Exact quantum theory of a time-dependent bound quadratic Hamiltonian system

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(Received 14 May 1993)

Using the path-integral method, the propagator, the wave function, and the expectation values are evaluated explicitly for a time-dependent bound quadratic Hamiltonian system. We also have derived the relation between the wave function and a dynamical invariant which determines whether or not the system is bound. The expectation value of the quantum-mechanical invariant obeys the uncertainty relation with an auxiliary condition as the solution of the classical equation of the system.

PACS number(s): 03.65.Ge

I. INTRODUCTION

During the past few decades there has been a surge of interest in the quantum-mechanical solutions for oscillator systems with time-dependent Hamiltonians. It appears to be possible to solve the equations describing the quantum-mechanical behavior of physical systems with quadratic Hamiltonians. Lewis and Riesenfeld [1] have derived the relation between the dynamical invariant and solution of the Schrödinger equation for time-dependent oscillators. Camitz *et al.* [2] have considered the harmonic oscillator with a time-dependent frequency. Landovitz *et al.* [3] have obtained solutions, which allow the calculations of interesting quantum quantities such as the Green's functions and transition amplitudes, for a time-dependent linear quantum system and quantum-mechanical linearly damped harmonic oscillator. Recently attention has been paid to the exact solutions and coherent states for damped [4] or damped driven harmonic oscillators [5] and for Duffing oscillators [6].

In previous papers, making use of the path-integral method, we have obtained wave functions, energy expectation values, uncertainty relations, and transition amplitudes for a quantum damped driven harmonic oscillator [7], coupled forced harmonic oscillator [8], and forced time-dependent harmonic oscillator [9], and have also evaluated the coherent states for the damped harmonic oscillator [10] and harmonic oscillator with time-dependent frequency [11].

The purpose of this paper is to evaluate the exact solution of a general time-dependent quadratic Hamiltonian system and find the relation between the classical and quantum-mechanical solutions through the path-integral method. In Sec. II we derive and consider two explicit time-dependent invariant quantities that show whether or not the system is bound. In Sec. III we evaluate the propagator of the bound system and then the wave function by using the result obtained in Sec. II. Section IV gives the expectation values of the position, momentum, and the squares. The quantum average of the invariant operator and the uncertainty relations of the system are determined. In Sec. V we summarize our results.

II. CLASSICAL INVARIANTS

We first consider a system with the Hamiltonian

$$H=\tfrac{1}{2}[A(t)p^2+B(t)(xp+px)+C(t)x^2]\ , \tag{2.1}$$

where x and p are a canonical coordinate and its conjugate momentum, respectively, $A(t)$ is a nonzero time-dependent function, and $B(t)$ and $C(t)$ are time-dependent functions of arbitrary form. These are piecewise continuously differentiable (with respect to time t) functions. From Hamilton's equation of motion, we obtain the classical equation of motion:

$$\ddot{x}-\frac{\dot{A}(t)}{A(t)}\dot{x}+\left[A(t)C(t)+\frac{\dot{A}(t)B(t)}{A(t)}-B^2(t)-\dot{B}(t)\right]x=0\ . \tag{2.2}$$

If we introduce the new variable

$$x=q\exp\left\{\int B(t)dt\right\}\ , \tag{2.3}$$

Eq. (2.2) can be simplified to

$$\ddot{q}+\left[2B(t)-\frac{\dot{A}(t)}{A(t)}\right]\dot{q}+A(t)C(t)q=0\ . \tag{2.4}$$

However, we cannot find the solution with a general form for arbitrary time-dependent coefficients. For simplicity we can express Eq. (2.2) as

$$\ddot{x}+\zeta(t)\dot{x}+\xi(t)x=0 \ . \tag{2.5}$$

Here, the new time-dependent functions are

$$\zeta(t)\equiv-\frac{\dot{A}(t)}{A(t)} \tag{2.6}$$

and

$$\xi(t)\equiv\left[A(t)C(t)+\frac{\dot{A}(t)B(t)}{A(t)}-B^2(t)-\dot{B}(t)\right] . \tag{2.7}$$

Equation (2.5) does not have a general solution but can be expressed in the form

$$x=\eta(t)e^{i\gamma(t)} , \tag{2.8}$$

where the functions $\eta(t)$ and $\gamma(t)$ must be determined from Eq. (2.5); these are real and depend only on time. Substitution of Eq. (2.8) into (2.5) gives the real and imaginary parts of this equation as

$$\ddot{\eta}-\eta\dot{\gamma}^2+\zeta(t)\dot{\eta}+\xi(t)\eta=0 , \tag{2.9}$$

$$\eta\ddot{\gamma}+2\dot{\eta}\dot{\gamma}+\zeta(t)\eta\dot{\gamma}=0 . \tag{2.10}$$

The invariant quantity can be found from Eq. (2.10) in the form

$$\Omega=\frac{\eta^2\dot{\gamma}}{A(t)} , \tag{2.11}$$

which is a time-invariant quantity with an auxiliary condition given by the classical Eq. (2.5). If the invariant quantity Ω is not equal to zero, then γ is not constant and the position x has the form of a complex function of time. Since the particle of this system will pass through more than two points on the trajectory, the motion of the system will be bound in some restricted region. If Ω is equal to zero, the motion of the system will be unbound.

We can find another classical invariant quantity with the auxiliary condition by classical Eq. (2.5). Let us assume that this invariant quantity be given as

$$I(t)=\tfrac{1}{2}[\alpha(t)p^2+2\beta(t)xp+\delta(t)x^2] , \tag{2.12}$$

where $\alpha(t)$, $\beta(t)$, and $\delta(t)$ are all real time-dependent functions. The multiplicative factors have been chosen for convenience. From Hamilton's equation of motion, the time derivative of $I(t)$ becomes

$$\frac{dI}{dt}=\frac{\partial I}{\partial t}+\frac{\partial I}{\partial x}\frac{\partial H}{\partial p}-\frac{\partial I}{\partial p}\frac{\partial H}{\partial x}=0 . \tag{2.13}$$

Combining Eqs. (2.1) and (2.12) with (2.13) we obtain the coupled differential equations of $\alpha(t)$, $\beta(t)$, and $\delta(t)$:

$$\dot{\alpha}=2B(t)\alpha-2A(t)\beta , \tag{2.14}$$

$$\dot{\beta}=C(t)\alpha-A(t)\delta , \tag{2.15}$$

$$\dot{\delta}=2C(t)\beta-2B(t)\delta . \tag{2.16}$$

Solving the above three equations, we get

$$\alpha=\eta^2(t) , \tag{2.17}$$

$$\beta=\frac{B(t)}{A(t)}\eta^2-\frac{1}{A(t)}\eta\dot{\eta} , \tag{2.18}$$

$$\delta=\left[\frac{B(t)}{A(t)}\eta-\frac{1}{A(t)}\dot{\eta}\right]+\frac{\Omega^2}{\eta^2} , \tag{2.19}$$

together with the subsidiary conditions Eqs. (2.9) and (2.10). Therefore substituting Eqs. (2.17)–(2.19) into (2.12), we obtain the invariant quantity as

$$I(t)=\left\{\left[\frac{\Omega}{\eta}x\right]^2+\left[\left[\frac{B(t)}{A(t)}\eta-\frac{1}{A(t)}\dot{\eta}\right]x+\eta p\right]^2\right\} . \tag{2.20}$$

In Eq. (2.20) we may easily confirm that $I(t)$ is always positive in the unbound system, while $I(t)$ is not positive in the bound system.

III. PROPAGATOR AND WAVE FUNCTION

Since the system has a quadratic Hamiltonian, the propagator has the form [12]

$$\begin{aligned}K(x,t;x',t')&=\left[\frac{1}{2\pi i\hbar}\frac{\partial^2 S_c}{\partial x\,\partial x'}\right]^{1/2}e^{(i/\hbar)S_c}\\&=\exp\left\{\frac{i}{\hbar}[a(t,t')x^2+b(t,t')xx'\right.\\&\qquad\left.+c(t,t')x'^2+d(t,t')]\right\} ,\end{aligned} \tag{3.1}$$

where S_c is the classical action. Instead of Eq. (3.1) we may introduce the definition of the propagator for the bound system as

$$K(x,t;,x',t')=\sum_n\psi_n(x,t)\psi_n^*(x',t') , \tag{3.2}$$

where $\psi_n(x,t)$ is the solution of the Schrödinger equation. For the unbound system, the propagator is given as

$$K(x,t;x',t')=\int dk\ \psi_k(x,t)\psi_k^*(x',t') . \tag{3.3}$$

The above two propagators Eqs. (3.2) and (3.3), must satisfy the following Schrödinger equations, respectively:

$$i\hbar\frac{\partial K}{\partial t}=H(x,p,t)K , \tag{3.4}$$

$$-i\hbar\frac{\partial K^*}{\partial t'}=H^\dagger(x',p',t')K^* . \tag{3.5}$$

Substitution of Eq. (3.1) into (3.4) yields the three coupled differential equations

$$\dot{a}=2i\hbar A(t)a^2-2B(t)a-\frac{i}{\hbar}c(t) , \tag{3.6}$$

$$\dot{b}=2\hbar iA(t)abx'-B(t)bx' , \tag{3.7}$$

$$\dot{c}x'^2+\dot{d}=\frac{i\hbar}{2}A(t)b^2x'^2+i\hbar A(t)a-\tfrac{1}{2}B(t) , \tag{3.8}$$

which can be solved to give

$$a(t)=\frac{i}{2\hbar}\frac{1}{A(t)}\frac{\dot{q}}{q}-\frac{i}{2\hbar}\frac{1}{A(t)}\frac{B(t)}{A(t)} , \tag{3.9}$$

$$b(t)=\frac{b_0}{q}\ ,\tag{3.10}$$

$$c(t)x'^2+d(t)-\frac{i\hbar}{2}b_0^2x'^2\int^t\frac{A(s)}{q^2(s)}ds+\ln q^{-1/2}+d_0\ .\tag{3.11}$$

Solving the above three equations, we have the following form with the auxiliary condition, i.e., the classical solution. Here, q is obviously a solution of Eq. (2.5). If $\eta(t)e^{-i\gamma(t)}$ is the classical solution at time t in the bound system, then $\eta(t)e^{i\gamma(t)}$ is also a solution of that system. The classical solution may be rewritten in the form

$$q(t,t')=\eta\eta'\sin(\gamma-\gamma')\ ,\tag{3.12}$$

where $\eta=\eta(t)$, $\eta'=\eta(t')$, and so on. With the help of Eq. (3.12), we may integrate Eq. (3.11). We can determine the integral constants in Eqs. (3.9)–(3.11). Then the coefficients $a(t)$, $b(t)$, and $c(t)$ with the auxiliary conditions Eqs. (2.9) and (2.10) are given by

$$a(t)=\frac{1}{2A}\left[\frac{\dot\eta}{\eta}+\dot\gamma\cot(\gamma-\gamma')-\frac{B}{A}\right]\ ,\tag{3.13}$$

$$b(t)=\left[\frac{\dot\gamma\dot\gamma'}{AA'}\right]^{1/2}\frac{1}{\sin(\gamma-\gamma')}\ ,\tag{3.14}$$

$$c(t)=\frac{1}{A'}\left[-\frac{\dot\eta'}{\eta'}+\dot\gamma'\cot(\gamma-\gamma')+\frac{B'}{A'}\right]\ ,\tag{3.15}$$

where $A=A(t)$, $A'=A(t')$, and so on. Combining Eqs. (3.13)–(3.15) with the first expression of Eq. (3.1), we obtain $d(t)$ as

$$e^{(i/\hbar)d(t)}=\left[\frac{1}{2\pi i\hbar}\frac{\partial^2 S_c}{\partial x\,\partial x'}\right]^{1/2}=\left[\frac{\dot\gamma^{1/2}\dot\gamma'^{1/2}}{2\pi i\hbar\sin(\gamma-\gamma')A^{1/2}A'^{1/2}}\right]^{1/2}\ ,\tag{3.16}$$

and we obtain the propagator of this system as

$$K(x,t;x',t')=\left[\frac{\dot\gamma^{1/2}\dot\gamma'^{1/2}}{2\pi i\hbar\sin(\gamma-\gamma')A^{1/2}A'^{1/2}}\right]^{1/2}\times\exp\left\{\frac{i}{2\hbar A}\left[\frac{\dot\eta}{\eta}+\dot\gamma\cot(\gamma-\gamma')-B\right]x^2+\frac{i}{2\hbar A'}\left[-\frac{\dot\eta'}{\eta'}+\dot\gamma'\cot(\gamma-\gamma')+B'\right]x'^2+\frac{i}{\hbar}\left[\frac{\dot\gamma\dot\gamma'}{AA'}\right]^{1/2}\frac{xx'}{\sin(\gamma-\gamma')}\right\}\ .\tag{3.17}$$

Introducing the new variables

$$X=\left[\frac{\dot\gamma}{\hbar A}\right]^{1/2}x\ ,\tag{3.18}$$

$$Y=\left[\frac{\dot\gamma'}{\hbar A'}\right]^{1/2}x'\ ,\tag{3.19}$$

$$Z=e^{-i(\gamma-\gamma')}\ ,\tag{3.20}$$

we may reexpress the propagator in the simple form

$$K(x,t;x',t')=\left[\frac{1}{\pi\hbar}\left[\frac{\dot\gamma\dot\gamma'}{AA'}\right]^{1/2}e^{-i(\gamma-\gamma')}\right]^{1/2}\times\frac{1}{\sqrt{1-Z^2}}e^{(X^2+Y^2)/2}\times\exp\left\{\frac{-(X^2+Y^2)+2XYZ}{1-Z^2}\right\}\ .\tag{3.21}$$

To find the explicit form of the wave function of the system, we make use of Mehler's formula [13], which is expressed in terms of the nth order of the Hermite polynomial, $H_n(X)$,

$$\sqrt{1-Z^2}\exp\left\{\frac{2XYZ-X^2-Y^2}{1-Z^2}\right\}=e^{-(X^2+Y^2)}\sum_{n=0}^{\infty}\frac{Z^n}{2^n n!}H_n(X)H_n(Y)\ .\tag{3.22}$$

Therefore comparison of Eqs. (3.21) and (3.22) with (3.2) gives the exact wave function of the system:

$$\psi_n(x,t)=\left[\frac{1}{\pi\hbar}\frac{\dot\gamma}{A}\right]^{1/4}\left[\frac{1}{2^n n!}\right]^{1/2}e^{-i[(1/2)+n]\gamma}\times H_n\left\{\left[\frac{\dot\gamma}{\hbar A}\right]^{1/2}x\right\}\times\exp\left\{-\frac{1}{2\hbar A}\left[\dot\gamma-i\left[\frac{\dot\eta}{\eta}-B\right]\right]x^2\right\}\ .\tag{3.23}$$

We note that Eq. (3.23) is the wave function of the bound system with the auxiliary condition of classical solution. A similar procedure can be followed for the unbound system.

IV. UNCERTAINTY RELATION AND EXPECTATION VALUES

The uncertainty is

$$(\Delta x \Delta p)_{m,n} = \{[(\langle m|x^2|n\rangle - \langle m|x|n\rangle^2)^*(\langle m|x^2|n\rangle - \langle m|x|n\rangle^2)]^{1/2} \times [(\langle m|p^2|n\rangle - \langle m|p|n\rangle^2)^*(\langle m|p^2|n\rangle - \langle m|p|n\rangle^2)]^{1/2}\}^{1/2} . \quad (4.1)$$

Using Eq. (3.23) and performing the integral over x yields

$$\langle m|x|n\rangle = \sqrt{n+1}\mu\delta_{m,n+1} + \sqrt{n}\,\mu^*\delta_{m,n-1} , \quad (4.2)$$

$$\langle m|p|n\rangle = \sqrt{n+1}\nu\delta_{m,n+1} + \sqrt{n}\,\nu^*\delta_{m,n-1} , \quad (4.3)$$

$$\langle m|x^2|n\rangle = \sqrt{(n+1)(n+2)}\mu^2\delta_{m,n+1} + (2n+1)\mu\mu^*\delta_{m,n} + \sqrt{n(n-1)}\mu^{*2}\delta_{m,n-2} , \quad (4.4)$$

$$\langle m|p^2|n\rangle = \sqrt{(n+1)(n+2)}\nu^2\delta_{m,n+2} + (2n+1)\nu\nu^*\delta_{m,n} + \sqrt{n(n-1)}\nu^{*2}\delta_{m,n-2} , \quad (4.5)$$

$$\langle m|\tfrac{1}{2}(xp+px)|n\rangle = \sqrt{(n+1)(n+2)}\mu\nu\delta_{m,n+2} + (2n+1)\left[\frac{\mu\nu^*+\nu\mu^*}{2}\right]\delta_{m,n} + \sqrt{n(n-1)}\mu^*\nu^*\delta_{m,n-2} , \quad (4.6)$$

where $\mu(t)$ and $\nu(t)$ are given by

$$\mu = \mu(t) = \left[\frac{\hbar A(t)}{2\dot{\gamma}}\right]^{1/2} e^{i\gamma} , \quad (4.7)$$

$$\nu = \nu(t) = \left[\frac{\hbar}{2A(t)\dot{\gamma}}\right]^{1/2} \left[\left[\frac{\dot{\eta}}{\eta} - B(t)\right] + i\dot{\gamma}\right] e^{i\gamma} . \quad (4.8)$$

Substituting Eqs. (4.2)–(4.5) into (4.1), we get the uncertainty relations for various states:

$$(\Delta x \Delta p)_{n+2,n} = \sqrt{(n+1)(n+2)}|\mu||\nu| = \frac{\hbar}{2}\sqrt{(n+1)(n+2)} \times \left[1 + \frac{1}{\dot{\gamma}^2}\left[\frac{\dot{\eta}}{\eta} - B(t)\right]^2\right]^{1/2} , \quad (4.9)$$

$$(\Delta x \Delta p)_{n+1,n} = \frac{\hbar}{2}(n+1)\left[1 + \frac{1}{\dot{\gamma}^2}\left[\frac{\dot{\eta}}{\eta} - B(t)\right]^2\right]^{1/2} , \quad (4.10)$$

$$(\Delta x \Delta p)_{n,n} = (n+\tfrac{1}{2})\hbar\left[1 + \frac{1}{\dot{\gamma}^2}\left[\frac{\dot{\eta}}{\eta} - B(t)\right]^2\right]^{1/2} . \quad (4.11)$$

We define the quantum invariant operator corresponding to Eq. (2.20), i.e., the classical invariant quantity, as

$$I = \frac{1}{2}\left\{\left[\frac{\Omega^2}{\eta^2} + \frac{1}{A^2(t)}[B(t)\eta - \dot{\eta}]^2\right]x^2 + \frac{\eta}{A(t)}[B(t)\eta - \dot{\eta}](xp+px) + \eta^2 p^2\right\} , \quad (4.12)$$

where x and p are the position and momentum operators, respectively. I should naturally satisfy the condition Eq. (2.13),

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + \frac{1}{i\hbar}[I,H] = 0 , \quad (4.13)$$

with the Hamiltonian given by Eq. (2.1). Substitution of Eqs. (4.4)–(4.6) into (4.12) gives the expectation value

$$\langle m|I|n\rangle = (n+\tfrac{1}{2})\hbar\frac{\eta^2\dot{\gamma}}{A}\delta_{m,n} = (n+\tfrac{1}{2})\hbar\Omega\delta_{m,n} , \quad (4.14)$$

which is a time-invariant quantity corresponding to the classical invariant quantity.

V. SUMMARY

In this section we summarize the results obtained in the previous sections. We have obtained an explicit time-dependent invariant for the classical time-dependent quadratic Hamiltonian system. Even though this system is not closed, we can confirm whether or not the system is bound by using this invariant quantity. For example, the solution of the equation for a simple harmonic oscillator is a linear combination of $e^{i\omega_0 t}$ and $e^{-i\omega_0 t}$, and thus $\gamma = \omega_0 t$ and $\eta = \text{const}$. The invariant quantity Ω for the harmonic oscillator is $m\omega_0\eta^2$. Therefore this system is bound. For the case of a free particle, the classical solution is given by $C_1 e^{-\gamma t}$ or $C_2 t$. This means that the invariant quantity Ω is equal to zero, and thus this system is bound. With the same arguments, we can easily affirm that the damped harmonic oscillator is a bound system, where overdamped and underdamped oscillators are unbound.

We have obtained the propagator and wave function for the bound system. This wave function is discrete and expressed by the classical solution as a subsidiary condition. To obtain the expectation values and uncertainty relations, we have used the wave function [Eq. (3.23)] together with the invariant operator, which is inferred from classical one. The expectation values of the quantum mechanical invariant operator I also satisfy the uncertainty relation, which is time dependent. While these results are for the bound system, the unbound system can be solved in a similar fashion. A task for future work will be to evaluate the coherent states and squeezing relations for the systems.

ACKNOWLEDGMENTS

This work was supported by the Center for Thermal and Statistical Physics, KOSEF, under Contract No. 93-08-00-05, by the BSRI Program, Ministry of Education, Republic of Korea, and by the U.S. National Science Foundation under Grant No. CHE-9196214.

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