PHYSICAL REVIEW A VOLUME 48, NUMBER 4 OCTOBER 1993

Generalized Coleman-Hepp model and quantum coherence

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(Received 3 May 1993)

The modified Coleman-Hepp model proposed recently by Nakazato and Pascazio [Phys. Rev. Lett. **70**, 1 (1993)] is further generalized so that an ultrarelativistic particle interacts with a linear array of N spins, each of generic magnitude s rather than $\frac{1}{2}$. We study the evolution of an appropriately defined measure $C(t)$ of quantum coherence, with t being time, and elucidate the roles played by N and s on the process of decoherence. It is shown that N and s appear in the form of the product Ns in $C(\infty)$ at zero temperature, but not in general. It is also noted that the effect of temperature shows up only for $s \geq 1$. We study the temperature dependence of $C(\infty)$ in detail, and point out in particular that it is not necessarily a monotonic function of temperature. The limiting cases of $s \to \infty$ and/or $N \to \infty$ are also considered to investigate how the present model tends to a boson model, that is, a model in which the spins are replaced by harmonic oscillators.

PACS number(s): 03.65.Bz, 03.65.Fd

I. INTRODUCTION

In a paper [1] dedicated to the sixtieth birthday of M. Fierz, Hepp introduced and analyzed "the Coleman model" which "can be considered as a caricature of an electron in the one-dimensional motion, whose spin is measured by the result of a local interaction with an infinite spin array." The magnitude of each spin was taken to be $\frac{1}{2}$. This model, hereafter to be called the Coleman-Hepp (CH) model, was intended also to mimic the development of a photoemulsion in terms of a Hamiltonian which does not depend on time. Exploiting the infinite size of the system, Hepp identified a variable which may be regarded as corresponding to "the macroscopic pointer position," and derived "a reduction of the wave packet with respect to all local observables." Hepp's C^*-algebraic language was translated into a common physicists' language by Bell [2], who critically analyzed the meaning of the reduction of the wave packet as derived by Hepp. In particular, Bell pointed out that the reduction claimed by Hepp occurs only after an infinite duration of time, which is unphysical. Later, Cini [3] introduced a model which replaces the above infinite array of spins by a single harmonic oscillator (or a single spin of large magnitude), and discussed its measurement-theoretical implication. In particular, he emphasized that the so-called reduction of the wave packet is "a consequence—though not an exact one but valid to a very high degree of accuracy—of the laws of quantum mechanics," the degree of accuracy being the higher the greater the number of degrees of freedom (e.g., the magnitude of spin) involved. (The Cini model can also be regarded as a modified version of the model considered by Cini *et al.* [4], which in turn is a modified version of the Haake-Weidlich model [5] consisting of a harmonic oscillator interacting with a large number of two-level atoms.) A finite-array version of the CH model was studied by Kudaka, Matsumoto, and Kakazu [6] from a generalized-coherent-state point of view, and also by Machida and Namiki [7], Namiki [8], Namiki and Pascazio [9], and Nakazato and Pascazio [10], who discussed its relationship with Cini's model and also studied it from the point of view of " a many-Hilbert-space theory" put forward by Machida and Namiki [11] with its mathematical aspect explained by Araki [12]. Very recently, Nakazato and Pascazio [13,14] proposed a significant modification of the CH model, where an energy exchange between the electron and the spin array is taken into account, and discussed its relation with the quantum measurement process. In these works [6–14] particular attention was paid to the limit of infinite array and/or to the asymptotic feature as embodied in the S matrix.

In spite of these excellent works, there are more to be fruitfully learned from the CH model or its variant. In the present paper we generalize the Nakazato-Pascazio [13] version of the CH model so that the magnitude of each spin is arbitrary, and study it from a somewhat different point of view. That is, we analyze the time evolution of an appropriately defined measure of quantum coherence paying due attention to the effect of the initial state of the system, namely, the initial wave packet of the electron and the initial ensemble of the spin array. We consider the case of finite array consisting of N spins each of magnitude s, although we briefly touch upon the limits of $N \to \infty$ or $s \to \infty$ as well. We shall obtain an explicit picture on how the *decoherence* process proceeds. In particular we investigate the role played by N and s as well as by the initial state. Although the S matrix depends only on the product Ns, N and s play independent roles in the docoherence process. It is also shown that the temperature of the initial spin array is significant if (and only if) $s \geq 1$.

II. GENERALIZED COLEMAN-HEPP MODEL

The purpose of this section is to introduce our model, which is motivated by the Coleman-Hepp (CH) model and more directly by its extended version proposed by Nakazato and Pascazio [13]. Let us begin with reviewing these models. We shall somewhat change the standard notation into the one which is more appealing to physical intuition.

In the CH model, a particle with spin $\frac{1}{2}$ moves in the one-dimensional space with a positive constant velocity v in the x direction. It interacts with a *detector* consisting of N spins. The lth spin ($l=l,\ldots,N$) is located at the spatial position x_l, has magnitude $\frac{1}{2}$, and is represented by Pauli matrices σ_l^α ($\alpha=1,2,3$). In our convention the superscript and the subscript denote the spin component and the spatial location, respectively. (The term detector is used here just for brevity without any further implication.) The interaction is such that the particle at the position x exerts a magnetic field $B(x-x_l)$ on the lth spin if and only if particle's spin is up. (By convension B has the dimension of frequency.) Let X and P be the position and the wave-number operators of the particle, respectively, so that

$$[X,P]=i \ , \tag{2.1}$$

and τ^α ($\alpha=1,2,3$) be the particle's spin operator. The Hamiltonian, whose dimension is taken to be that of frequency, of the CH model is then given by

$$H_{\rm CH}=vP+\frac{1}{2}\mathcal{P}_+\sum_l B(X-x_l)\sigma_l^2 \ , \tag{2.2}$$

where $\mathcal{P}_+$ is the projector onto the particle-spin-up subspace, to be called the *plus subspace* for short:

$$\mathcal{P}_\pm=(1\pm\tau^3)/2 \ . \tag{2.3}$$

Throughout this paper, it is to be understood that

$$\sum_l=\sum_{l=1}^N \tag{2.4}$$

unless the range of the summation is stipulated otherwise. The choice of σ^2, rather than σ^1, in Eq. (2.2) is merely a matter of convention; it affects unobservable phases only. In the CH model there is no exchange of energy between the particle and the detector. In order to remedy this somewhat unphysical feature Nakazato and Pascazio (NP) [13] proposed the following Hamiltonian, where ω is a positive constant;

$$H_{\rm NP}=vP+\frac{1}{2}\omega\sum_l(\sigma_l^3+1) + \frac{i}{4}\mathcal{P}_+\sum_l B(X-x_l)\sigma_l^-\exp(i\omega X/v)+\text{H.c.} \ , \tag{2.5}$$

where $\sigma^-=\sigma^1-i\sigma^2$ and H.c. stands for the Hermitian conjugate. This is identical to Eq. (4) of Ref. [13] apart from the projector $\mathcal{P}_+$, which we prefer to be present in order to facilitate the study of coherence [cf. Eq. (3.21)]. This is the starting point of our generalization. In the last expression it is straightforward to replace $\sigma_l^\alpha/2$ by the spin operator S_l^α of magnitude s_l. There is no difficulty either in making $B(x)$ and ω depend on l. For the sake of definiteness we suppose that each $B_l(x)$ has a compact support:

$$B_l(x)=0 \quad \text{for } |x|>\delta \ , \tag{2.6}$$

where δ is a positive constant. We do not stipulate the value of x_l except that

$$\Delta+\delta<x_1\le\cdots\le x_N<L-\delta \ , \tag{2.7}$$

where Δ and L are positive constants. In addition we suppose that the particle's spin undergoes free precession with frequency Ω in the absence of interaction with the detector. With these generalizations, we arrive at our model described by the Hamiltonian

$$H=H_\omega^0+\mathcal{P}_+H_\omega^1 \ , \tag{2.8a}$$

where

$$H_\omega^0=H^0+\sum_l\omega_l(S_l^3+s_l) \ , \tag{2.8b}$$

$$H^0=vP+\tfrac{1}{2}\Omega\tau^3 \ , \tag{2.8c}$$

$$H_\omega^1=\frac{i}{2}\sum_l B_l(X-x_l)S_l^-\exp(i\omega_lX/v)+\text{H.c.} = \sum_l \mathbf{S}_l\cdot\mathbf{B}_l(X,x_l) \ . \tag{2.8d}$$

In the last expression we have introduced the notation

$$\mathbf{S}=(S^1,S^2,S^3) \ , \tag{2.9}$$

$$\mathbf{B}(X,x)=B(X-x)\left[-\sin\frac{\omega X}{v},\cos\frac{\omega X}{v},0\right] \ . \tag{2.10}$$

By construction, this model contains the NP model (and *a fortiori* the CH model) as a special case.

The above generalization is mathematically so straightforward that it does not spoil the exact solvability of the problem. As we shall see, however, it allows extra parameters leading to richer physics, and it also clarifies the condition of appearance of some particular mathematical structures (e.g., generalized coherent states) encountered in the NP (or CH) model.

III. BASIC PROPERTIES OF THE MODEL

Let $|x\rangle, |p\rangle$, and $|\pm\rangle$ be eigenstates of X, P, and τ^3, respectively, such that

$$\begin{aligned} &X|x\rangle=x|x\rangle, \quad \langle x|x'\rangle=\delta(x-x') \ , \\ &P|p\rangle=p|p\rangle, \quad \langle p|p'\rangle=\delta(p-p') \ , \\ &\tau^3|\pm\rangle=\pm|\pm\rangle, \quad \tau^\pm|\mp\rangle=2|\pm\rangle, \quad \langle+|+\rangle=1 \ , \end{aligned} \tag{3.1}$$

and let $\{|s,m\rangle|m=-s,\ldots,s\}$ be the standard basis of the spin-s representation of SU(2) associated with the spin operator $\mathbf{S}$,

$$\begin{aligned} &S^3|s,m\rangle=m|s,m\rangle \ , \\ &S^\pm|s,m\rangle=\sqrt{(s\pm m+1)(s\mp m)}|s,m\pm1\rangle \ . \end{aligned} \tag{3.2}$$

When we refer to the lth spin $\mathbf{S}_l$, we shall affix subscript l to the ket; for instance $|s_l,m\rangle_l$. For the moment let us focus on one of the spins, omit writing the subscript, and introduce the notation

$$|p;s,m\rangle = |p\rangle \otimes |s,m\rangle \ . \tag{3.3}$$

Then

$$S^{\pm}\exp(\mp i\omega X/v)|p;s,m\rangle \propto |p\mp\omega/v;s,m\pm1\rangle \ . \tag{3.4}$$

Hence, for any s, H^1_ω describes an energy-exchange process, as observed by NP in the case of $s=\frac{1}{2}$.

By virtue of the relation

$$[X/v,H^0]=i \tag{3.5}$$

the operator X/v plays the role of the *time operator.* Hence the vector $\mathbf{B}(X,x)$ defined by Eq. (2.10) represents a resonant rotating field in the usual sense used in the theory of magnetic resonance or quantum optics; interaction Hamiltonian (2.8d) is the so-called rotating-wave approximant to a more general one in which ω_l are different from those in free Hamiltonian (2.8b). In turns out to be convenient to define the time-dependent *tipping angle*

$$\theta_l(x,t)=\int_0^t dt' B_l(x-x_l+vt') \ , \tag{3.6a}$$

as well as

$$\theta_l(x)=\frac{1}{v}\int_{-\infty}^{x} dx' B_l(x'-x_l) \tag{3.6b}$$

and the *asymptotic tipping angle*

$$\theta_l=\theta_l(\infty)=\frac{1}{v}\int_{-\infty}^{\infty} dx\, B_l(x) \ . \tag{3.6c}$$

As the particle passes by, the lth spin is subjected to the magnetic-field pulse and is rotated by the above amount. Note that under conditions (2.6) and (2.7) we have the identity

$$\theta_l(x,t)=\theta_l(x+vt) \quad \text{if } x<\Delta \ . \tag{3.7}$$

The fundamental quantity of interest is the time-evolution operator. Since $\mathcal{P}_\pm$ commute with H^0_ω and H^1_ω, it is obvious that

$$\exp(-iHt)=\exp(-iH^0_\omega t)\{\mathcal{P}_- + U_\omega(t)\mathcal{P}_+\} \ , \tag{3.8}$$

where

$$U_\omega(t)=\exp(iH^0_\omega t)\exp[-i(H^0_\omega+H^1_\omega)t] \tag{3.9}$$

is the time-evolution operator (in the interaction picture) restricted to the plus subspace. The observation made in the preceding paragraph motivates us to work in the *rotating frame,* a somewhat different procedure from that of NP. With the aid of the unitary operator

$$\mathcal{R}_l(X)=\exp(-i\omega_l S_l^3 X/v) \ , \tag{3.10a}$$

$$\mathcal{R}(X)=\prod_l \mathcal{R}_l(X) \ , \tag{3.10b}$$

one sees that

$$U_\omega(t)=\mathcal{R}(X)U(t)\mathcal{R}(X)^\dagger \ , \tag{3.11}$$

$$\exp(-iHt)=\mathcal{R}(X)\exp(-iH^0 t) \times\{\mathcal{P}_- + U(t)\mathcal{P}_+\}\mathcal{R}(X)^\dagger \ . \tag{3.12}$$

Here H^0 is defined by Eq. (2.8c) and

$$U(t)=\exp(iH^0 t)\exp[-i(H^0+H^1)t] =\exp(ivPt)\exp[-i(vP+H^1)t] \ , \tag{3.13}$$

where

$$H^1=\sum_l B_l(X-x_l)S_l^2 \ . \tag{3.14}$$

This $U(t)$ is nothing but the plus-subspace time-evolution operator for the CH model generalized to arbitrary spin and l-dependent $B(x)$. [To be precise, we have omitted a physically insignificant overall c-number phase factor $\exp(-i\sum_l \omega_l s_l t)$ on the right-hand side of Eq. (3.12).]

A particularly useful physical quantity is the expectation value of τ^-:

$$\langle\tau^-\rangle(t)=\langle\Psi|e^{iHt}\tau^- e^{-iHt}|\Psi\rangle \ , \tag{3.15}$$

where $|\Psi\rangle$ is the normalized initial state (at $t=0$) of the entire system. Let us suppose that

$$|\Psi\rangle=\frac{1}{\sqrt{2}}(|+\rangle+|-\rangle)\otimes|\psi\rangle\otimes|D\rangle \ , \tag{3.16}$$

where $|\psi\rangle$ is the initial orbital state of the particle and $|D\rangle$ is the initial state of the detector. We shall use the notation

$$|\psi\rangle\otimes|D\rangle=|\psi;D\rangle=\int dx\,\psi(x)|x;D\rangle =\int dp\,\tilde{\psi}(p)|p;D\rangle \ , \tag{3.17}$$

where the initial orbital wave function

$$\psi(x)=\langle x|\psi\rangle \tag{3.18}$$

is supposed to be localized around the origin with width Δ, which we take to be equal to the constant introduced at Eq. (2.7):

$$\psi(x)=0 \quad \text{for } |x|>\Delta \ . \tag{3.19}$$

The corresponding momentum amplitude has been written as $\tilde{\psi}(p)$. By use of Eq. (3.8) and the relation $\tau^-\mathcal{P}_-=0$, we find

$$\langle\tau^-\rangle(t)=C(t)e^{-i\Omega t} \ , \tag{3.20}$$

where

$$C(t)=\langle\Psi|U_\omega(t)\tau^-|\Psi\rangle =\langle\psi;D|U_\omega(t)|\psi;D\rangle \ . \tag{3.21}$$

It has the property that

$$C(0)=1, \quad |C(t)|\le 1 \ . \tag{3.22}$$

Thus, $|C(t)|$ may be regarded as a degree of quantum coherence at time t. For brevity we shall call $C(t)$ *the coherency.* (This is *not* a standard terminology.) It is not the unique measure of coherence but a compact and appealing one both physically and mathematically. In a more formal discussion one may consider the S matrix. Its nontrivial operation is closed within the plus subspace. The S matrix restricted to the plus subspace, denoted by $\mathcal{S}$, is defined by

$$\mathcal{S}=U_\omega(\infty) \ . \tag{3.23}$$

The asymptotic value of $C(t)$ is therefore given by

$$C(\infty)=\langle\psi;D|\mathcal{S}|\psi;D\rangle\ . \tag{3.24}$$

This quantity is essentially the same as "the visibility" discussed by NP. For the latter to be given by the right-hand side of Eq. (3.24), however, a few assumptions and some arguments are needed (cf. the Appendix of Ref. [14]), while our formulas for $C(t)$ are exact and easily obtainable; this is the virtue of having the projector $\mathcal{P}_+$ in Eq. (2.8a).

We conclude this section by slightly generalizing the definition (3.21) of the coherency; if the detector was not in a pure state initially but in a mixture described by a density operator ρ_D, then

$$C(t)=\langle\psi|\mathrm{tr}\rho_D U_\omega(t)|\psi\rangle\ , \tag{3.25}$$

where the trace denoted by tr is to be taken over the detector's degrees of freedom alone. This formula shall be used in Sec. VII.

IV. TIME EVOLUTION OF COHERENCY

The time-evolution operator $U(t)$ can be evaluated by the standard procedure [1]; solving the equation

$$i\frac{\partial}{\partial t}U(t)=H^1(t)U(t)\ , \tag{4.1}$$

$$H^1(t)=\exp(ivPt)H^1\exp(-ivPt)$$
$$=\sum_l B_l(X+vt-x_l)S_l^2\ , \tag{4.2}$$

one finds

$$U(t)=\exp\left[-i\sum_l\theta_l(X,t)S_l^2\right]\ , \tag{4.3}$$

where the function $\theta_l(x,t)$ is defined by Eq. (3.6a). Note that its argument in Eq. (4.3) is the operator X. Equations (3.10), (3.11), and (4.3) then lead to

$$U_\omega(t)|\psi;D\rangle=\int dx\ \psi(x)R(x,t)|x;D\rangle\ , \tag{4.4a}$$

where

$$R(x,t)=\prod_l\mathcal{R}_l(x)\exp[-i\theta_l(x+vt)S_l^2]\mathcal{R}_l(x)^\dagger\ , \tag{4.4b}$$

and we have taken account of condition (3.19) to use identity (3.7). Asymptotically (i.e., $t\to\infty$), in particular, $\theta_l(x+vt)$ can be replaced by the constant θ_l defined by Eq. (3.6c), because in Eq. (4.4) x is restricted to the range $|x|<\Delta$. Thus

$$\mathcal{S}|\psi;D\rangle=\int dx\ \psi(x)R(x)|x;D\rangle\ , \tag{4.5a}$$

where

$$R(x)=\prod_l\mathcal{R}_l(x)\exp(-i\theta_lS_l^2)\mathcal{R}_l(x)^\dagger\ . \tag{4.5b}$$

In fact, under conditions (2.6), (2.7), and (3.19) $\mathcal{S}$ in Eq. (4.5) can be replaced by $U_\omega(t)$ provided that $vt>L+\Delta$.

In the rest of this section we suppose that the initial state of the detector is an eigenstate with eigenvalue m_l of S_l^3 for each l. That is,

$$|D\rangle=|M\rangle\equiv|s_1,m_1\rangle_1\otimes|s_2,m_2\rangle_2\otimes\cdots\otimes|s_N,m_N\rangle_N\ , \tag{4.6}$$

where M stands for the N-tuple $(m_1,\ldots,m_N)$. We shall use the notation

$$\sum_{M'}=\sum_{m_1'=-s_1}^{s_1}\cdots\sum_{m_N'=-s_N}^{s_N}\ . \tag{4.7}$$

Equation (4.4) then reduces to

$$U_\omega(t)|\psi;M\rangle=\int dx\ \psi(x)\sum_{M'}\exp(-iE_{M'M}x)\times r_{M'M}(x,t)|x,M'\rangle\ , \tag{4.8}$$

where

$$E_{M'M}=E_{M'}-E_M \tag{4.9}$$

and

$$E_{M'}=\sum_l\omega_l(m_l'+s_l)/v \tag{4.10}$$

is the energy (in units of wave number) of the detector in state $|M'\rangle$. We have also introduced

$$r_{M'M}(x,t)=\prod_l r(s_l,m_l',m_l;\theta_l(x+vt))\ , \tag{4.11}$$

where

$$r(s,m',m;\theta)=\langle s,m'|\exp(-i\theta S^2)|s,m\rangle$$
$$=\left[\cos\frac{\theta}{2}\right]^{2s}\sum_k(-1)^k\left[\tan\frac{\theta}{2}\right]^{2k-m'+m}\frac{\sqrt{(s+m')!(s-m')!(s-m)!(s+m)!}}{k!(k-m'+m)!(s+m'-k)!(s-m-k)!}\ . \tag{4.12}$$

In the last formula (i.e., the Wigner formula [15]), the summation is retricted to the range where the arguments of all factorials are non-negative. The coherency can now be evaluated as

$$C(t)=\langle\psi;M|U_\omega(t)|\psi;M\rangle$$
$$=\int dx\,|\psi(x-vt)|^2\prod_l r(s_l,m_l,m_l;\theta_l(x))\ , \tag{4.13}$$

where

$$r(s,m,m;\theta)=\left[\cos\frac{\theta}{2}\right]^{2s}\sum_k\left[i\tan\frac{\theta}{2}\right]^{2k}\frac{(s+m)!(s-m)!}{(k!)^2(s+m-k)!(s-m-k)!}\ . \tag{4.14}$$

We see that the coherency is independent of ω_l. Given values of s_l, m_l and functions $B_l(x)$ and $\psi(x)$, we can work out $C(t)$ explicitly for any t. A concrete example shall be given in Sec. VI.

The S matrix can be evaluated similarly. Taking the limit $t \to \infty$ in Eq. (4.8), we find

$$\mathcal{S}|\psi;M\rangle = \sum_{M'} \exp(-iE_{M'M}X) r_{M'M} |\psi;M'\rangle \ , \tag{4.15}$$

where

$$r_{M'M} = \prod_l r(s_l, m_l', m_l; \theta_l) \tag{4.16}$$

is independent of x and t. One may go over to the momentum representation via Eq. (3.17) to arrive at the formula

$$\mathcal{S}|\psi;M\rangle = \int dp\, \tilde{\psi}(p) \sum_{M'} r_{M'M} |p+E_M - E_{M'};M'\rangle \ . \tag{4.17}$$

The argument of each ket on the right-hand side clearly expresses the asymptotic energy conservation as in the NP model.

V. CASE OF DETECTOR INITIALLY IN GROUND STATE

Suppose that the detector was initially in its ground state, namely,

$$|D\rangle = |G\rangle, \quad G \equiv (-s_1, -s_2, \ldots, -s_N) \ , \tag{5.1}$$

with the notation of Eq. (4.6). In this case Eqs. (4.8) and (4.13) are simplified considerably, because

$$r(s,m,-s;\theta) = \begin{bmatrix} 2s \\ s+m \end{bmatrix}^{1/2} \left[-\sin\frac{\theta}{2}\right]^{s+m} \left[\cos\frac{\theta}{2}\right]^{s-m} \tag{5.2}$$

and in particular

$$r(s,-s,-s;\theta) = \left[\cos\frac{\theta}{2}\right]^{2s} \ . \tag{5.3}$$

In the rest of this section we discuss only $C(\infty)$ and the S matrix. Equations (3.24), (4.15), and (5.3) give

$$C(\infty) = \prod_l \left[\cos\frac{\theta_l}{2}\right]^{2s_l} \ , \tag{5.4}$$

which vanishes if at least one of θ_l's is equal to π; a 180° pulse completely flips a spin and renders the entire state orthogonal to the ground state. Let us further specialize to the case where ω_l and θ_l defined by Eq. (3.6c) are independent of l; accordingly we drop the subscript l from them. [Note, however, that $\theta_l(x)$ still depends on l through x_l even if $B_l(x)$ is also independent of l.] Then, operators $\mathcal{R}(x)$ and $R(x)$ defined by Eqs. (3.10b) and (4.5b), respectively, can be expressed in terms of the total spin

$$J^\alpha = \sum_l S_l^\alpha \tag{5.5}$$

as

$$\mathcal{R}(x) = \exp(-i\omega J^3 x/v) \ , \tag{5.6}$$

$$R(x) = \mathcal{R}(x)\exp(-i\theta J^2)\mathcal{R}(x)^\dagger \ . \tag{5.7}$$

Since $|G\rangle$ is the common eigenstate of J^3 and $\mathbf{J}^2$ with eigenvalue $-j$ and $j(j+1)$, respectively, where

$$j \equiv \sum_l s_l \ , \tag{5.8}$$

it follows that

$$\begin{aligned} R(x)|G\rangle = R(x)|j,-j\rangle &= \sum_{\mu=-j}^{j} \exp[-i(\mu+j)\omega x/v] r(j,\mu,-j;\theta)|j,\mu\rangle \\ &= \sum_{n=0}^{2j} \exp(-in\omega x/v) r(j,n-j,-j;\theta)|j,n-j\rangle \ , \end{aligned} \tag{5.9}$$

where the notation of Eq. (3.2) has been used with (s,m) replaced by (j,μ). Consequently Eq. (4.17) takes the form

$$\mathcal{S}|\psi;G\rangle = \left[\cos\frac{\theta}{2}\right]^{2j} \int dp\, \tilde{\psi}(p) \sum_{n=0}^{2j} \begin{bmatrix} 2j \\ n \end{bmatrix}^{1/2} \left[-\tan\frac{\theta}{2}\right]^n |p - n\omega/v; j, n-j\rangle \ . \tag{5.10}$$

This corresponds to Eq. (14) of NP [13]. What they called a generalized coherent state has thus been generalized to the case of arbitrary $\{s_l\}$. In addition, it has been shown that such a state occurs if ω_l and θ_l are independent of l; it is, however, *not* necessary that s_l is independent of l.

It is to be noted that $\{s_l\}$ and N appear in Eq. (5.9) only in the form of j. If s_l is also independent of l, then

$$j = Ns \ ; \tag{5.11}$$

as far as the operation of the S matrix on the ground state (of the detector) is concerned, the role of N (i.e., the number of distinct spins) and that of s (i.e., the magnitude of each spin) are indistinguishable. This result may be intuitively understood by imagining the case of all N spins located at the same position ($x_1 = x_2 = \cdots = x_N$); then one would effectively have a single spin of magnitude Ns. This is essentially what was called "big bang" by Hepp [1] in his "example 5." (He was concerned with the case of $s=\frac{1}{2}$ and $N=\infty$). Indistinguishability of N and s as well as its intuitive interpretation given above, however, breaks down if the initial state of the detector does not belong to the spin-j representation of $\mathbf{J}$. An example is the case of initially thermal detector to be discussed in Sec. VII. Parameters N and s will play independent roles also in a more detailed property than the mere S matrix, in the time evolution of coherency, for instance. We give some explicit examples in the next section.

We conclude this section by considering $C(\infty)$ for the case where s_l is independent of l and each θ_l deviates only slightly from the mean value

$$\theta \equiv N^{-1} \sum_l \theta_l \ . \tag{5.12}$$

Equation (5.4) may then be evaluated as

$$\begin{aligned} C(\infty) &= \exp\left[2s \sum_l \ln \cos\frac{\theta_l}{2}\right] \\ &\simeq \left\{ \left[\cos\frac{\theta}{2}\right] \exp\left[-\frac{(\delta\theta)^2}{8\left[\cos\frac{\theta}{2}\right]^2}\right]\right\}^{2Ns} , \end{aligned} \tag{5.13}$$

where

$$(\delta\theta)^2 \equiv N^{-1} \sum_l (\theta_l - \theta)^2 \ . \tag{5.14}$$

The asymptotic coherency is seen to be depressed by the weak l dependence of θ_l; it is not necessarily the randomness but simply any l dependence that counts in this connection.

VI. EXPLICIT EXAMPLES OF $C(t)$

Let us illustrate the separate N and s dependence of $C(t)$ by considering a simple model in which $B_l(x)$ is independent of l and

$$x_l = \Delta + la, \quad a > 2\delta \ , \tag{6.1}$$

where a is a constant and δ is the half-width of $B(x)$ [cf. Eq. (2.6)]; Eq. (6.1) ensures that $\psi(x)$ and $B(x - x_l)$ do not overlap each other. We consider two typical situations, namely, that of $\Delta \ll \delta$ and $\Delta \gg \delta$. When $\Delta \ll \delta$, we may replace $|\psi(x)|^2$ in Eq. (4.13) by $\delta(x)$ and obtain

$$C(t) = \left[\prod_l \cos\left[\frac{1}{2}\theta_l(vt)\right]\right]^{2s} . \tag{6.2}$$

This is illustrated in Fig. 1(a) for $(N,s) = (2,\frac{1}{2})$ and in Fig. 1(b) for $(N,s) = (1,1)$, where the abscissa refers to the dimensionless time vt/a. Details of the profile depend on the shape of $B(x)$; the figures correspond to the case of $B(x)$ being a Gaussian pulse of half-width δ, truncated at $|x| = \delta$;

$$B(x)/v \propto \delta^{-1} \exp(-x^2/\delta^2)\Theta(\delta - |x|) \ , \tag{6.3}$$

where Θ is the step function and the proportionality constant is so chosen as to satisfy the condition (3.6c). Oscillatory behaviors occur for large θ, which corresponds to large $B(x)$ or small particle velocity v. When $\Delta \gg \delta$, on the other hand, we may replace $B(x)$ by $v\theta\delta(x)$. Accordingly

$$C(t) = \int dx\, |\psi(x - vt)|^2 \left[\prod_l \cos\frac{\theta\Theta(x - x_l)}{2}\right]^{2s} . \tag{6.4}$$

[In this case θ may be restricted to the interval $(0, 2\pi)$ without loss of generality.] This is illustrated in Fig. 2(a) for $(N,s) = (2,\frac{1}{2})$ and in Fig. 2(b) for $(N,s) = (1,1)$. Details of the profile depend on the shape of $|\psi(x)|$; the figures correspond to the case of $|\psi(x)|^2$ being a Gaussian packet of half-width Δ, truncated at $|x| = \Delta$;

$$|\psi(x)|^2 \propto \Delta^{-1} \exp(-x^2/\Delta^2)\Theta(\Delta - |x|) \ . \tag{6.5}$$

It would be instructive, although we shall not do so explicitly, to depict $\langle\tau^-\rangle(t)$ on the complex plane. It gives a clear picture of how the precession of the particle's spin is affected by the interaction with the detector. In each of the cases shown in Figs. 1 and 2, two qualitatively different regimes exist, namely, the regimes $\Omega \gg a/v$ and $\Omega \ll a/v$; in the latter regime the magnitude of $\langle\tau^-\rangle(t)$ diminishes before it has precessed appreciably.

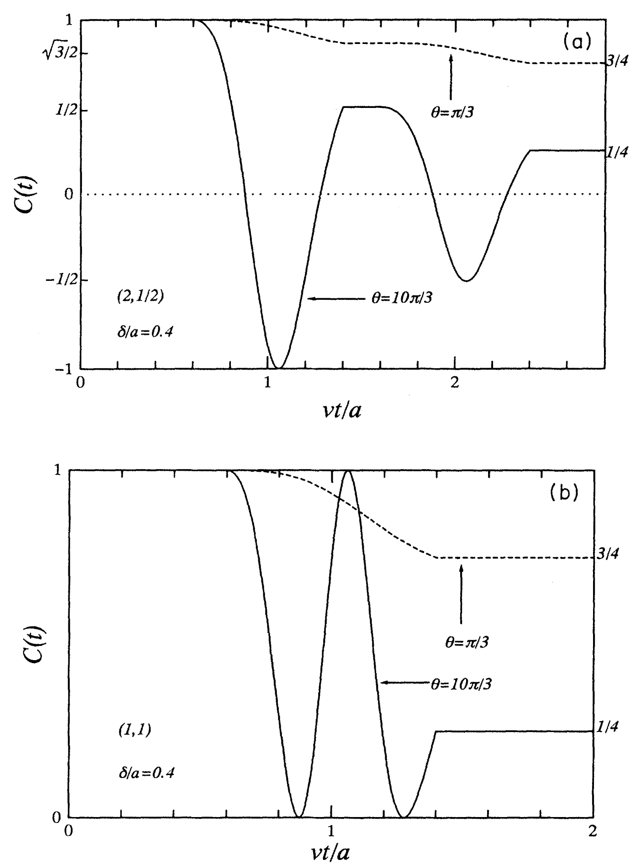

FIG. 1. (a),(b) Examples of $C(t)$ given by Eq. (6.2). The pair of numbers in the lower left corner denotes (N,s).

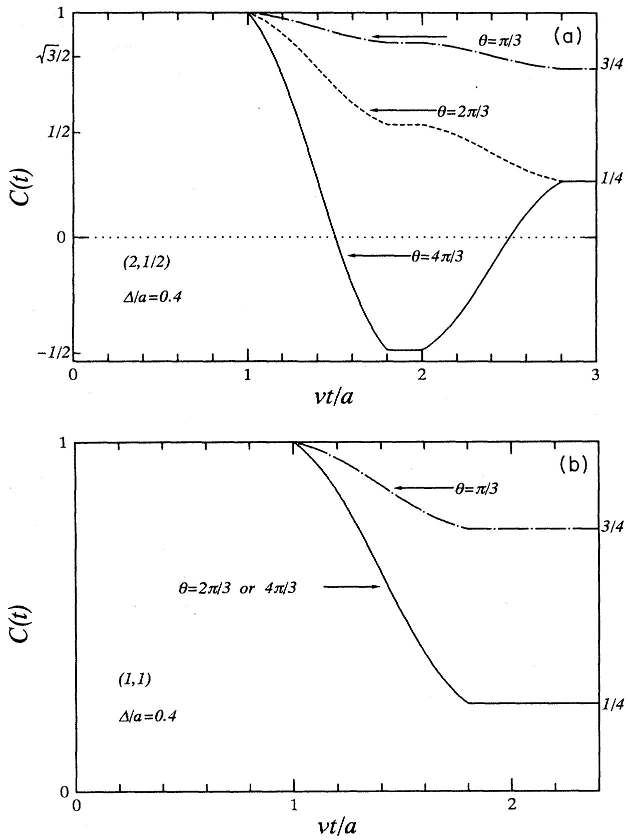

FIG. 2. (a)(b) Examples of $C(t)$ given by Eq. (6.4). The pair of numbers in the lower left corner denotes (N,s).

VII. CASE OF DETECTOR INITIALLY IN THERMAL EQUILIBRIUM

Let us now suppose that the detector had been in thermal contact with a heat bath and had attained the thermal equilibrium of temperature T by the time $t=0$, when the detector was decoupled from the heat bath. We focus our attention on the coherency as defined by Eq. (3.25). In the present case

$$\rho_D=\prod_l \rho(s_l,\beta_l)\ , \tag{7.1}$$

where

$$\beta_l=\hbar\omega_l/k_BT\ , \tag{7.2}$$

$$\rho(s,\beta)=[Z(s,\beta)]^{-1}\exp(-\beta S^3)\ , \tag{7.3}$$

$$Z(s,\beta)=\mathrm{tr}_s\exp(-\beta S^3)=\frac{\sinh(s+\frac{1}{2})\beta}{\sinh\frac{1}{2}\beta}\ , \tag{7.4}$$

with tr_s being the trace over the $(2s+1)$-dimensional spin space. Putting (7.1), (3.11), and (4.3) in (3.25), we obtain

$$C(t)=\int dx\,|\psi(x-vt)|^2\prod_l W(s_l,\beta_l;\theta_l(x))\ , \tag{7.5}$$

where

$$W(s,\beta;\theta)=\langle\exp(-i\theta S^2)\rangle_\beta\ , \tag{7.6}$$

with the notation

$$\langle A\rangle_\beta=\mathrm{tr}_s\rho(s,\beta)A\ . \tag{7.7}$$

In particular,

$$C(\infty)=\prod_l W(s_l,\beta_l;\theta_l)\ . \tag{7.8}$$

Rotation invariance of $\rho(s,\beta)$ around the S^3 axis implies that

$$\begin{aligned}W(s,\beta;\theta)&=W(s,\beta;-\theta)\ ,\\ W(s,\beta;\pi+\theta)&=(-1)^{2s}W(s,\beta;\pi-\theta)\ .\end{aligned} \tag{7.9}$$

With notation (4.14) we have

$$\mathrm{tr}_s\exp(-\beta S^3)\exp(-i\theta S^2)=\sum_m e^{-m\beta}r(s,m,m;\theta)\ . \tag{7.10}$$

Again, rotational symmetry implies

$$r(s,m,m;\theta)=r(s,-m,-m;\theta)\ . \tag{7.11}$$

In the special case of $s=\frac{1}{2}$, therefore, $W(s,\beta;\theta)$ is independent of temperature.

One might intuitively guess that $|C(\infty)|$ should decrease as the temperature goes up. However, this is not so in general. The case of $s_l=\frac{1}{2}$ for each l gives a counter example in which $|C(\infty)|$ is independent of temperature. A more drastic one is the case of $\theta_l=\pi$ for each l. A 180° pulse reverses the direction of spin, that is, the state $|s,m\rangle$ is transformed into $|s,-m\rangle$, which is orthogonal to the original state unless $m=0$. This property manifests itself in the formula

$$r(s,m,m;\pi)=(-1)^s\delta_{m,0}\ . \tag{7.12}$$

It follows from Eqs. (7,8), (7.10), and (7.12) that $C(\infty)=0$ if at least one of the s_l's is a half odd integer, and that

$$|C(\infty)|=1\Big/\prod_l Z(s_l,\beta_l)\ , \tag{7.13}$$

if all s_l are integers. This is a monotonically *increasing* function of temperature. On the other hand, one's intuition is confirmed when θ is small and $s\geq 1$; see Fig. 3 below and also Secs. VIII and IX.

In order to evaluate $W(s,\beta;\theta)$ for a generic case, we recall the well-known relationship among the rotation operators; let γ, θ, and γ' be the Euler angles corresponding to the rotation through an angle φ about the axis specified by a unit vector $\mathbf{n}$, then

$$\exp(-i\gamma S^3)\exp(-i\theta S^2)\exp(-i\gamma' S^3)=\exp(-i\varphi\mathbf{n}\cdot\mathbf{S})\ , \tag{7.14a}$$

where

$$\cos\frac{\theta}{2}\cos\frac{\gamma+\gamma'}{2}=\cos\frac{\varphi}{2}\ . \tag{7.14b}$$

Taking the trace of both sides, we find

$$\mathrm{tr}_s\exp[-i(\gamma+\gamma')S^3]\exp(-i\theta S^2)=Z(s,i\varphi)\ . \tag{7.15}$$

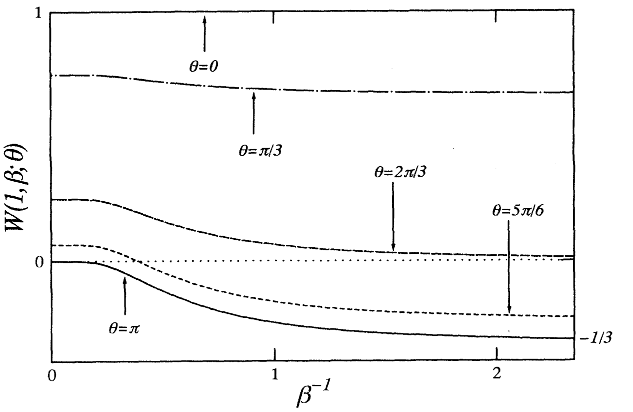

FIG. 3. $W(1,\beta;\theta)$ as a function of temperature.

Since both sides are analytic functions of $\gamma+\gamma'$, we may continue the latter to $-i\beta$, to find

$$W(s,\beta;\theta)=Z(s,\zeta)/Z(s,\beta)\ , \tag{7.16a}$$

where

$$\cosh\frac{\zeta}{2}=\cos\frac{\theta}{2}\cosh\frac{\beta}{2}\ . \tag{7.16b}$$

In the evaluation of $Z(s,\zeta)$, one can adopt without loss of generality the convention that $\sinh(\zeta/2)$ lies either on the positive real axis or positive imaginary axis of the complex plane. Explicit forms of $W(s,\beta;\theta)$ for small values of s may be worked out by use of Eqs. (7.4) and (7.16). For example,

$$\begin{aligned} W\left[\frac{1}{2},\beta;\theta\right]&=\cos\frac{\theta}{2}\ ,\\ W(1,\beta;\theta)&=\frac{\cosh\beta+(1+\cosh\beta)\cos\theta}{1+2\cosh\beta} \\ W\left[\frac{3}{2},\beta;\theta\right]&=\left[\cos^2\frac{\theta}{2}-(\cosh\beta)^{-1}\sin^2\frac{\theta}{2}\right]\cos\frac{\theta}{2}\ . \end{aligned} \tag{7.17}$$

In particular, if ω_l, θ_l, and s_l are independent of l, we obtain

$$C(\infty)=[W(s,\beta;\theta)]^N \tag{7.18}$$

which depends separately on N and s. We depict $W(1,\beta;\theta)$ as a function of temperature for a few values of θ in Fig. 3; by virtue of the symmetry relation (7.9), θ may be restricted to the interval $[0,\pi]$.

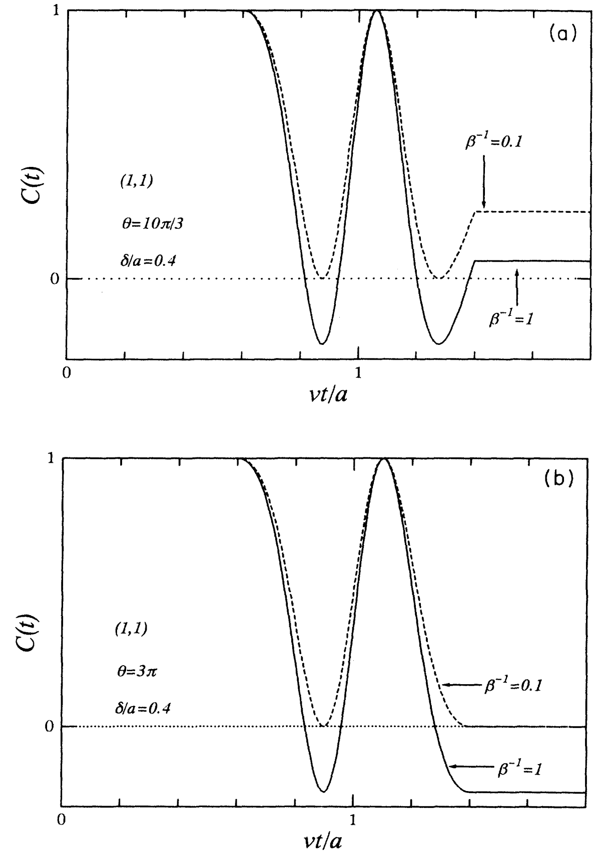

FIG. 4. (a),(b) Examples of $C(t)$ at a finite temperature; the system considered is the same as in Fig. 1(b).

This figure shows explicitly that $|W|$ is not necessarily a monotonically decreasing function of temperature. Figure 4 illustrates $C(t)$ at a finite temperature in comparison with the zero-temperature case.

Nakazato and Pascazio [14] studied "the case in which the detector is initially in a thermal state" with the NP model. Their "thermal state," however, is not the true thermal state described by (7.1) with $s=\frac{1}{2}$; they restricted themselves to the total-spin-$N/2$ subspace. Thus, so far as their investigation of S matrix is concerned, their detector consists effectively of a single spin of magnitude $N/2$ initially in thermal equilibrium. Indeed, "the visibility" given by their Eq. (3.20) reads, in our notation, $W(N/2,\beta;\theta)$. In their rather "involved" calculation, they followed related calculations in Refs. [6,10] to employ the normal-ordering formula;

$$\exp[-i\theta(S^+e^{-i\varphi}+S^-e^{i\varphi})]=\exp(-iS^+e^{-i\varphi}\tan\theta)\exp(-2S^3\ln\cos\theta)\exp(-iS^-e^{i\varphi}\tan\theta)\ , \tag{7.19}$$

which follows from the observation that all the exponential factors in it belong to a representation of $Gl(2,\mathbb{C})$ [16]. By contrast, our derivation of (7.15) on the basis of (7.13) is simple and elementary.

VIII. CASE OF LARGE SPIN

The case of $s_l \gg 1$ for all l is of some interest; so long as low-lying excited states are concerned, each spin may be replaced by a boson and one is left with a detector with an array of boson sources (or of harmonic oscillators). In order to investigate this limit, we begin with the definition

$$|n\rangle_s = |s, n-s\rangle, \quad n=0,1,\ldots,2s \ . \tag{8.1}$$

With this notation, Eq. (3.2) reads as

$$\begin{aligned} S^3|n\rangle_s &= (n-s)|n\rangle_s \ , \\ S^+|n\rangle_s &= \sqrt{(n+1)(2s-n)}\,|n+1\rangle_s \ , \\ S^-|n\rangle_s &= \sqrt{[2s-(n-1)]n}\,|n-1\rangle_s \ . \end{aligned} \tag{8.2}$$

Therefore the spin operators can be expressed in terms of the operators a and $a^\dagger$ defined by

$$\begin{aligned} a|n\rangle_s &= \sqrt{n}\,|n-1\rangle_s \ , \\ a^\dagger|n\rangle_s &= \sqrt{n+1}\,|n+1\rangle_s \ , \end{aligned} \tag{8.3}$$

as follows (the Holstein-Primakoff transformation [17]):

$$\begin{aligned} S^3 &= a^\dagger a - s \ , \\ S^- &= (2s - a^\dagger a)^{1/2} a \ , \\ S^+ &= a^\dagger (2s - a^\dagger a)^{1/2} \ . \end{aligned} \tag{8.4}$$

[To be precise, $a^\dagger|2s\rangle_s$ is not defined by Eq. (8.3). However factors are so carefully arranged in the definition (8.4) of **S** that $a^\dagger|2s\rangle_s$ is not needed.] We also define

$$f(x) = s^{1/2} B(x) \ . \tag{8.5}$$

Putting Eqs. (8.4) and (8.5) in (2.8), and formally letting $s \to \infty$ while keeping $f(x)$ fixed, we find

$$H^0_\omega = H^0 + \sum_l \omega_l a_l^\dagger a_l \ , \tag{8.6a}$$

$$H^1_\omega = \frac{i}{\sqrt{2}} \sum_l f_l(X - x_l) a_l \exp(i\omega_l X/v) + \text{H.c.} \tag{8.6b}$$

At the same time, we define

$$|n\rangle = |n\rangle_\infty \ . \tag{8.7}$$

Hence a and $a^\dagger$ simply become boson annihilation and creation operators on the space spanned by $\{|n\rangle | n=0,1,\ldots\}$. The physical significance of $f_l(x)$ is clear; in the *rotating frame* (cf. Sec. III) H^1_ω reduces to

$$-\sum_l f_l(X - x_l) q_l, \quad q_l \equiv (a_l - a_l^\dagger)/\sqrt{2}i \ , \tag{8.8}$$

and $f_l(x)$ is a force exerted on the lth harmonic oscillator.

Let us examine the above formal procedure of letting $s \to \infty$ more closely. To be specific we shall deal with $C(t)$ at finite temperature and the S matrix as operated on the ground state.

First, consider $C(t)$ defined by Eqs. (7.5) and (7.6). In accordance with (8.5), we define

$$\begin{aligned} \alpha_l(x) &= -(s_l/2)^{1/2} \theta_l(x) \\ &= -\frac{1}{\sqrt{2}v} \int_{-\infty}^{x} dx' f_l(x' - x_l) \ , \end{aligned} \tag{8.9a}$$

$$\alpha_l = \alpha_l(\infty) = -\frac{1}{\sqrt{2}v} \int_{-\infty}^{\infty} dx\, f_l(x) \ . \tag{8.9b}$$

Supposing that α is independent of s, we find up to order s^{-2} that

$$\exp(\pm\zeta/2) = \left[1 \mp \frac{\alpha^2}{4s} z \pm \frac{1}{6} \left[\frac{\alpha^2}{4s} \right]^2 (1 \pm 3z - 3z^2) z \right] e^{\pm\beta/2} \ , \tag{8.10}$$

where $z = \coth(\beta/2)$. Consequently

$$W(s,\beta;\theta) = \left[1 + \frac{\alpha^2}{2s}(2n_\beta + 1) \left\{ n_\beta - \frac{\alpha^2}{2} \left[n_\beta^2 + n_\beta + \frac{1}{6} \right] \right\} \right] \times \exp\left[-\frac{\alpha^2}{2}(2n_\beta + 1) \right] \ , \tag{8.11}$$

where

$$n_\beta = (e^\beta - 1)^{-1} \ , \tag{8.12}$$

and we have discarded terms of $O(s^{-2})$ or $O(e^{-(2s+1)\beta})$. The last exponential factor is the familiar persistence amplitude for a harmonic oscillator driven by an external force, which can also be obtained by working directly with Hamiltonian (8.6); $\alpha^2 n$ and $\alpha^2(n+1)$ are one-boson absorption and emission probability, respectively, in the lowest-order perturbation theory when the initial boson (i.e., harmonic-oscillator) state is $|n\rangle$. The prefactor exhibits how large s should be for the boson limit to be effectively attained. The coherency $C(t)$ itself is obtained by substituting Eq. (8.11) into (7.5). In particular, if ω_l and α_l are independent of l, then

$$C(\infty) = \exp[-N\alpha^2(n_\beta + \tfrac{1}{2})] \tag{8.13}$$

in the boson limit. Let us illustrate the behavior of $C(t)$ in the boson limit with the model introduced in Sec. VI; Eqs. (6.2) and (6.4) are valid if $[\cos\frac{1}{2}\theta_l(vt)]^{2s}$ and $[\cos\{(\theta/2)\Theta(x - x_l)\}]^{2s}$ are replaced by $\exp[-(n_\beta + \frac{1}{2})\alpha_l^2(vt)]$ and $\exp[-(n_\beta + \frac{1}{2})\alpha^2\Theta(x - x_l)]$, respectively. In contrast to the case of finite s, such oscillatory behaviors as in Figs. 1 or 2 do not occur; $C(t)$ is monotonically suppressed as $f(x)$ increases or v decreases.

Next, we consider Eq. (5.10) for the S matrix in the special case of ω_l, θ_l, and s_l being l independent. We use notation (8.2) with **S** and s replaced by **J** and j defined by Eqs. (5.5) and (5.11), respectively; accordingly

$$|G\rangle = |j, -j\rangle = |0\rangle_j \ , \tag{8.14}$$

$$|j, n-j\rangle = |n\rangle_j = \frac{1}{n!} \begin{bmatrix} 2j \\ n \end{bmatrix}^{-1/2} (J^+)^n |0\rangle_j \ . \tag{8.15}$$

It follows that

$$\mathcal{S}|\psi;0\rangle_j=\left[\cos\frac{\theta}{2}\right]^{2j}\int dp\,\tilde{\psi}(p)\sum_{n=0}^{2j}\frac{1}{n!}\left[-J^+\tan\frac{\theta}{2}\right]^n|p-n\omega/v;0\rangle_j$$
$$=\left[\cos\frac{\theta}{2}\right]^{2j}\exp\left[-e^{-i\omega X/v}J^+\tan\frac{\theta}{2}\right]|\psi;0\rangle_j\ . \tag{8.16}$$

This formula can be obtained also by use of Eq. (7.19); cf. Ref. [14]. For our purpose, however, the knowledge of the particular case (5.2) of the Wigner formula was sufficient. We also introduce operators b and $b^\dagger$ which are related to $\mathbf{J}$ via Eqs. (8.3) and (8.4) with $(\mathbf{S},s,a)$ replaced by $(\mathbf{J},j,b)$; in particular

$$J^+=b^\dagger(2j-b^\dagger b)^{1/2}\ . \tag{8.17}$$

Putting Eqs. (8.9) and (8.17) in (5.10), and letting $s\rightarrow\infty$ (recall that $j=Ns$), we find

$$\mathcal{S}|\psi;0\rangle=\exp\left[-\frac{N}{2}\alpha^2+\sqrt{N}\,\alpha e^{-i\omega X/v}b^\dagger\right]|\psi;0\rangle$$
$$=e^{-N\alpha^2/2}\sum_{n=0}^{\infty}\int dp\,\tilde{\psi}(p)\frac{(\sqrt{N}\,\alpha)^n}{(n!)^{1/2}}|p-n\omega/v;n\rangle\ , \tag{8.18}$$

where we have dropped the subscript j on the ket in accordance with notation (8.7). This is of course a coherent state as far as the boson is concerned; it can be written as

$$\mathcal{S}|\psi,0\rangle=\int dx\,\psi(x)|x\rangle\otimes|\sqrt{N}\,\alpha(x)\rangle\ , \tag{8.19}$$

with the notation

$$|\alpha\rangle=\exp(-|\alpha|^2/2+\alpha b^\dagger)|0\rangle\ , \tag{8.20}$$

$$\alpha(x)=\alpha\exp(-i\omega x/v)\ . \tag{8.21}$$

In the limit $s\rightarrow\infty$, b is related to the individual bosons a_l via

$$b=N^{-1/2}\sum_l a_l\ . \tag{8.22}$$

IX. CASE OF LARGE N

The S matrix in the limit $N\rightarrow\infty$ was studied by Nakazato and Pascazio [14] for the NP model of $s=\frac{1}{2}$ under the condition that both

$$\tilde{\theta}=N^{1/2}\theta \tag{9.1}$$

and L in Eq. (2.8) are independent of N. Let us generalize their result to the case of arbitrary s, assuming for simplicity that ω_l, θ_l, and s_l are independent of l.

The $N\rightarrow\infty$ limit of Eq. (5.10) for the S matrix is formally the same as the $s\rightarrow\infty$ limit, since N and s occur there only in the combination $j=Ns$. The result is simply Eq. (8.18) with $\sqrt{N}\,\alpha$ replaced by $-(s/2)^{1/2}\tilde{\theta}$.

Next, we consider $C(\infty)$ at finite temperature. Since θ is of order $N^{-1/2}$, we can expand each factor on the right-hand side of Eq. (7.8) as

$$W(s,\beta;\theta)=1-\frac{\kappa}{2N}\tilde{\theta}^2+O(N^{-2})\ , \tag{9.2}$$

where

$$\kappa=\langle(S^2)^2\rangle_\beta=\tfrac{1}{2}\{s(s+1)-\mu_2\}\ , \tag{9.3}$$

with the notation

$$\mu_k=\langle(S^3)^k\rangle_\beta\ . \tag{9.4}$$

Consequently

$$C(\infty)=\exp(-\kappa\tilde{\theta}^2/2)\ . \tag{9.5}$$

It can be shown (see the Appendix) that

$$\frac{\partial\mu_2}{\partial\beta}=\mu_1\mu_2-\mu_3>0\ , \tag{9.6}$$

except for the case of $s=\frac{1}{2}$ or $\beta=\infty$, where the right-hand side vanishes. Hence κ is a monotonically increasing function of temperature for any $s\geq 1$. It follows readily from Eq. (9.3) that

$$\kappa=\frac{s}{2}+\left[s-\frac{1}{2}\right]e^{-\beta}+O(e^{-2\beta})\ , \tag{9.7}$$

in the low-temperature regime, and that

$$\kappa=\frac{1}{3}s(s+1)\left[1-\frac{1}{15}\left[s-\frac{1}{2}\right]\left[s+\frac{3}{2}\right]\beta^2\right]+O(\beta^4) \tag{9.8}$$

in the high-temperature regime. A closed-form expression valid for any temperature may be obtained by noting that

$$\mu_2=\mu_1^2+\chi\ , \tag{9.9}$$

where

$$\mu_1=-\frac{\partial}{\partial\beta}\ln Z(s,\beta)$$
$$=\frac{1}{2}\coth\frac{\beta}{2}-\left[s+\frac{1}{2}\right]\coth\left[s+\frac{1}{2}\right]\beta$$

is the mean spin polarization given by the Brillouin function, and

$$\chi=-\frac{\partial\mu_1}{\partial\beta} \tag{9.10}$$

is its fluctuation (or the susceptibility). The result is

$$\kappa=-\frac{1}{2}\mu_1\coth\frac{\beta}{2}\ , \tag{9.11}$$

which is depicted in Fig. 5 as a function of β for a few values of s.

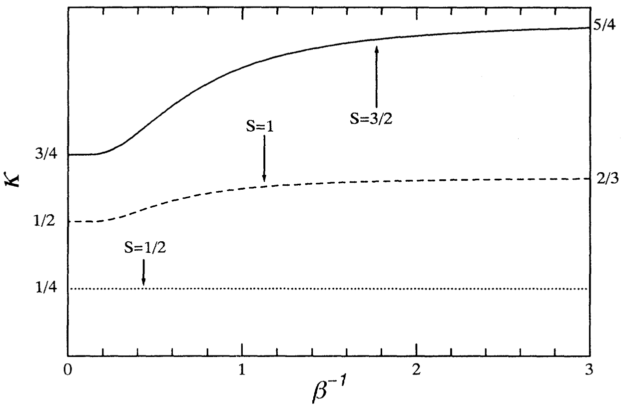

FIG. 5. Temperature dependence of κ.

In Hepp's original paper [1], where $x_l = la$ and a is a constant independent of N, the S matrix and $C(\infty)$ were not physically relevant unless the limit $N\to\infty$ was taken after the limit $t\to\infty$; the latter, however, is "an unattainable mathematical limit" [2]. By constrast, in the present paper, we have supposed $a \simeq L/N$ with L being N independent as in Refs. [7,8]. Therefore S matrix and $C(\infty)$ make clear sense;

$$C(\infty)=C(t), \quad \mathcal{S}|\psi;D\rangle = U_\omega(t)|\psi;D\rangle \tag{9.12}$$

for any t greater than $v^{-1}(L+\Delta)$, as already mentioned in Sec. IV.

X. CONCLUDING REMARKS

Since S matrix depends on Ns only, most of asymptotic properties discussed in this article can be read off NP [13,14]; only the derivations of various formulas have been somewhat simplified. Also, *the boson model* (8.6) obtained by letting $s\to\infty$ is merely a lattice generalization of the NP version of "the maser model," a variant of the Jaynes-Cummings model [18]. When it comes to a finite-time property, however, N and s play separate roles. Even the restriction to the total-spin-Ns subspace would not render the detector equivalent to one with a single spin of magnitude Ns (as in the Cini model [3]); such an "equivalence" mentioned by NP [14] holds only for asymptotic properties.

The time-dependent physical quantity explicitly evaluated in this paper is the coherency $C(t)$ defined by (3.25). There are of course other physical quantities of interest, such as the energy of the particle $\langle vP\rangle(t)$, the energy of the detector $\langle \sum_l \omega_l S_l^3\rangle(t)$, individual detector spins $\langle S_l^\alpha\rangle(t)$, fluctuations of these quantities, and so on. These shall be subjects for a future publication.

ACKNOWLEDGMENTS

This work is a direct result of a stimulating seminar kindly given by S. Pascazio at Tohoku University. We thank him for an intensive discussion. We have also benefited greatly from other participants of the seminar and discussion, T. Komatsu, T. Matsumura, N. Saito, N. Yamada, and especially K. Niizeki whose penetrating remarks on the finite-temperature effects and the Holstein-Primakoff transformation, in particular, are more or less heavily reflected in our presentation of Secs. VII and VIII. We are grateful to T. Nakamura for valuable comments on the manuscript. Many thanks are also due to H. Nakazato and K. Kakazu for providing us with information prior to publication.

APPENDIX

The identity in (9.6) is a special case of

$$\frac{\partial}{\partial\beta}\mu_k = \mu_1\mu_k - \mu_{k+1} , \tag{A1}$$

which follows from direct differentiation. Likewise, the inequality in (9.6) is a special case of the more general one

$$(-1)^{k+l}\{\mu_{k+l}-\mu_k\mu_l\} \ge 0 , \tag{A2}$$

which is valid for arbitrary positive integers k and l, the equality holding only if "$\beta=\infty$" or "k or l (or both) is even *and* $s=\frac{1}{2}$." Perhaps the simplest proof proceeds as follows. Consider the quantity

$$(\mu_{k+l}-\mu_k\mu_l)[Z(s,\beta)]^2 = \sum_{m,n} g(m,n) , \tag{A3a}$$

where

$$g(m,n) \equiv m^k(m^l-n^l)e^{-\beta(m+n)} . \tag{A3b}$$

The right-hand side of Eq. (A3a) is equal to

$$\sum_{m,n}\tilde{g}(m,n), \quad \tilde{g}(m,n)\equiv\frac{1}{2}\{g(m,n)+g(n,m)\} , \tag{A4}$$

which in turn is equal to

$$\frac{1}{2}\sum_{m,n} f(m,n) , \tag{A5a}$$

where

$$f(m,n)\equiv\tilde{g}(m,n)+\tilde{g}(-m,-n)$$
$$=(m^k-n^k)(m^l-n^l)h_{k+l}(m+n) , \tag{A5b}$$

with

$$h_{k+l}(x)=\begin{cases}\cosh\beta x & \text{if } k+l \text{ is even ,}\\ -\sinh\beta x & \text{if } k+l \text{ is odd .}\end{cases} \tag{A6}$$

It follows from the elementary property

$$\mathrm{sgn}(m^k-n^k)=\begin{cases}\mathrm{sgn}(m^2-n^2) & \text{if } k \text{ is even}\\ \mathrm{sgn}(m-n) & \text{if } k \text{ is odd ,}\end{cases} \tag{A7}$$

that

$$(-1)^{k+l}f(m,n)\ge 0 , \tag{A8}$$

where, for finite β, the equality holds only when $m=n$ if both k and l are odd, and only when $m=n$ or $m=-n$ otherwise. This proves inequality (A2). For (A2) to be an equality, (A8) should be an equality for all (m,n). This is possible only if k or l (or both) is even *and* $s=\frac{1}{2}$.

Incidentally, if both k and l are odd, inequality (A2) is a special case of

$$\langle K(A)L(A)\rangle \geq \langle K(A)\rangle\langle L(A)\rangle\ , \tag{A9}$$

where

$$\langle K\rangle \equiv \mathrm{tr}\rho K\ . \tag{A10}$$

This inequality holds for an arbitrary density operator ρ and a Hermitian operator A, provided that both K and L are real nondecreasing functions. Its proof is analogous to, but much simpler than, the proof of (A2); simply repeat the procedures corresponding to (A3) and (A4) with m^k, m^l, and $e^{-\beta m}$ replaced by $\langle m|K|m\rangle$, $\langle m|L|m\rangle$, and $\langle m|\rho|m\rangle$, respectively, where $\{|m\rangle\}$ is the orthonormal basis diagonalizing A.

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