Lower bound on the ground-state energy and one-dimensional N-fermion problem

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We derive an expression for a lower bound of the ground-state energy of N identical fermions. In principle, this relation can describe any order of multiplicity of many-body forces and represents a generalization of a Hall inequality. Applying the Hall lower-bound inequality to the Calogero and Sutherland potentials, we show the great importance of spin states. Furthermore, using a generalized formula, we obtain a lower bound for a system with three-body forces.

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I. INTRODUCTION

The only one-dimensional quantum-mechanical problem of N identical particles for several forms of potentials has been solved exactly [1]. Although almost none of these potentials is realistic, they can be used for insights into real physical problems. On the other hand, they could be employed for the evaluation of different approaches to the many-body problem. Here we use them for judging the "equivalent one-particle problem" [2,3].

Let the Hamiltonian of an N-particle homogeneous system read

$$H=\sum_{i}^{N}\frac{p_i^2}{2m}+\sum_{\substack{i,j\\(i<j)}}^{N}V_{ij}(r_{ij})+\sum_{\substack{i,j,k\\(i<j<k)}}^{N}V_{ijk}(r_{ij},r_{jk},r_{ki})+\cdots , \tag{1}$$

where V_{ij} is the pair potential, V_{ijk} is the three-body potential, etc., and $\mathbf{r}$ $(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_N)$ are position vectors (spin coordinates are not written explicitly).

In the "center mass" of kinetic energy $(\sum_i \mathbf{p}_i=\mathbf{0})$ Hamiltonian (1) becomes

$$H=\sum_{\substack{i,j\\(i<j)}}^{N}\frac{(\mathbf{p}_i-\mathbf{p}_j)^2}{2mN}+\sum_{\substack{i,j\\(i<j)}}^{N}V_{ij}(r_{ij})+\sum_{\substack{i,j,k\\(i<j<k)}}^{N}V_{ijk}(r_{ij},r_{jk},r_{ki})+\cdots . \tag{2}$$

Let us separate the coordinates of the center of mass introducing a new set of coordinates $\boldsymbol{\rho}$ $(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,..,\boldsymbol{\rho}_N)$ using the transformation

$$\boldsymbol{\rho}=B\mathbf{r} . \tag{3}$$

Matrix B is real and nonsingular. It can be shown that if $\tilde{B}$ is a transpose of B, the new momenta $\boldsymbol{\pi}$ $(\boldsymbol{\pi}_1,\boldsymbol{\pi}_2,\ldots,\boldsymbol{\pi}_N)$ are given by

$$\boldsymbol{\pi}=\tilde{B}^{-1}\mathbf{p} , \tag{4}$$

where $\tilde{B}^{-1}$ is the inverse of $\tilde{B}$, $\boldsymbol{\pi}_j=-i\hbar\nabla_{\rho_j}$, and $\mathbf{p}_j=-i\hbar\nabla_{r_j}$.

Hall and Post (HP) [2,3] have obtained the lower-bound energy

$$E_L=\frac{1}{N-1}\sum_{\substack{\text{over first}\\(N-1)\text{ states}}}\epsilon_j\le E_0 , \tag{5}$$

where ϵ_j are eigenstates of the reduced two-body Hamiltonian given by

$$\mathcal{H}(\rho_2)=(N-1)\left\{\frac{-\hbar^2}{2m\lambda}\Delta_{\rho_2}+\frac{1}{2}NV(\sqrt{2}\rho_2)\right\} ,$$

which includes only the two-body correlation V_{ij}; λ is a parameter, and E_0 is a ground-state energy of system (2) with only the two-body potential V_{ij}. It should be mentioned that there is also a similar Dyson-Lenard (DL) inequality for the lower bound of the ground-state energy [4].

In Sec. II we present a generalization of the HP result, which takes into account three- and many-body forces. Section III describes the application of the HP results to Calogero [5–7], Sutherland [8–10], and the confining [11,12] potentials which have not been studied on this basis so far. In Sec. IV our result is applied to the calculation of a lower bound of the ground-state energy which contains three-body potential as well [13,14]. Some comments are given in Sec. V. An important part of the proof of our generalized expression is carried out in Appendix A. In Appendix B the solutions of reduced eigenvalue problems for Calogero, Sutherland, and confining potentials are presented, respectively. Appendix C describes the solution of the eigenvalue problem of the three-body reduced Hamiltonian.

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II. GENERALIZED FORMULA FOR THE LOWER BOUND

Let us assume that $\boldsymbol{\rho}_1$ is coordinate of the center of mass. For a trial function $\chi(\boldsymbol{\rho}_2,\boldsymbol{\rho}_3,\ldots,\boldsymbol{\rho}_N)$, which is normalized and antisymmetric (in particle indices $1,2,\ldots,N$), we find the expectation value of the Hamiltonian (2)

$$\langle\chi|H|\chi\rangle=\langle x|\mathcal{H}_n|\chi\rangle\ , \tag{6}$$

where

$$\mathcal{H}_n=\sum_{k=2}^{n}\frac{k!(n-k)!}{n!}\sum_{j_k=k}^{n}\cdots\sum_{j_2=2}^{j_3-1}\sum_{j_1=1}^{j_2-1}H_{j_1j_2\cdots j_k}\ . \tag{7}$$

n is a parameter which may be $2,3,\ldots,N$ and $k=2,3,\ldots,n$,

$$H_{j_1j_2}=\frac{N(N-1)}{4mN}(\mathbf{p}_{j_1}-\mathbf{p}_{j_2})^2+\tfrac{1}{2}N(N-1)V_{j_1j_2}\ , \tag{8}$$

$$H_{j_1j_2\cdots j_k}=V_{j_1j_2\cdots j_k},\quad k=3,4,\ldots,n\ . \tag{9}$$

A lower bound on the ground-state energy E_0 can be obtained in the following way. Assuming that after introducing new coordinates $\boldsymbol{\rho}$ and momenta $\boldsymbol{\pi}$, the Hamiltonian is transformed into

$$\mathcal{H}_n\to\mathcal{H}(\boldsymbol{\rho}_2,\ldots,\boldsymbol{\rho}_n)\ , \tag{10}$$

and that also the eigenvalue problem

$$\mathcal{H}(\mathbf{p}_2,\ldots,\boldsymbol{\rho}_n)\Phi_i(\boldsymbol{\rho}_2,\ldots,\boldsymbol{\rho}_n)=\epsilon_i\Phi_i(\boldsymbol{\rho}_2,..,\boldsymbol{\rho}_n) \tag{11}$$

can be solved. Let us expand unknown ground-state wave function $\Psi_0(\boldsymbol{\rho}_2,\ldots,\boldsymbol{\rho}_N)$ in terms of normalized eigenstates $\Phi_i(\boldsymbol{\rho}_2,..,\boldsymbol{\rho}_n)$,

$$\Psi_0(\boldsymbol{\rho}_2,\ldots,\boldsymbol{\rho}_N)=\sum_i C_i\Phi_i(\boldsymbol{\rho}_2,..,\boldsymbol{\rho}_n)\Psi_i(\boldsymbol{\rho}_{n+1},\ldots,\boldsymbol{\rho}_N)\ . \tag{12}$$

C_i are constant coefficients and Ψ_i are antisymmetric functions which satisfy the relation

$$A(n+1,\ldots,N)\Psi_i(\boldsymbol{\rho}_{n+1},\ldots,\boldsymbol{\rho}_N)=\Psi_i(\boldsymbol{\rho}_{n+1},\ldots,\boldsymbol{\rho}_N)\ ; \tag{13}$$

the operator A is an antisymmetrizer. By assumption Ψ_0 is antisymmetric and normalized, i.e.,

$$\begin{aligned}&A(1,2,\ldots,N)\Psi_0=\Psi_0\ ,\\&\langle\Psi_0|\Psi_0\rangle=1\ .\end{aligned} \tag{14}$$

The ground-state energy is

$$E_0=\langle\Psi_0|H|\Psi_0\rangle=\langle\Psi_0|\mathcal{H}|\Psi_0\rangle=\sum_i|C_i|\epsilon_i\ . \tag{15}$$

Inequality for the coefficients C_i can be determined using the Shwartz inequality and Hermiticity, the projection property, and decomposition of A:

$$A(n,n+1,\ldots,N)=\frac{1}{N-n+1}[1-P_{n,n+1}-\cdots-P_{n,N}]\ ,$$

where $P_{j,l}$ is the exchange operator in individual particle indices j and l, namely

$$\begin{aligned}|C_i|^2=|\langle\Phi_i\Psi_i|\Psi_0\rangle|^2&=|\langle\Phi_i\Psi_i|A(n,n+1,\ldots,N)\Psi_0\rangle|^2\\&=|\langle A(n,n+1,\ldots,N)\Phi_i\Psi_i|\Psi_0\rangle|^2\\&\le\langle A(n,n+1,\ldots,N)\Phi_i\Psi_i|A(n,n+1,\ldots,N)\Phi_i\Psi_i\rangle\\&\le\frac{1}{N-n+1}\langle\Phi_i\Psi_i|[1-P_{n,n+1}-\cdots-P_{n,N}]\Phi_i\Psi_i\rangle\\&\le\frac{1}{N-n+1}\{1-[N-(n+1)]\delta\}\ ,\end{aligned} \tag{16}$$

where

$$\delta=\int\cdots\int\Phi_i^*(\rho_2,\ldots,\rho^n)\Psi_i^*(\rho_{n+1},\ldots,\rho_N)P_{n,n+1}\Phi_i(\rho_2,\ldots,\rho_n)\Psi_i(\rho_{n+1},\ldots,\rho_N)d\boldsymbol{\rho}_2d\boldsymbol{\rho}_3\cdots d\boldsymbol{\rho}_N\ . \tag{17}$$

It can be shown that δ is a non-negative number for transformation (4) which satisfies

$$P_{n,n+1}\boldsymbol{\rho}_n=\boldsymbol{\rho}_{n+1},\quad P_{n,n+1}\boldsymbol{\rho}_{n+1}=\boldsymbol{\rho}_n,\quad P_{n,n+1}\boldsymbol{\rho}_l=\boldsymbol{\rho}_l,\quad l\ne n,n+1\ . \tag{18}$$

Now δ becomes

$$\begin{aligned}\delta&=\int\cdots\int\Phi_i^*(\rho_2,\ldots,\rho_n)\Psi_i^*(\rho_{n+1},\ldots,\rho_N)\Phi_i(\rho_2,\ldots,\rho_{n-1},\rho_{n+1})\Psi_i(\rho_n,\rho_{n+2},\ldots,\rho_N)d\boldsymbol{\rho}_2d\boldsymbol{\rho}_3\cdots d\boldsymbol{\rho}_N\\&=\int\cdots\int[\Phi_i^*(\rho_2,\ldots,\rho_n)\Psi_i(\rho_n,\rho_{n+2},\ldots,\rho_N)d\boldsymbol{\rho}_n]\\&\qquad\times[\Phi_i(\rho_2,\ldots,\rho_{n-1},\rho_n)\Psi_i^*(\rho_{n+1},\rho_{n+2},\ldots,\rho_N)d\boldsymbol{\rho}_{n+1}]d\rho_2\cdots d\boldsymbol{\rho}_{n-1}d\boldsymbol{\rho}_{n+2}\cdots d\boldsymbol{\rho}_N\ge0\ .\end{aligned} \tag{19}$$

Hence from (16) it follows that

$$|C_i|^2 \le \frac{1}{N-n+1} . \tag{20}$$

Combining this relation with (15) we find (Appendix A)

$$E_L = \frac{1}{N-n+1} \sum_{i}^{N-n+1} \epsilon_i \le E_0 ; \tag{21}$$

the summation is taken over the first $N-n+1$ energy states of the n-body reduced Hamiltonian (11). Notice that relation (21) describes n-body correlations. For $n=2$ it reproduces Hall result [3] and for $n \to N$ we come back to the original N-body problem as well.

III. SOME APPLICATIONS OF THE HP FORMULA

As already mentioned, there are several exactly solvable one-dimensional N-body systems in which particles interact through pair potentials. From these systems we consider two of them, those with Calogero and those with Sutherland pair interaction, respectively. They are chosen because of their discrete spectrum.

The Calogero potential has the form

$$V_{ij} = \tfrac{1}{2} \sum_{i,j}^{N} \tfrac{1}{4} m\omega^2 (x_i - x_j)^2 + \tfrac{1}{2} \sum_{i,j}^{N} \frac{g}{(x_i - x_j)^2} . \tag{22}$$

The exact ground-state energy is [7]

$$E_0 = \tfrac{1}{8}\hbar\omega (N-1)\sqrt{2N} \left[N+2+N \left[1+\frac{4mg}{\hbar^2} \right]^{1/2} \right] . \tag{23}$$

The lower bound on E_0 that we found here is (Appendix B)

$$E_L = \hbar\omega (N-1) \left[\frac{N}{2\lambda} \right]^{1/2} \left[\frac{N-4}{9} + 1 + \frac{1}{2} \left[1 + \frac{2mg\lambda N}{\hbar^2} \right]^{1/2} \right] . \tag{24}$$

The Sutherland potential is

$$V_{ij} = \tfrac{1}{2} \sum_{i,j}^{N} \frac{a^2 g^2}{\sin^2 a (x_i - x_j)} . \tag{25}$$

In this case the exact ground-state energy is given by [1,8]

$$E_0 = \frac{\hbar^2 a^2}{12m} N(N^2-1) \left[1 + \frac{2mg^2}{\hbar^2} + \left[1 + \frac{4mg^2}{\hbar^2} \right]^{1/2} \right] . \tag{26}$$

A complete spectrum of the two-body reduced Hamiltonian with this potential is obtained in Appendix B as well, and a lower bound on the ground-state energy E_0 reads

$$E_L = \frac{\hbar^2 a^2}{m\lambda} \frac{N-1}{3^4} \left\{ (N-4)(N-2) + \tfrac{9}{2}(N-4) \left[1 + \left[1 + \frac{2m\lambda g^2 N}{\hbar^2} \right]^{1/2} \right] + \frac{81}{2} \left[1 + \frac{m\lambda g^2}{\hbar^2} + \left[1 + \frac{2m\lambda g^2 N}{\hbar^2} \right]^{1/2} \right] \right\} . \tag{27}$$

The ratio E_0/E_L from (23) and (24) is (the C index represents Calogero)

$$\frac{E_0}{E_{LC}} = \frac{\sqrt{\lambda}}{4} \frac{N \left[1 + \left[1 + \frac{4mg}{\hbar^2} \right]^{1/2} \right] + 2}{\frac{N-4}{9} + 1 + \frac{1}{2} \left[1 + \frac{2m\lambda g^2 N}{\hbar^2} \right]^{1/2}} , \tag{28}$$

and from (26) and (27) (the S index represents Sutherland)

$$\frac{E_0}{E_{LS}} = \frac{27\lambda N(N+1)}{4} \left[1 + \frac{2mg^2}{\hbar^2} + \left[1 + \frac{4mg^2}{\hbar^2} \right]^{1/2} \right] \times \left\{ (N-4)(N-2) + \frac{9}{2}(N-4) \left[1 + \left[1 + \frac{2m\lambda g^2 N}{\hbar^2} \right]^{1/2} \right] + \frac{81}{2} \left[1 + \frac{m\lambda g^2 N}{\hbar^2} + \left[1 + \frac{2m\lambda g^2 N}{\hbar^2} \right]^{1/2} \right] \right\}^{-1} \tag{29}$$

In the limit $N \to \infty$

$$\frac{E_0}{E_{LC}} = \tfrac{9}{4}\sqrt{\lambda} \left[1 + \left[1 + \frac{4mg}{\hbar^2} \right]^{1/2} \right] \tag{30}$$

and

$$\frac{E_0}{E_{LS}} = \frac{27\lambda}{4} \left[1 + \frac{2mg^2}{\hbar^2} + \left[1 + \frac{4mg^2}{\hbar^2} \right]^{1/2} \right] . \tag{31}$$

It is important to mention that, in contrast to harmonic potential case [3], the inequality (5) is not satisfied if spin states are not taken into account.

In addition, let us analyze the confining potential [11,12]

$$V = \tfrac{1}{2}\gamma \sum_{i,j}^{N} |x_i - x_j| , \tag{32}$$

giving the discrete spectrum as well. Only the leading term of the ground-state energy is known exactly [12],

$$E_0=\frac{9}{28}\left[\frac{\sqrt{3}}{\pi}\right]^{1/3}[\Gamma(\tfrac{2}{3})]^3N^{7/3}\left[\frac{\hbar^2\gamma^2}{m}\right]^{1/3}$$
$$=0.654N^{7/3}\left[\frac{\hbar^2\gamma^2}{m}\right]^{1/3} . \quad (33)$$

In Appendix B a lower bound, for large N, is determined,

$$E_L=\frac{\pi^{2/3}3^{5/3}}{2^{7/3}5}\left[\frac{2}{\lambda}\right]^{1/3}N^{7/3}\left[\frac{\hbar^2\gamma^2}{m}\right]^{1/3}$$
$$=0.531\left[\frac{2}{\lambda}\right]^{1/3}N^{7/3}\left[\frac{\hbar^2\gamma^2}{m}\right]^{1/3} , \quad (34)$$

which is consistent with (33). Introducing the value $\lambda=\frac{4}{3}$, we find $E_L=1.1447E_{L_{AB}}$, where $E_{L_{AB}}$ is the value from the work [12], which is closer to the real value (33). The value E_L is a better estimate of the exact value (33).

IV. A PROBLEM WITH THREE-BODY FORCES

In this section we consider the one-dimensional N-body problem with three-body forces. Assuming that potentials in the Hamiltonian of the system (2) are [13,14]

$$V_{ij}=\tfrac{1}{2}K(x_i-x_j)^2 , \quad (35)$$

$$V_{ijk}=\frac{f}{[(x_i-x_k)+(x_j-x_k)]^2}=\frac{f}{(x_i+x_j-2x_k)^2} , \quad (36)$$

The Hamiltonian (2) now reads

$$H=\tfrac{1}{2}\sum_{i,j}^{N}\left[\frac{(\mathbf{p}_i-\mathbf{p}_j)^2}{2mN}+V_{ij}\right]+\tfrac{1}{6}\sum_{i,j,k}^{N}V_{ijk} . \quad (37)$$

We choose matrix B in the form

$$\begin{bmatrix}
\frac{1}{\sqrt{N}} & \frac{1}{\sqrt{N}} & \cdots & & & & & & \frac{1}{\sqrt{N}}\\
\frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & \cdots & & & & & 0\\
0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & \cdots & & & & 0\\
0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 & \cdots & & & 0\\
\frac{1}{\sqrt{4\times5}} & \frac{1}{\sqrt{4\times5}} & \frac{1}{\sqrt{4\times5}} & \frac{1}{\sqrt{4\times5}} & \frac{-4}{\sqrt{4\times5}} & 0 & \cdots & & 0\\
\frac{1}{\sqrt{5\times6}} & \frac{1}{\sqrt{5\times6}} & \frac{1}{\sqrt{5\times6}} & \frac{1}{\sqrt{5\times6}} & \frac{1}{\sqrt{5\times6}} & \frac{-5}{\sqrt{5\times6}} & 0 & \cdots & 0\\
\vdots & & & & & & & & \\
\frac{1}{\sqrt{(N-1)N}} & \cdots & & & & & & \frac{1}{\sqrt{(N-1)N}} & \frac{-(N-1)}{\sqrt{(N-1)N}}
\end{bmatrix} .$$

Using Eq. (4), it is easy to show that

$$\tilde{B}^{-1}=\begin{bmatrix}
\frac{1}{\sqrt{N}} & \frac{1}{\sqrt{N}} & \cdots & & & & & & & \frac{1}{\sqrt{N}}\\
\frac{-3}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & 0 & \cdots & & & & 0\\
\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{-3}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & 0 & \cdots & & & & 0\\
\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{-3}{2\sqrt{2}} & 0 & \cdots & & & & 0\\
\frac{1}{\sqrt{4\times5}} & \frac{1}{\sqrt{4\times5}} & \frac{1}{\sqrt{4\times5}} & \frac{1}{\sqrt{4\times5}} & \frac{-4}{\sqrt{4\times5}} & 0 & \cdots & & & 0\\
\frac{1}{\sqrt{5\times6}} & \frac{1}{\sqrt{5\times6}} & \frac{1}{\sqrt{5\times6}} & \frac{1}{\sqrt{5\times6}} & \frac{1}{\sqrt{5\times6}} & \frac{-5}{\sqrt{5\times6}} & 0 & \cdots & & 0\\
\vdots & & & & & & & & & \\
\frac{1}{\sqrt{(N-1)N}} & \cdots & & & & & & & \frac{1}{\sqrt{(N-1)N}} & \frac{-(N-1)}{\sqrt{(N-1)N}}
\end{bmatrix} .$$

The expectation value (6) now becomes

$$\langle\chi|H|\chi\rangle=\left\langle\chi\left|\frac{1}{2}\sum_{i,j}^{N}\frac{1}{2mN}(\mathbf{p}_i-\mathbf{p}_j)^2\right|\chi\right\rangle+\left\langle\chi\left|\frac{1}{2}\sum_{i,j}^{N}V_{i,j}\right|\chi\right\rangle+\left\langle\chi\left|\frac{1}{6}\sum_{i,j,k}^{N}V_{ijk}\right|\chi\right\rangle$$

$$=\frac{(N-1)}{4m}\langle\chi|(\mathbf{p}_1-\mathbf{p}_2)^2|\chi\rangle+\langle\chi|\tfrac{1}{2}N(N-1)V_{12}|\chi\rangle+\langle\chi|\tfrac{1}{6}N(N-1)(N-2)V_{123}|\chi\rangle\ . \tag{38}$$

At this point we denote elements of the second and third row of matrix $\tilde{B}^{-1}$ by λ_i and γ_i, respectively, i.e.,

$$\pi_2=\sum_i\beta_i\mathbf{p}_i,\quad \pi_3=\sum_i\gamma_i\mathbf{p}_i\ .$$

Since in our case $\sum_i\beta_i=0$ and $\sum_i\gamma_i=0$, it is possible to define the two quantities

$$\lambda\equiv\sum_i\beta_i^2=-2\sum_{\substack{i,j=1\\(i<j)}}^{N}\beta_i\beta_j\ ,$$

$$\mu\equiv\sum_i\gamma_i^2=-2\sum_{\substack{i,j=1\\(i<j)}}^{N}\gamma_i\gamma_j\ .$$

From the definition of matrix $\tilde{B}^{-1}$ it follows that $\lambda=\mu=\frac{3}{2}$. The average values are

$$\langle\chi|\pi_2^2|\chi\rangle=\lambda[\langle\chi|\mathbf{p}_1^2|\chi\rangle-\langle\chi|\mathbf{p}_1\cdot\mathbf{p}_2|\chi\rangle]\ ,$$

$$\langle\chi|\pi_3^2|\chi\rangle=\mu[\langle\chi|\mathbf{p}_1^2|\chi\rangle-\langle\chi|\mathbf{p}_1\cdot\mathbf{p}_2|\chi\rangle]\ .$$

Then from (38) we have

$$\langle\chi|H|\chi\rangle=N(N-1)\left\{\frac{1}{4m\lambda N}[\langle\chi|\pi_2^2|\chi\rangle+\langle\chi|\pi_3^2|\chi\rangle]+\tfrac{1}{4}[\langle\chi|V(\rho_2)+V(\rho_3)|\chi\rangle]+\frac{N-2}{6}\langle\chi|V(\rho_2,\rho_3)|\chi\rangle\right\}.$$

Therefore, the reduced Hamiltonian in the eigenvalue problem (11) reads

$$\mathcal{H}=N(N-1)\left[\frac{-\hbar^2}{4m\lambda N}\left[\frac{\partial^2}{\partial\rho_2^2}+\frac{\partial^2}{\partial\rho_3^2}\right]+\frac{K}{4}(\rho_2^2+\rho_3^2)+\frac{(N-2)f}{12}\frac{1}{(\rho_2+\rho_3)^2}\right]. \tag{39}$$

The eigenvalue problem is solved in Appendix C. A lower limit of energy is obtained using the formula (21)

$$E_L=\frac{1}{N-2}\sum_i^{N-2}(\epsilon_{nl})_i\ . \tag{40}$$

The quantity ϵ_{nl} is given in (C14). Performing a summation over the first $(N-2)$ states, without spin degeneracy, we find

$$E_L=\hbar\sqrt{K/m}\,\frac{N-1}{4(N-2)}\sqrt{N/\lambda}\times[\tfrac{2}{3}k(k+1)(8k+1)+(N-2)(3+\sqrt{1+2F})]\ , \tag{41}$$

where $k=\frac{1}{2}(\sqrt{4N-7}-1)$, or

$$E_L=\frac{\hbar}{4}\sqrt{K/m}\,(N-1)\sqrt{N/\lambda}[\tfrac{8}{3}\sqrt{4N-7}+\sqrt{1+2F}+1]\ . \tag{42}$$

The leading term in the large-N limit for the lower bound of the ground state is $E_L=(N^{5/7}/4)\sqrt{2fK/3}$.

V. DISCUSSION

It is interesting to observe the N dependence of the ground-state energy of one-dimensional systems, which have been solved exactly so far. The power of N is greater than 2. This means that they are not extensive.

Comparing the results, obtained by using the HP formula, with those found by the DL inequality, it is noticed that the former are closer to exact solutions. This is simply acceptable if we recognize that the Hall approach is a kind of "optimization" procedure. For instance, the ratios E_0/E_L for HP and DL are 1.155 and 1.414 for harmonic and 1.075 and 1.23 for confining potentials, respectively.

The lower bound (42), which includes three-body forces, depends on N, for large N, as $N^{5/2}$. Although the exact solution is not known, we argue that the leading term of the exact solution has the same N dependence. Again it is found that this system is not extensive.

We emphasize that both inequalities HP and DL are satisfied for Calogero and Sutherland potentials if only spin states are included. Since ground-state energies, obtained from antisymmetric wave functions in r space, are the only ones available, we found "lower bounds" for these potentials summing over symmetric spin states (spin $\frac{1}{2}$). The bounds are definitely below the exact values. This suggests that the spin states should be included in future treatment of one-dimensional systems. Furthermore, the same energy relation (21) is valid for bosons.

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APPENDIX A

In order to prove inequality (21), notice that the summation in (15) is taken over all states, and from (20) it follows that

$$|C_i|^2+\delta_i=\frac{1}{N-n+1},\quad \delta_i\geq 0\ .$$

Then from (15)

$$E_0=\frac{1}{N-n+1}\sum_{i=1}^{N-n+1}\epsilon_i+\Delta\ , \tag{A1}$$

where

$$\Delta=\frac{1}{N-n+1}\sum_{N-n+2}^{\infty}\epsilon_i-\sum_{i=1}^{\infty}\epsilon_i\delta_i$$

$$=\sum_{N-n+2}^{\infty}\left[\frac{1}{N-n+1}-\delta_i\right]\epsilon_i-\sum_{i=1}^{N-n+1}\epsilon_i\delta i\ .$$

Let us define a constant η, which satisfies

$$\epsilon_{N-n+1}\leq\eta\leq\epsilon_{N-n+2}\ ,$$

and a quantity

$$\Delta_a=\eta\sum_{N-n+2}^{\infty}\left[\frac{1}{N-n+1}-\delta_i\right]-\eta\sum_{i=1}^{N-n+1}\delta_i$$

$$=\eta\left[\sum_{N-n+2}^{\infty}\frac{1}{N-n+1}-\sum_{i=1}^{\infty}\delta_i\right]$$

$$=\eta\left[\sum_{i=1}^{\infty}\left[\frac{1}{N-n+1}-\delta i\right]-\sum_{i=1}^{N-n+1}\frac{1}{N-n+1}\right]$$

$$=\eta(1-1)=0\ .$$

Since $\Delta\geq\Delta_a$, it follows that $\Delta\geq 0$ and from (A1), consequently, the inequality (21).

APPENDIX B

In this appendix Eqs. (24) and (27) are proved first. We chose the same matrix as Hall [3],

$$B=\begin{bmatrix}
\frac{1}{\sqrt{N}} & \frac{1}{\sqrt{N}} & \cdots & & & & & & \frac{1}{\sqrt{N}}\\
\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & \cdots & & & & & 0\\
\frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 & \cdots & & & & 0\\
\frac{1}{\sqrt{3\times4}} & \frac{1}{\sqrt{3\times4}} & \frac{1}{\sqrt{3\times4}} & \frac{-3}{\sqrt{3\times4}} & 0 & \cdots & & & 0\\
\frac{1}{\sqrt{4\times5}} & \frac{1}{\sqrt{4\times5}} & \frac{1}{\sqrt{4\times5}} & \frac{1}{\sqrt{4\times5}} & \frac{-4}{\sqrt{4\times5}} & 0 & \cdots & & 0\\
\frac{1}{\sqrt{5\times6}} & \frac{1}{\sqrt{5\times6}} & \frac{1}{\sqrt{5\times6}} & \frac{1}{\sqrt{5\times6}} & \frac{1}{\sqrt{5\times6}} & \frac{-5}{\sqrt{5\times6}} & 0 & \cdots & 0\\
\vdots & & & & & & & & \\
\frac{1}{\sqrt{(N-1)N}} & \cdots & & & & & & \frac{1}{\sqrt{(N-1)N}} & \frac{-(N-1)}{\sqrt{(N-1)N}}
\end{bmatrix}$$

and found

$$\tilde{B}^{-1}=\begin{bmatrix}
\frac{1}{\sqrt{N}} & \frac{1}{\sqrt{N}} & \cdots & & & & & \frac{1}{\sqrt{N}}\\
\frac{\sqrt{2}}{3} & \frac{-2\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & 0 & \cdots & & & 0\\
\frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & \frac{-2\sqrt{2}}{3} & 0 & \cdots & & & 0\\
\frac{1}{\sqrt{3\times4}} & \frac{1}{\sqrt{3\times4}} & \frac{1}{\sqrt{3\times4}} & \frac{-3}{\sqrt{3\times4}} & 0 & \cdots & & 0\\
\frac{1}{\sqrt{4\times5}} & \frac{1}{\sqrt{4\times5}} & \frac{1}{\sqrt{4\times5}} & \frac{1}{\sqrt{4\times5}} & \frac{-4}{\sqrt{4\times5}} & 0 & \cdots & 0\\
\vdots & & & & & & & \\
\frac{1}{\sqrt{(N-1)N}} & \cdots & & & & & \frac{1}{\sqrt{(N-1)N}} & \frac{-(N-1)}{\sqrt{(N-1)N}}
\end{bmatrix}.$$

The two-body reduced Hamiltonian

$$\mathcal{H}(\rho_2)=(N-1)\frac{-\hbar^2}{2m\lambda}\frac{d^2}{d\rho_2^2}+\frac{1}{2}N(N-1)V(\sqrt{2}\rho_2)\ , \tag{B1}$$

where $\lambda=\frac{4}{3}$, $\rho_2=(1/\sqrt{2})(x_1-x_2)$, contains the potential

$$V(\sqrt{2}\rho_2)=\frac{m\omega^2}{4}(\sqrt{2}\rho_2)^2+\frac{g}{(\sqrt{2}\rho_2)^2} \tag{B2}$$

or

$$V(\sqrt{2}\rho_2)=\frac{a^2g^2}{\sin^2(\sqrt{2}ax)} \tag{B3}$$

for the Calogero and Sutherland cases, respectively. Now we solve the eigenvalue problem

$$\mathcal{H}(x)\phi_n(x)=\epsilon_n\phi_n(x)\ . \tag{B4}$$

Equation (B4) can be resolved for the potential (B2) using Calogero's result [7] for ($N=2$). In this particular case the energy reads

$$\epsilon_n=\hbar\omega(N-1)\times\left[\frac{N}{2\lambda}\right]^{1/2}\left[1+\frac{1}{2}\left[1+\frac{2\lambda mgN}{\hbar^2}\right]^{1/2}+2n\right]\ , \tag{B5}$$

where $n=0,1,2,3,\ldots$. Eigenvalues of Eq. (B4) for Sutherland's potential are formally the same as in Ref. [5] and we have

$$\epsilon_l=(N-1)\frac{\hbar^2a^2}{m\lambda}\left[l+\frac{1}{2}\left[1+\frac{2m\lambda g^2N}{\hbar^2}\right]^{1/2}+\frac{1}{2}\right]^2\ , \tag{B6}$$

where $l=0,1,2,3,\ldots$.

Knowing that the above energy spectrum corresponds to the antisymmetric states in r space, let us suppose that we deal with the particles of spin $\frac{1}{2}$. Therefore, spin states are symmetric and each term has the triplet degeneracy. Summing over the first $N-1$ states, i.e., up to $(N-1)/3$, we find Eqs. (24) and (27).

The two-body confining potential in our approach reads

$$V(\sqrt{2}x)=\frac{\sqrt{2}}{2}\gamma|x|\ . \tag{B7}$$

The spectrum of (B4) for this potential, exploiting the results of paper [12], is given by

$$\epsilon_{2n}=(N-1)\frac{N^{2/3}}{2}\left[\frac{2}{\lambda}\right]^{1/3}\left[\frac{\hbar^2\gamma^2}{m}\right]^{1/3}\mu_n\ ,$$

$$\epsilon_{2n+1}=(N-1)\frac{N^{2/3}}{2}\left[\frac{2}{\lambda}\right]^{1/3}\left[\frac{\hbar^2\gamma^2}{m}\right]^{1/3}\tilde{\mu}_n\ ,$$

where $n=0,1,2,\ldots$, and μ_n and $\tilde{\mu}_n$ are the zeros of derivative of Airy functions for even and odd levels, respectively. Lower bounds are given by (5), i.e.,

$$E_L=\frac{N^{2/3}}{2}\left[\frac{2}{\lambda}\right]^{1/3}\left[\frac{\hbar^2\gamma^2}{m}\right]^{1/3}\sum_{n=0}^{(N-3)/2}(\mu_n+\tilde{\mu}_n) \tag{B8}$$

for N odd and

$$E_L=\frac{N^{2/3}}{2}\left[\frac{2}{\lambda}\right]^{1/3}\left[\frac{\hbar^2\gamma^2}{m}\right]^{1/3}\times\left[\sum_{n=0}^{(N-2)/2}\mu_n+\sum_{n=0}^{(N-4)/2}\tilde{\mu}_n\right] \tag{B9}$$

for N even, where for large N

$$\mu_n=\left[\frac{3\pi}{2}\right]^{2/3}(n+\tfrac{3}{4})^{2/3}\ , \tag{B10}$$

$$\tilde{\mu}_n=\left[\frac{3\pi}{2}\right]^{2/3}(n+\tfrac{1}{4})^{2/3}\ . \tag{B11}$$

Summing (B8) or (B9), with (B10) and (B11), we produce the bound energy (34).

APPENDIX C

In this appendix we solve the eigenvalue problem with the Hamiltonian (39)

$$\left[-\frac{d^2}{dx^2}-\frac{d^2}{dy^2}+\Omega^2(x^2+y^2)+\frac{F}{(x+y)^2}\right]\varphi=\varepsilon\varphi(x)\ , \tag{C1}$$

where in our case

$$\Omega^2=\frac{Km\lambda N}{\hbar^2},\quad F=\frac{m\lambda N(N-2)}{3\hbar^2}f,\quad \varepsilon=\frac{4m\lambda}{\hbar^2(N-1)}\epsilon\ . \tag{C2}$$

Introducing the polar coordinates

$$x=r\cos\vartheta=\frac{1}{\sqrt{2}}(x_2-x_1)\ ,$$

$$y=r\sin\vartheta=\frac{1}{\sqrt{2}}(x_2-x_3)\ ,$$

we find

$$\left[-\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}+\Omega^2r^2-\varepsilon+\frac{1}{r^2}\left[-\frac{\partial^2}{\partial\vartheta^2}+\frac{F}{1+\sin 2\vartheta}\right]\right]\phi=0\ . \tag{C3}$$

Factorizing $\phi=\xi(\vartheta)\chi(r)$ we get two independent equations:

$$\left[-\frac{\partial^2}{\partial\vartheta^2}+\frac{F}{1+\sin(2\vartheta)}\right]\xi=\Lambda\xi\ , \tag{C4}$$

$$\left[-\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}+\Omega^2r^2+\frac{\Lambda}{r^2}-\varepsilon\right]\chi=0\ , \tag{C5}$$

where Λ is the separation constant. Equations (C4) and

(C5) can be solved exactly. Furthermore, the solution of Eq. (C5) is known [13], and the energy eigenvalue is

$$\varepsilon=2\Omega(2n+\sqrt{\Lambda}+1), \quad n=0,1,2,\ldots . \tag{C6}$$

In order to solve (C4) we introduce a new variable

$$z=\tfrac{1}{2}[1+\sin(2\vartheta)] ,$$

by which (C4) is transformed into

$$\left[z(1-z)\frac{d^2}{dz^2}-\left[z-\frac{1}{2}\right]\frac{d}{dz}-\frac{F}{8z}+\frac{\Lambda}{4}\right]\xi=0 . \tag{C7}$$

Substitution $\xi=z^{\alpha}H(z)$ gives the hypergeometric equation

$$\left[z(1-z)\frac{d^2}{dz^2}+[2\alpha+\tfrac{1}{2}-z(1+2\alpha)]\frac{d}{dz}+\left[\frac{\Lambda}{4}-\alpha^2\right]\right]H=0 , \tag{C8}$$

where

$$\alpha=\tfrac{1}{4}(1\pm\sqrt{1+2F}) . \tag{C9}$$

The complete solution of the Eq. (C8) is

$$H(z)=AH(\delta,\beta,\gamma;z)+Bz^{1-\gamma}H(\delta+1-\gamma,\beta+1-\gamma,2-\gamma;z) , \tag{C10}$$

where

$$\delta=\alpha-\tfrac{1}{2}\sqrt{\Lambda}, \quad \beta=\alpha+\tfrac{1}{2}\sqrt{\Lambda} ; \tag{C11}$$

$$\gamma=\tfrac{1}{2}+2\alpha . \tag{C12}$$

The physical solution includes $B=0$ and a positive sign of α in (C9). To cut the infinite series of H we impose

$$\delta=-l, \quad l=0,1,2,\ldots$$

or equivalently

$$\sqrt{\Lambda}=2l+\tfrac{1}{2}(1+\sqrt{1+2F}) . \tag{C13}$$

From relations (C2) and (C6) we find the energy

$$\epsilon_{nl}=\frac{(N-1)\hbar^2}{2m\lambda}\Omega[2(n+l)+\tfrac{1}{2}(3+\sqrt{1+2F})] , \quad n=0,1,2,3,\ldots, \quad l=1,3,5,\ldots \tag{C14}$$

for indistinguishable fermions [14].

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