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Contributions to the binding, two-loop correction to the Lamb shift

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In this paper, I present an evaluation of the two-loop diagrams, including the closed electron loop, which contribute to the Lamb shift in $(\alpha/\pi)^2(Z\alpha)^5/n^3$ order. These corrections sum to 2.710 61(1) for S states, which gives a 37.41-kHz shift for the $1S$ state of the hydrogen atom.

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I. INTRODUCTION

Although the binding, two-loop correction to the Lamb shift has not yet been calculated in full, some partial results have been already obtained [1–4]. This correction is expected to be several kilohertz for the $2S$ hydrogen state, and hence its evaluation is necessary to match the precision of the latest experimental measurements [5,6]

$$L^1_{\rm expt}(2S-2P_{1/2}) = 1\,057\,845(9)\ \text{kHz}\,,$$

$$L^2_{\rm expt}(2S-2P_{1/2}) = 1\,057\,851(2)\ \text{kHz}\,, \qquad (1)$$

while a recent measurement [7], of the $1S$ Lamb shift,

$$L_{\rm expt}(1S) = 8172.84(9)\,\text{MHz}, \qquad (2)$$

is also on the verge of being sensitive to the two-loop binding correction.

The other higher-order corrections $\alpha(Z\alpha)^6$ [8], $\alpha(Z\alpha)^5 m/M$ [9], $(Z\alpha)^6 m/M$ [10] have been already calculated, except for the three-loop contribution determined by the slope of electron form factor at $q^2=0$, which is expected to give about 1 kHz. There remains, however, the problem of the proton charge radius, for which two experiments [11,12] give results that differ in the predicted $2S$ Lamb shift by about 18 kHz. This is the main source of uncertainty in the theoretical result, and should be removed either by a further electron scattering experiment or by a measurement of the Lamb shift for muonic hydrogen which is currently being prepared by Taqqu [13].

Experience from the higher-order QED calculation shows [14] that such results should be relied upon only after they have been performed by two independent groups. The results presented here are in agreement with the recent result of Eides and co-workers. My calculation relies upon a mixed analytical and numerical approach, and a program for symbolic manipulation [15] has been used extensively. To check the calculations I performed several tests for intermediate results, and all the more sophisticated analytical transformations have been performed by writing symbolic programs.

II. EVALUATION

All two-loop diagrams that contribute to the Lamb shift in order $(\alpha/\pi)^2(Z\alpha)^5$ are presented in Fig. 1. Groups I and IV contribute also to the lower $(\alpha/\pi)^2(Z\alpha)^4$ order. We start the evaluation with group VI, the simplest of the diagrams.

For the first one (the last diagram in Fig. 1), the energy shift E can be expressed as

$$E = \int \frac{d^3p}{(2\pi)^3}\,\rho(p)\,V^{(2)}(p), \qquad (3)$$

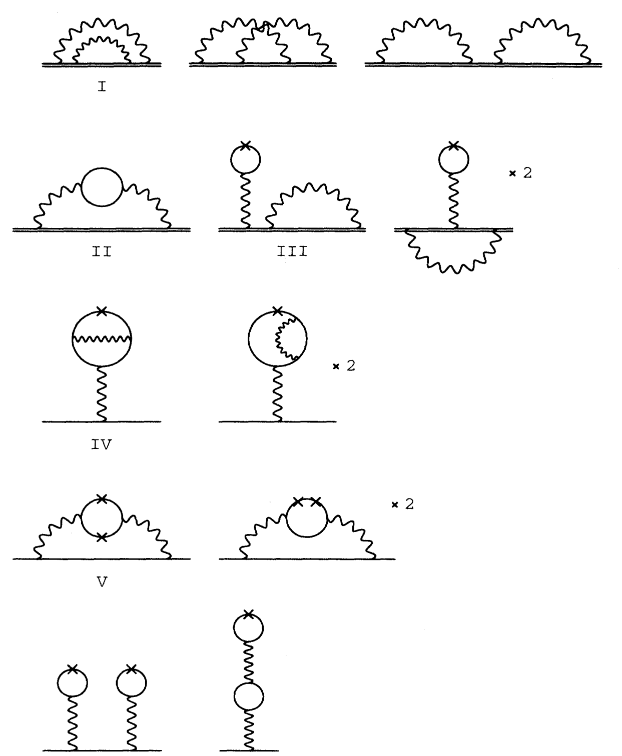

FIG. 1. Feynman diagrams representing two-loop corrections to the Lamb shift in $\left(\frac{\alpha}{\pi}\right)^2(Z\alpha)^5$ order; the double line means a Dirac propagator in a Coulomb field.

where ρ is a Schrödinger charge density and $V^{(2)}$ is a second-order (in α) correction to the Coulomb potential V (in the one-loop approximation),

$$V(p) = -\frac{4\pi\alpha}{p^2}\,\frac{1}{1+\bar\omega(-p^2)} = -\frac{4\pi\alpha}{p^2}\,[1-\bar\omega(-p^2)+\bar\omega^2(-p^2)-\cdots]\,. \tag{4}$$

We introduce here some functions that are also used for the evaluation of other diagrams:

$$\bar\omega(k^2) = \frac{\alpha}{\pi}\,k^2\,\bar u(-k^2) = \frac{\alpha}{\pi}\,k^2\int_4^\infty d(q^2)\,\frac{1}{q^2\,(q^2-k^2)}\,u(q^2)\,, \tag{5}$$

$$\bar u(p^2) = \frac{1}{3\,p^2}\left\{\frac{1}{3}+2\left(1-\frac{2}{p^2}\right)\times\left[\sqrt{1+\frac{4}{p^2}}\,\mathrm{arccoth}\left(\sqrt{1+\frac{4}{p^2}}\right)-1\right]\right\}, \tag{6}$$

$$u(q^2) = \frac{1}{3}\sqrt{1-\frac{4}{q^2}}\left(1+\frac{2}{q^2}\right)\,. \tag{7}$$

The energy shift can now be written as

$$E = -\int\frac{d^3p}{(2\pi)^3}\,\rho(p)\,\frac{4\pi\alpha}{p^2}\,\bar\omega^2(-p^2)\,. \tag{8}$$

Since for small p the function $\bar\omega$ behaves like p^2, the charge density can be approximated by

$$\rho(p) = \frac{(2\,\lambda)^4}{[p^2+(2\,\lambda)^2]^2} \approx \frac{(2\,\lambda)^4}{p^4}\,, \tag{9}$$

where $\lambda = m\,\alpha$.

Using (3) and (5), the contribution of this diagram to the energy shift becomes

$$E = -\left(\frac{\alpha}{\pi}\right)^2 (Z\,\alpha)^5\,\frac{32}{\pi}\int_0^\infty dp\,\bar u^2(p^2)\,. \tag{10}$$

The contribution from the second diagram can be written as follows:

$$E = \left\langle\phi\middle|V^{(1)}\frac{1}{\not p - m}V^{(1)}\middle|\phi\right\rangle. \tag{11}$$

In this order we may approximate the wave function $\phi(r)$ by $\phi(0)$, and obtain

$$E = \rho(0)\int\frac{d^3p}{(2\pi)^3}\,[V^{(1)}(p)]^2\,\mathrm{Tr}\left[\frac{1}{\not p-m}\,\frac{(\gamma_0+I)}{4}\right] = -\left(\frac{\alpha}{\pi}\right)^2 (Z\,\alpha)^5\,\frac{16}{\pi}\int_0^\infty dp\,\bar u^2(p^2)\,. \tag{12}$$

After the summation of terms (10) and (12), and an analytical integration [16], we obtain

$$E_{\rm VI} = -\left(\frac{\alpha}{\pi}\right)^2 (Z\,\alpha)^5\,\frac{48}{\pi}\int_0^\infty dp\,\bar u^2(p^2) = -\left(\frac{\alpha}{\pi}\right)^2 (Z\,\alpha)^5\,\frac{23}{378}\pi = -\left(\frac{\alpha}{\pi}\right)^2 (Z\,\alpha)^5\,0.191\,155\,. \tag{13}$$

We consider now the diagrams from group IV. They sum to a gauge-independent quantity which can be expressed by the two-loop vacuum polarization $\bar\omega^{(2)}$,

$$E_{\rm IV} = \int\frac{d^3p}{(2\pi)^3}\,\rho(p)\,\frac{4\pi\alpha}{p^2}\,\bar\omega^{(2)}(-p^2)\,. \tag{14}$$

$\bar\omega^{(2)}$ has been calculated analytically in [17] with the following result:

$$\bar\omega^{(2)}(-p^2) = \left(\frac{\alpha}{\pi}\right)^2 (-p^2)\int_4^\infty d(q^2)\,\frac{1}{q^2(q^2+p^2)}\,u^{(2)}(q^2)\,, \tag{15}$$

$$\begin{aligned} u^{(2)}(q^2) = \frac{1}{3}\Bigg\{&\left[\frac{11}{16}\,(3-\delta^2)\,(1+\delta^2)+\frac{\delta^4}{4}\right]\ln\left(\frac{1+\delta}{1-\delta}\right) \\ &+(3-\delta^2)\,(1+\delta^2)\left[L_2\left(-\frac{1-\delta}{1+\delta}\right)+2L_2\left(\frac{1-\delta}{1+\delta}\right)+\frac{3}{2}\ln\left(\frac{1+\delta}{1-\delta}\right)-\ln\left(\frac{1+\delta}{1-\delta}\right)\ln(\delta)\right] \\ &+\frac{3}{8}\,\delta\,(5-3\delta^2)+\frac{3}{2}\,\delta(3-\delta^2)\ln\left(\frac{1-\delta^2}{4}\right)-2\,\delta\,(3-\delta^2)\ln(\delta)\Bigg\}\,, \end{aligned} \tag{16}$$

where

$$\delta = \sqrt{1-\frac{4}{q^2}}\,. \tag{17}$$

The first term in the expansion of $\bar\omega^{(2)}$ in p^2 is proportional to p^2 and gives exactly the $\alpha^2\,(Z\alpha)^4$ order contribution to the energy. We separate it out by making the following replacement in (15):

$$\frac{1}{q^2+p^2} \to \frac{1}{q^2+p^2}-\frac{1}{q^2}\,, \tag{18}$$

so that we can make the approximation (9) and obtain

$$E_{\rm IV} = \left(\frac{\alpha}{\pi}\right)^2 (Z\,\alpha)^5 \,\frac{16}{\pi} \int_{-\infty}^{\infty} dp\, \frac{1}{p^4}\,(-p^2) \int_4^{\infty} \frac{d(q^2)}{q^2} \times \left(\frac{1}{q^2+p^2} - \frac{1}{q^2}\right) u^{(2)}(q^2)\,. \tag{19}$$

After integration with respect to p, $E_{\rm IV}$ becomes

$$E_{\rm IV} = \left(\frac{\alpha}{\pi}\right)^2 (Z\,\alpha)^5\, 16 \int_4^{\infty} d(q^2)\,\frac{1}{q^5}\,u^{(2)}$$

$$= \left(\frac{\alpha}{\pi}\right)^2 (Z\,\alpha)^5\, 4 \int_0^1 d\delta\,\delta\,\sqrt{1-\delta^2}\,u^{(2)}(\delta)\,, \tag{20}$$

where δ is defined in (17). The remaining integral is done analytically [16] with the result

$$E_{\rm IV} = \left(\frac{\alpha}{\pi}\right)^2 (Z\,\alpha)^5 \left(\frac{15\,647}{13\,230}\pi - \frac{25}{63}\pi^2 + \frac{52}{63}\pi\ln(2)\right)$$

$$= \left(\frac{\alpha}{\pi}\right)^2 (Z\,\alpha)^5\; 1.596\,396. \tag{21}$$

The expression for the energy shift related to diagram II is the following:

$$E_{\rm II} = -e^2 \int \frac{d^4k}{(2\pi)^4 i}\,\frac{\bar\omega(k^2)}{k^2} \left\langle \bar\phi \left| \gamma^\mu \frac{1}{\not p - \not k - V\gamma^0 - m} \right| \phi \right\rangle, \tag{22}$$

where $V = -\alpha/r$. The above matrix element has been calculated analytically in [8] by direct integration with respect to the electron momentum. The result in α^5 order denoted by f is the following:

$$\left\langle \bar\phi \left| \gamma^\mu \frac{1}{\not p - \not k - V\gamma^0 - m} \right| \phi \right\rangle^{(5)} = \frac{\alpha^5}{m} f(\omega, k)\,, \tag{23}$$

$$\begin{aligned} f(\omega,k) = {} & \frac{512\,X^5\,(1+\omega)}{3\,(k^2+X^2)^6} - \frac{896\,X^3}{3\,(k^2+X^2)^5} - \frac{512\,\omega\,X^3}{3\,(k^2+X^2)^5} + \frac{128\,\omega^2\,X^3}{(k^2+X^2)^5} - \frac{256\,X^5}{3\,(k^2+X^2)^5} + \frac{128\,X}{(k^2+X^2)^4} - \frac{16\,\omega\,X}{(k^2+X^2)^4} \\ & - \frac{40\,\omega^2\,X}{(k^2+X^2)^4} + \frac{8\,\omega^3\,X}{(k^2+X^2)^4} + \frac{520\,X^3}{3\,(k^2+X^2)^4} - \frac{72\,\omega\,X^3}{(k^2+X^2)^4} - \frac{56\,X}{(k^2+X^2)^3} + \frac{40\,\omega\,X}{(k^2+X^2)^3} \\ & + \frac{4\,i\,(1+\omega)}{k\,(k^2+X^2)^2}\,\ln\left(\frac{-i\,k+X}{i\,k+X}\right), \end{aligned} \tag{24}$$

where $\omega = k^0, k = \sqrt{\mathbf{k}^2}, X = \sqrt{2\,\omega - \omega^2}$ and we set the electron mass $m = 1$. Using (22) and (23) $E_{\rm II}$ becomes

$$E_{\rm II} = -\left(\frac{\alpha}{\pi}\right)^2 (Z\,\alpha)^5\,\frac{1}{i\,\pi} \int_{-\infty}^{\infty} d\omega \int_0^{\infty} dk\,k^2\,\bar u(-\omega^2+k^2)\,f(\omega,k)\,, \tag{25}$$

where $\bar u$ is given in (6). To make the further numerical evaluation easier we change the contour of ω integration to

$$\omega = 1 + i\,w\,, \tag{26}$$

and therefore

$$X^2 = 1 + w^2\,, \tag{27}$$

and obtain

$$E_{\rm II} = \left(\frac{\alpha}{\pi}\right)^2 (Z\,\alpha)^5\; 0.229\,053\,. \tag{28}$$

The diagrams from the III group have an interesting physical interpretation. The energy contribution $E_{\rm III}$ can be written as

$$E_{\rm III} = \int \frac{d^3p}{(2\pi)^3}\,\rho^{(1)}(p)\,V^{(1)}(p)\,, \tag{29}$$

i.e., as a product of the one-loop correction to the electron charge density and the one-loop correction to the vacuum polarization. This correction to the charge density has been evaluated in [18] and is given by the following expression:

$$\rho^{(1)}(p) = \frac{8\,\alpha^5}{\pi\,p^2} \int_0^{\infty} d(q^2)\,\frac{f(q^2)}{q^2+p^2}\,, \tag{30}$$

$$f(q^2) = J + \frac{q^2}{4}\left(\frac{1}{1+q^2} - J\right) + \frac{4}{q^2}(J-1) - 1 + \Theta(q-2)\left(\frac{4}{q^2}+1\right)\left(\frac{1}{q^2\sqrt{1-\frac{4}{q^2}}} + \sqrt{1-\frac{4}{q^2}}\right), \tag{31}$$

$$J = \frac{1}{q}\left[\arctan(q) - \Theta(q-2)\,\arccos\left(\frac{2}{q}\right)\right]. \tag{32}$$

After using (29) and (30), $E_{\rm III}$ becomes

$$E_{\rm III} = \int \frac{d^3p}{(2\pi)^3}\,\frac{8\,\alpha^5}{\pi\,p^2} \int_0^{\infty} d(q^2)\,\frac{f(q^2)}{q^2+p^2}\,\frac{4\,\pi\,\alpha}{p^2}\,\frac{(-\alpha\,p^2)}{\pi} \times \int_4^{\infty} d(k^2)\,\frac{1}{k^2\,(k^2+p^2)}\,u(k^2)\,. \tag{33}$$

Integration with respect to p is performed by means of the formula

$$\int \frac{d^3p}{(2\pi)^3} \frac{4\pi}{p^2\,(p^2+k^2)\,(p^2+q^2)} = \frac{1}{q\,k\,(q+k)}, \quad (34)$$

and $E_{\rm III}$ becomes

$$E_{\rm III} = -\left(\frac{\alpha}{\pi}\right)^2 (Z\,\alpha)^5\; 32 \int_0^\infty dq\, f(q^2)\, h(q)\,, \quad (35)$$

where

$$h(q) = \int_2^\infty dk\, \frac{u(k^2)}{k^2\,(q+k)}\,. \quad (36)$$

Taking u from (7) the k integration gives

$$h(q) = \frac{1}{6\,q}\left\{ \frac{\pi}{32}\left(9 - \frac{128}{q^4} - \frac{48}{q^2}\right) + \frac{1}{3\,q}\left(5 + \frac{12}{q^2}\right) \right. \quad (37)$$

$$\left. -\frac{1}{q}\left(1+\frac{2}{q^2}\right)\sqrt{1-\frac{4}{q^2}}\,\ln\left(\frac{1+\sqrt{1-\frac{4}{q^2}}}{1-\sqrt{1-\frac{4}{q^2}}}\right)\right\}. \quad (38)$$

The final q integration is done numerically with the result

$$E_{\rm III} = \left(\frac{\alpha}{\pi}\right)^2 (Z\,\alpha)^5\, 1.920\,576\,. \quad (39)$$

We pass now to the last diagrams with the closed electron loop, i.e., to group V (of Fig. 1). Because their evaluation is more difficult, we present these calculations in more detail. The energy shift E_V is given by the following expression:

$$E_V = e^4 \int \frac{d^4k}{(2\pi)^4} \int \frac{d^3q}{(2\pi)^3}$$

$$\times \left\langle \bar\phi \middle| \gamma^\mu \frac{1}{\not p - \not k - m}\gamma^\nu \middle| \phi \right\rangle \frac{1}{k^4}\, \frac{(4\,\pi\,\alpha)^2}{q^4}\, \Pi_{\mu\nu}\,, \quad (40)$$

where

$$\Pi_{\mu\nu} = \int \frac{d^4p}{(2\pi)^4} {\rm Tr}\left\{ \gamma_\mu \frac{1}{\not p - m}\gamma_0 \frac{1}{\not p + \not q - m}\gamma_\nu \frac{1}{\not p + \not k + \not q - m}\gamma_0 \frac{1}{\not p + \not k - m} \right.$$
$$+\gamma_\mu \frac{1}{\not p - m}\gamma_\nu \frac{1}{\not p + \not k - m}\gamma_0 \frac{1}{\not p + \not k + \not q - \gamma_0}\, \frac{1}{\not p + \not k - m}$$
$$\left. +\gamma_\mu \frac{1}{\not p - m}\gamma_0 \frac{1}{\not p + \not q - m}\gamma_0 \frac{1}{\not p - m}\gamma_\nu \frac{1}{\not p + \not k - m}\right\}. \quad (41)$$

The general expression for the $\Pi_{\mu\nu\rho\sigma}$ was derived in the classical paper [19], and a similar problem of the forward scattering of light by a Coulomb field has been considered in [20]. Our problem differs from the light scattering by the fact that the photon is not on mass shell $k^2 \neq 0$, and the former general expression is too complicated to be sure that we will not make an error during its rewriting. Thus, we calculate our expression from the beginning. The matrix element in (40) can be evaluated by taking the wave function ϕ on mass shell. After symmetrization with respect to the sign of k, one obtains

$$\left\langle \bar\phi \middle| \gamma^\mu \frac{1}{\not p - \not k - m}\gamma^\nu \middle| \phi \right\rangle_{\rm sym}$$

$$= \phi(0)^2\, \frac{2\,k^2}{k^4 - 4(kt)^2}\left(t^\mu\, t^\nu + g^{\mu\nu}\frac{(kt)^2}{k^2}\right), \quad (42)$$

where t is a timelike vector $t = (m, 0, 0, 0)$.

After evaluating the traces of (41) we pass to Euclidean space by changing the integration contour. The integration with respect to p in (41) is carried out by introducing Feynman parameters and the Pauli-Villars mass regulator M. For the first term in (41) this requires three parameters a, b, c. Since the denominators here are linearly dependent upon p, after the p integration one integrates with respect to the parameter whose square does not appear in the denominator; for example, c. The denominators in these three terms are then the same and have the form

$$\frac{\Theta(1-a-b)}{k^2\,a(1-a) + q^2\,b(1-b) + 2(kq)\,ab}\,. \quad (43)$$

The terms in the numerator can thus be added together and the M-dependent terms cancel out, leaving a polynomial of k, q, a, b. A problem arises in its evaluation, because of the presence of k^4, q^4 in the denominator in (40). Before the integration with respect to a, b the whole expression is divergent in k, q. This is the reason why we do not now continue the integration by introducing another Feynman parameter. After integration over a and b, it is apparent from the effective Heisenberg-Euler Lagrangian that the numerator of our expression should behave for small k or q like k^2 or q^2. However, this requires a subtraction of the finite term at $k = q = 0$. This interesting additional subtraction is required by the renormalization procedure. We have checked by analytical integration that in the case of $k = 0$ or $q = 0$, the expression vanishes after this subtraction. We solve the divergence problem by dividing our expression into two parts. The first is obtained by putting $k \cdot q = 0$ and the second is the remainder.

In the second part we symmetrize in q and obtain terms of the following form:

$$\left[\frac{1}{[1+k^2\,a(1-a)+q^2\,b(1-b)]^2-4(k\cdot q)^2\,a^2\,b^2} - \frac{1}{[1+k^2\,a(1-a)+q^2\,b(1-b)]^2}\right]\mathcal{P}(k,q,a,b), \qquad (44)$$

where $\mathcal{P}$ denotes a polynomial. Next, we integrate with respect to all angles by means of the formula

$$\int \frac{d\Omega_u}{2\pi^2}\,\frac{1}{\alpha^2+u_x^2}\,\frac{1}{\beta^2-u_y^2} = \frac{4}{\alpha\,\beta}\,\mathrm{arctanh}\left(\frac{1}{\beta+\sqrt{\beta^2-1}}\,\frac{1}{\alpha+\sqrt{1+\alpha^2}}\right), \qquad (45)$$

where u is a unit vector with u_x, u_y, x, and y components, and $d\Omega_u$ denotes the integration over the three-dimensional u-sphere. The remaining integration with respect to k, q, a, b is done numerically. The separation of the terms with $k\cdot q=0$ allows the divergence in k and q to be removed separately from each term, thus avoiding the numerical instability in the evaluation of this integral.

TABLE I. Results of the calculation of binding two-loop corrections.

No.	This paper	Eides and co-workers
I	?	?
II	-0.229 053(1)	-0.2290
III	1.920 576(1)	1.920
IV	1.596 396	1.5959
V	-0.386 15(1)	?
VI	-0.191 155	-0.1913

The first part, because the absence of $k\cdot q$, is open to further analytical integration. We integrate first with respect to q, then b, and finally with respect to the (k,t) angle. The remaining two-dimensional integral with respect to k and a is performed numerically. From the sum of the two parts we obtain

$$E_{\mathrm{V}} = -\left(\frac{\alpha}{\pi}\right)^2 (Z\,\alpha)^5\;0.386\,15(1)\,. \qquad (46)$$

The evaluation of the remaining diagrams from the first group will be presented in a subsequent paper; these diagrams seem to be the most difficult in the evaluation. Our starting point will be the following formula, which we give without proof:

$$E_{\mathrm{I}} = \phi(0)^2\int\frac{d^3p}{(2\pi)^3}\,\frac{(4\,\pi\,\alpha)^2}{p^4}\mathrm{Tr}\left[\left(\Lambda_R(0,p,0)+2\,\Gamma_R(0,p)\frac{1}{\not p-m}+\Sigma_R(p)\frac{1}{(\not p-m)^2}\right)\frac{(\gamma_0+I)}{4}\right]_S, \qquad (47)$$

where Λ, Γ, and Σ are two-, one-, and zero-vertex functions with implicit indices equal to 0, which are generated by the expansion of the diagrams I in powers of the Coulomb field, and S means the separation of the constant term for $p=0$.

III. CONCLUSION

The result of our calculations is summarized in Table I. The evaluated corrections sum to

$$\delta E = \left(\frac{\alpha}{\pi}\right)^2 (Z\,\alpha)^5\;2.710\,61(1) = 37.41\ \mathrm{kHz} \qquad (48)$$

for the $1S$ state. In the third column of this table we give the known results obtained by Eides and co-workers [1–3] multiplied by π to match our convention. There are only small differences caused, we believe, by the numerical integration, and these results thus appear to be verified. The remaining uncalculated diagrams, which give a contribution of the same order in α, are diagrams from the first group in Fig. 1 and the three-loop contribution. The latter can be calculated using the on-mass shell approximation, and the numerical method developed by Kinoshita [21] or the analytical approach by Remiddi and co-workers [22] are suited for its evaluation.

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