!\!^{-} v$, where $\lambda\!\!\!^{-} \equiv \hbar/mv$ is the atomic de Broglie wavelength [7] and A the area enclosed by the arms. The phase signal is then given by $\varphi^{\text{signal}} = 2Am\Omega/\hbar$; independent of the interferometer shape as long as A is the total area enclosed by the arms. This expression holds for both atom and light interferometers, if, in the photon case, we define an effective photon mass m_γ implicitly by $m_\gamma c^2 = \hbar\omega$. Now, since the "mass" of a photon is governed by optical energies of a few electron volts—and atomic masses are of order 10^3 MeV—we see that matter-wave gyroscopes potentially have a signal that is enhanced by many orders of magnitude, compared to light (laser) gyroscopes. Thus motivated, we next consider a detailed analysis of phase sensitivity in matter-wave interferometry.

In accordance with current experiments [3], let us consider the model illustrated in Fig. 2. There, we see a stream of N atoms passing one-at-a-time through a beam

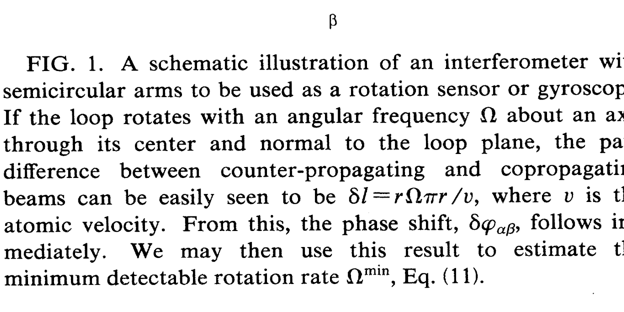

FIG. 1. A schematic illustration of an interferometer with semicircular arms to be used as a rotation sensor or gyroscope. If the loop rotates with an angular frequency Ω about an axis through its center and normal to the loop plane, the path difference between counter-propagating and copropagating beams can be easily seen to be $\delta l = r\Omega\pi r/v$, where v is the atomic velocity. From this, the phase shift, $\delta\varphi_{\alpha\beta}$, follows immediately. We may then use this result to estimate the minimum detectable rotation rate $\Omega^{\min}$, Eq. (11).

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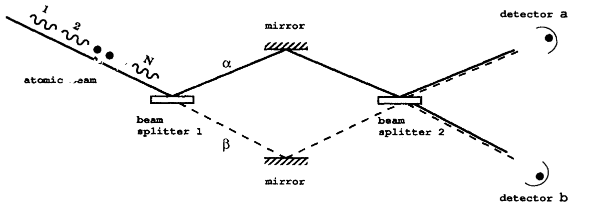

FIG. 2. We illustrate a scheme whereby a stream of N atoms are sent through a simple interferometer during a measurement time t_m. The atoms are split at beam splitter 1, follow paths α or β, are reflected off the mirrors, and are then recombined at beam splitter 2. The recombined atoms are detected at upper detector a or lower detector b where interference fringes are recorded.

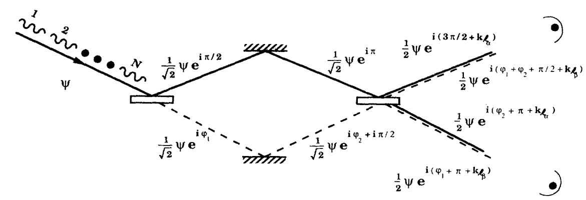

FIG. 3. Chasing phases through the interferometer accounts for accumulated phase shifts in the upper or lower branches. The phase shift upon reflection is arbitrary, but we choose it here to be $\pi/2$ for simplicity. Upon transmission, a phase shift of φ_1, or φ_2, is assumed for beam splitter 1 or 2, respectively

splitter into a simple interferometer with upper and lower branches labeled α and β, respectively. Upon recombining the two beams, we inspect the resultant interference pattern for phase shifts induced, say, by a gravitational potential between the two branches or a net rotation of the system. As in the optical dual [5], one might expect that the overall sensitivity of the device will be limited by the quantum limits imposed by particle-number fluctuations, ΔN, or the phase noise, $\Delta\varphi$, in the interferometer. It is often stated that ΔN is to be associated with the Poissonian fluctuations in the arrival time of atoms in the input beam, i.e., $\Delta N \sim \sqrt{\bar{n}}$ where $\bar{n}$ is the mean number of particles [8]. However, in the ideal experiment envisioned here, the total number of particles N is assumed to be known exactly obviating the need for Poisson statistics. We shall show that the *quantum* limit to particle-number noise arises not from fluctuations in the input-beam intensity but rather from beam-splitter uncertainties pertaining to the lack of knowledge of which path, α or β, the atom has taken through the interferometer.

To see this, let us continue developing our simple model depicted in Fig. 2. We assume that, upon reflection from a beam-splitter surface, the particles undergo an unimportant phase shift which we take to be $\pi/2$. The actual shift depends on specific properties of the beam splitter, but this does not affect the overall result [9]. Upon passage *through* a beam splitter, however, the atom undergoes a phase shift of φ_i, $i=1,2$, for the first and second beam splitter, respectively. The cumulative effect in the interferometer of these various processes on the atomic wave function ψ is depicted in Fig. 3 and leads to a wave function ψ_a corresponding to the upper detector a and ψ_b for the lower detector b, namely,

$$\begin{aligned}\psi_a &= \frac{\psi}{2} e^{i\theta_a}[1-e^{-ik(l_\alpha - l_\beta)}] \, , \\ \psi_b &= \frac{\psi}{2} e^{i\theta_b}[1+e^{-ik(l_\alpha - l_\beta)}] \, ,\end{aligned} \tag{1}$$

where $\theta_a \equiv \pi/2 + kl_\alpha + \varphi_2$ and $\theta_b \equiv kl_\alpha + \varphi_1 + \varphi_2$, and where, without loss of generality, we take $\varphi_1 = \varphi_2 = \pi$. Here, k is the atomic wave number and l_α and l_β are the path lengths through the upper and lower branches, respectively. We imagine now that the beam is recombined by the second beam splitter and then the detectors a and b shown in Fig. 2 count the number of atoms as they arrive in the recombined upper beam or lower beam, respectively. If we label N atoms with the index $i=1,\ldots,N$, as those sent through the interferometer during a measurement time t_m, then the appropriate state vector $|\varphi\rangle_i$ for the ith atom in the interferometer, after recombination, is given by

$$\begin{aligned}|\varphi\rangle_i &= \frac{e^{i\theta_a}}{2}(1-e^{-i\varphi_{\alpha\beta}})|1_a,0_b\rangle_i \\ &\quad + \frac{e^{i\theta_b}}{2}(1+e^{-i\varphi_{\alpha\beta}})|0_a,1_b\rangle_i \, ,\end{aligned} \tag{2}$$

where here $\varphi_{\alpha\beta} \equiv k(l_\alpha - l_\beta)$. We see that this state is an appropriate superposition of the number states $|1_a,0_b\rangle$ and $|0_a,1_b\rangle$ corresponding to an atom incident on the upper or lower detectors, a or b, respectively. The state vector $|\Phi\rangle_N$ for the N-atom state is then constructed via a direct product of the individual atomic states, namely

$$|\Phi\rangle_N \equiv \prod_{i=1}^{N} |\varphi\rangle_i \, . \tag{3}$$

Let $\hat{c}^\dagger_{\sigma,i}$ and $\hat{c}_{\sigma,i}$ where $\sigma = a,b$, be the creation and annihilation operators, respectively, for the number states $|n_a,n_b\rangle_i$, where, corresponding to number operators $\hat{n}_{\sigma,i} \equiv \hat{c}^\dagger_{\sigma,i}\hat{c}_{\sigma,i}$, the eigenvalues n_a and n_b are 0 or 1. Then the number operator $\hat{N}_\sigma$ for the number of a or b atoms is determined by

$$\hat{N}_\sigma = \sum_{i=1}^{N} \hat{n}_{\sigma,i} \quad (\sigma = a,b) \, , \tag{4}$$

and the operators $\hat{c}$ obey the commutation relations

$$[\hat{c}_{\sigma,i}\hat{c}^\dagger_{\sigma,j} \pm \hat{c}^\dagger_{\sigma,j}\hat{c}_{\sigma,i}] = \delta_{ij} \, , \tag{5}$$

where the plus or minus sign indicates Bose or Fermi statistics, respectively. The statistical nature of the atoms will be important in circumstances where the density of particles in the interferometer is so large that there is more than one atom at a time within a single coherence length. The expectation values $\langle \hat{N}_\sigma \rangle_N$ of these number operators, Eq. (4), are given by

$${}_N\langle\Phi|\hat{N}_a|\Phi\rangle_N = \sum_{i=1}^{N} \left| \frac{1-e^{-i\varphi_{\alpha\beta}}}{2} \right|^2 {}_i\langle 1_a,0_b,|\hat{n}_{a,i}|1_a,0_b\rangle_i \, , \tag{6a}$$

$$_N\langle\Phi|\hat{N}_b|\Phi\rangle_N=\sum_{i=1}^{N}\left|\frac{1+e^{-i\varphi_{\alpha\beta}}}{2}\right|^2{}_i\langle 0_a,1_b|\hat{n}_{b,i}|0_a,1_b\rangle_i\ . \tag{6b}$$

This yields the expression for the mean number of atoms in the a and b detectors as

$$\langle\hat{N}_a\rangle_N=N\sin^2\varphi_{\alpha\beta}/2\ ,\quad \langle\hat{N}_b\rangle_N=N\cos^2\varphi_{\alpha\beta}/2\ . \tag{7}$$

These expectations constitute the signal—what then is the minimum detectable noise?

To answer this question, we compute the quantum-noise fluctuations using the formalism developed earlier. Recalling the definitions for the number operator $\hat{N}$, Eq. (4), and the state vector $|\Phi\rangle_N$, Eq. (3), and using the commutation relations, Eq. (5), we may write

$$\begin{aligned}\langle\Delta\hat{N}_\sigma\rangle^2&={}_N\langle\Phi|\hat{N}_\sigma^2|\Phi\rangle_N-[{}_N\langle\Phi|\hat{N}_\sigma|\Phi\rangle_N]^2\\&={}_N\langle\Phi|\sum_{i=1}^{N}\hat{n}_{\sigma,i}\sum_{j=1}^{N}\hat{n}_{\sigma,j}|\Phi\rangle_N-\left[{}_N\langle\Phi|\sum_{i=1}^{N}\hat{n}_{\sigma,i}|\Phi\rangle_N\right]^2\\&=\sum_{i=1}^{N}{}_i\langle\varphi|\hat{n}_{\sigma,i}|\varphi\rangle_i\sum_{\substack{j=1\\j\neq i}}^{N}{}_j\langle\varphi|\hat{n}_{\sigma,j}|\varphi\rangle_j+\sum_{i=1}^{N}{}_i\langle\varphi|\hat{c}^\dagger_{\sigma,i}[1\pm\hat{n}_{\sigma,i}]\hat{c}_{\sigma,i}|\varphi\rangle_i-\left[\sum_{i=1}^{N}{}_i\langle\varphi|\hat{n}_{\sigma,i}|\varphi\rangle_i\right]^2\\&=\frac{N}{4}\sin^2\varphi_{\alpha\beta}\pm\sum_{i=1}^{N}{}_i\langle\varphi|\hat{c}^\dagger_{\sigma,i}\hat{c}^\dagger_{\sigma,i}\hat{c}_{\sigma,i}\hat{c}_{\sigma,i}|\varphi\rangle_i\quad(\sigma=a,b)\ ,\end{aligned} \tag{8}$$

where, as before, the upper and lower terms in braces correspond to $\sigma=a$ or b, respectively, and the plus or minus sign refers to the statistics of the particles: a plus sign for bosons and a minus sign for fermions. We note that the last statistics-dependent term of Eq. (8) is the sum of non-negative matrix elements and so itself is non-negative or nonpositive, according to the plus sign or minus sign, respectively. A quantitative analysis of the contribution of this statistics-dependent term requires a specific model of the coherences between atoms in a dense beam. However, one can qualitatively state that for sufficiently high densities, the use of fermionic atoms will tend to *lower* the quantum-noise limit—since the last term will be negative. Bosons will have the opposite effect. A detailed analysis of the statistics-dependent contribution is beyond the scope of this paper, and will be left to a later work. Hence, since in current experiments the beam intensity is so low that there is only one atom at a time within a single coherence length, the statistics-dependent second term in the last line of Eq. (8) is zero and we are left with the result

$$\langle\Delta N_\sigma\rangle=\frac{\sqrt{N}}{2}\sin\varphi_{\alpha\beta}\quad(\sigma=a,b)\ . \tag{9}$$

We notice that this result depends on the *total* number of atoms $N=N_a+N_b$ and *not* the mean number $\langle N\rangle$, nor the number in the branches, N_σ. Now, the signal in either branch N_σ is given by Eq. (7).

The quantum fluctuations in phase $\Delta\varphi_{\alpha\beta}$ in the measured phase difference $\varphi_{\alpha\beta}$ may be determined by [10]

$$\begin{aligned}|\Delta\varphi_{\alpha\beta}|&\equiv\frac{\langle\Delta N_\sigma\rangle}{|\partial\langle N_\sigma\rangle/\partial\varphi_{\alpha\beta}|}\\&=\frac{1}{\sqrt{N}}\ ,\end{aligned} \tag{10}$$

a result that is *independent* of both $\varphi_{\alpha\beta}$ and the detector a or b. This φ independence might appear surprising at first, but it is a direct result of the fact that the *quantum-number-state* noise $\langle\Delta N_\sigma\rangle$ is proportional to the slope of the signal $\langle N_\sigma\rangle$ for the upper and lower number states considered here. (See, in particular, Ref. [10].) Again, we stress that N is not the quantum expectation value $\langle N\rangle$ but rather the total number of atoms detected in the measurement time t_m. This is *not* then the expression one would expect from the application of the uncertainty principle, for in that case N would have to be replaced by $\langle N\rangle$. We reemphasize that it has not been clear what form of the uncertainty principle one should even use in an atom interferometer [6]. For light, the so-called number-phase uncertainty principle, $\Delta\varphi\Delta N\gtrsim 1$, yields for a coherent state $\Delta\varphi\cong 1/\langle N\rangle^{1/2}$—where only the expectation $\langle N\rangle$ and not the total number N is known. For atoms it is not obvious at all what the relationship should be, and we have shown that the result is unexpected in that Eq. (10) depends on the total number N that is precisely known for the atom interferometer, and where $\langle N\rangle$ has no meaning. In contradistinction, in a laser interferometer, it is impossible to know the total number of photons and only the mean can be specified. Hence, the matter-wave result, Eq. (10), is quantitatively, qualitatively, and philosophically different from the optical result.

We conclude by applying this result to the gyroscope problem. Let us note that the atom number N is given by jt_m, where j is the atomic flux (in atoms per second) hitting the detector. We have from Eq. (10) the minimum detectable phase shift, $\Delta\varphi_{\min}=1/\sqrt{jt_m}$, and equating this to the signal derived earlier, $\varphi^{\text{signal}}=2Am\Omega/\hbar$, we find the minimum detectable rotation rate $\Omega^{\min}$ is given by

$$\Omega^{\min}\cong\frac{\hbar}{2Am}(jt_m)^{-1/2}\quad(\text{matter})\ . \tag{11}$$

This should be compared to the same result obtained from using an optical interferometer in which the flux j is given by the power P divided by the photon energy $\hbar\omega$

TABLE I. Compared and contrasted are different properties of matter-wave and optical gyroscopes in terms of their sensitivity to phase differences—or equivalently—rotation rates. We see that the high mass of atoms initially contributes an increase of sensitivity of 10^{10}, but that the low atomic beam intensity, compared to photon beams, removes some of this advantage, as does the reduced number of round trips possible in an atom interferometer. Nevertheless, a typical factor of a 10^4 increase in rotation sensitivity can still be expected using atoms rather than photons.

	Matter	Laser	Matter-to-light sensitivity factor
Mass factor	$\sim 10^4$ MeV	~ 1 eV	$\sim 10^{10}$
Flux	$\rho v A \sim 10^{10} \times 10^4 \times 10^{-2}$ $= 10^{12}\,\frac{\text{particles}}{\text{sec}}$	$\frac{P}{\hbar\nu} \sim \frac{10^{-3}}{10^{-19}}$ $= 10^{16}\,\frac{\text{photons}}{\text{sec}}$	$\sim 10^{-2}$
Round trips	~ 1	$\sim 10^4$	$\sim 10^{-4}$

[5,7], in other words

$$\Omega^{\min} \cong \frac{\hbar}{A m_\gamma} \left[\frac{P}{\hbar\omega} t_m \right]^{-1/2} \quad \text{(light)} , \tag{12}$$

where m_γ is the effective photon mass, defined by $m_\gamma \equiv \hbar\omega/c^2$. In Table I we compare and contrast properties of the matter-wave and laser light interferometers in order to gauge their effectiveness in measuring $\Omega^{\min}$. As mentioned before, we note that the typical photon effective mass gives an increase in sensitivity of 10^{10}. This mass factor, however, is offset by the low particle flux available for atoms—this increases the laser gyroscope sensitivity over that of a matter-wave one by a factor of around 10^2. In addition, the atoms make about one "round trip" through an interferometer, whereas in a ring laser gyroscope the photons make many ($\approx 10^4$) circuits around the ring and yield an additional sensitivity factor of 10^4 in favor of the laser system. This still leaves the matter-wave device 10^4 times more sensitive.

In summary then, we conclude that the phase uncertainty arising in an atomic interferometer arises from atomic number fluctuations associated with the sorting of the particles between the two arms of the interferometer. Our result is different in that the minimum detectable phase $\Delta\varphi \sim 1/\sqrt{N}$ depends upon the total number N of atoms in the interferometer, and not the quantum expectation number $\langle N \rangle$ one would expect from a naive application of the uncertainty principle. In addition, by using a sufficiently dense atomic beam of fermions this limit can be improved upon, while the use of bosons has the opposite effect. Applying our results to an interferometer used as a gyroscope, we find that a matter-wave gyroscope can be expected to be more sensitive to rotation by some *four orders of magnitude* than present laser devices.

We should note that previous semiclassical arguments yield a statistical relation between phase and number noise of the form $\Delta\varphi\Delta n = 1$, "where Δn denotes the standard deviation of the total counting rate n registered at the detector, which obeys Poissonian statistics ($\Delta n = \sqrt{\langle n \rangle_{\text{Poisson}}}$) as a basic feature of the source emission process" [8]. However, the ideal experiment assumed here removes the essentially classical phase noise associated with the Poisson process, and uncovers the underlying quantum limit to phase detection, similar to that discussed in Ref. [10] by Wineland *et al.*, for N trapped atoms. Quantum-number noise, $\langle \Delta N_\sigma \rangle$, as well as the statistics-dependent fermion or boson contributions, are a direct result of the second-quantization procedure of the particle field ψ that introduces particle creation and annihilation operators for the upper and lower detectors of the interferometer. An ordinary first-quantized approach, aside from being intractable and inelegant, simply cannot provide the *quantum-number-phase* noise information.

Note added in proof. It has come to our attention that, for fermions, Yurke has obtained similar results as those presented here using a slightly different approach invoking spin-angular-momentum algebra techniques [11]. We believe our approach is more general in that it handles fermions and bosons with the same ease and on an equal footing, an important point when dealing with atoms that can just as likely be fermionic as bosonic. In addition, Yurke points out that the $N^{-1/2}$ limit can be surpassed by using correlated particles entering both input ports of the interferometer. We did not explicitly consider such a configuration here, but the noise-reducing properties of using correlated atoms could easily be included in the statistics-dependent, second-order, correlation function term that appears in our Eq. (8) for the quantum noise. We would like to thank P. Kumar for pointing out the work of Yurke to us.

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