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Series representation of quantum-field quasiprobabilities

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The Wigner and Glauber-Sudarshan P functions for quantum fields are usually defined in terms of integrals over the phase space of characteristic functions. We show that they and, in general, any s-parametrized quasiprobability distribution may be represented as a series in terms of displaced number states facilitating a straightforward evaluation.

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The representation of quantum fields in phase space in terms of quasiprobabilities is widely used in quantum optics, with particular emphasis being given to the Wigner function, the Glauber-Sudarshan P function, and the Husimi Q function [1,2]. The computation of quasiprobabilities, given a density matrix, is often a tedious task which involves integration over phase-space variables. The exception is the Q function which is simply expressed as the coherent expectation value of the field density matrix and is therefore widely adopted to describe field dynamics in situations where the density matrix is easily computed. However, the Wigner function has an interesting characteristic which makes it an excellent diagnostic of quantum properties: it can be negative in some areas of phase space when the field has nonclassical interferences (although of course not all nonclassical field states have negative contributions to their Wigner functions).

The Wigner function is usually expressed in an integral form which is not always easy to compute. Recently, Wünsche [3] has derived another form for the Wigner function, and in general, for any s-parameterized quasiprobability distribution. However, the method given in Ref. [3] is very formal and we believe the method we use to obtain it is simpler and gives more physical insight. In this Brief Report we present a series representation for the Wigner function, and, in an analogous way for any s-parameterized quasiprobability distribution. In order to do that, we use the fact that the Wigner, P, and Q functions may be expressed [1,2] in an integral form,

$$F(\alpha,s)=\frac{1}{\pi^2}\int C(\beta,s)\exp(\alpha\beta^*-\alpha^*\beta)d^2\beta\ , \tag{1}$$

where $C(\beta,s)$ is the s-ordered generalized characteristic function

$$C(\beta,s)=\mathrm{Tr}\{D(\beta)\rho\}\exp(s|\beta|^2/2)\ , \tag{2}$$

and s is a parameter which defines the relevant quasiprobability distribution. For $s=1$ we obtain the Glauber-Sudarshan P function, for $s=0$ we have the Wigner function, and for $s=-1$, we have the Q function. In Eq. (2), $D(\beta)$ is the Glauber displacement operator [4], and ρ is the density matrix of the field under investigation.

We shall concentrate first on the Wigner function. From Eq. (1), with $s=-1$, we can write the Q function as

$$Q(\alpha)=\int G(\beta)\exp(\alpha\beta^*-\alpha^*\beta)d^2\beta\ , \tag{3}$$

and the Wigner function as

$$W(\alpha)=\int G(\beta)\exp(\alpha\beta^*-\alpha^*\beta)\exp(|\beta|^2/2)d^2\beta\ , \tag{4}$$

where

$$G(\beta)=\frac{1}{\pi^2}\mathrm{Tr}\{D(\beta)\rho\}\exp(-|\beta|^2/2)\ . \tag{5}$$

From Eq. (4) we note that the Wigner function can be expressed as the infinite series

$$W(\alpha)=\sum_{n=0}^{\infty}\frac{2^{-n}}{n!}\int G(\beta)\exp(\alpha\beta^*-\alpha^*\beta)|\beta|^{2n}d^2\beta\ . \tag{6}$$

We now use the fact that

$$\frac{\partial}{\partial\alpha}\frac{\partial}{\partial\alpha^*}\exp(\alpha\beta^*-\alpha^*\beta)=-|\beta|^2\exp(\alpha\beta^*-\alpha^*\beta)\ ,$$

to express the Wigner function in terms of the Q function given in Eq. (3) as

$$W(\alpha)=\sum_{n=0}^{\infty}\frac{2^{-n}}{n!}\left[-\frac{\partial}{\partial\alpha}\frac{\partial}{\partial\alpha^*}\right]^n Q(\alpha)$$
$$=\exp\left[-\frac{1}{2}\frac{\partial}{\partial\alpha}\frac{\partial}{\partial\alpha^*}\right]Q(\alpha)\ . \tag{7}$$

To compute Eq. (7), we express the Q function in the usual form as the coherent-state expectation value of the field density operator

$$Q(\alpha)=\frac{1}{\pi}\langle\alpha|\rho|\alpha\rangle=\frac{1}{\pi}\mathrm{Tr}\{\rho|\alpha\rangle\langle\alpha|\}$$

so that

$$\left[-\frac{\partial}{\partial\alpha}\frac{\partial}{\partial\alpha^*}\right]^n Q(\alpha)=\frac{1}{\pi}\mathrm{Tr}\left\{\rho\left[-\frac{\partial}{\partial\alpha}\frac{\partial}{\partial\alpha^*}\right]^n|\alpha\rangle\langle\alpha|\right\}\ . \tag{8}$$

Our immediate task is to obtain an expression for

$$\left[-\frac{\partial}{\partial\alpha}\frac{\partial}{\partial\alpha^*}\right]^n |\alpha\rangle\langle\alpha| \ . \tag{9}$$

In order to do that, we note [5] that

$$\frac{\partial}{\partial\alpha}|\alpha\rangle\langle\alpha| = (a^\dagger - \alpha^*)|\alpha\rangle\langle\alpha| \equiv A^\dagger\rho(\alpha) \ , \tag{10a}$$

and

$$\frac{\partial}{\partial\alpha^*}|\alpha\rangle\langle\alpha| = |\alpha\rangle\langle\alpha|(a-\alpha) \equiv \rho(\alpha)A \ . \tag{10b}$$

From Eqs. (10), we obtain an expression for Eq. (8)

$$\left[-\frac{\partial}{\partial\alpha}\frac{\partial}{\partial\alpha^*}\right]^n |\alpha\rangle\langle\alpha| = \sum_{k=0}^{n}(-1)^k\begin{bmatrix} n \\ k\end{bmatrix}^2 (n-k)!(A^\dagger)^k\rho(\alpha)A^k \ . \tag{11}$$

In this form, Eq. (7) becomes

$$W(\alpha) = \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{2^{-n}}{n!}\sum_{k=0}^{n}(-1)^k\begin{bmatrix} n \\ k\end{bmatrix}^2 (n-k)! \times \mathrm{Tr}\{\rho(A^\dagger)^k\rho(\alpha)A^k\} \ . \tag{12}$$

Now we have to evaluate

$$\begin{aligned}\mathrm{Tr}\{\rho(A^\dagger)^k\rho(\alpha)A^k\} &= \langle\alpha|(a-\alpha)^k\rho(a^\dagger-\alpha^*)^k|\alpha\rangle = \langle 0|D^\dagger(\alpha)(a-\alpha)^k\rho(a^\dagger-\alpha^*)^k D(\alpha)|0\rangle \\ &= \langle 0|a^k D^\dagger(\alpha)\rho D(\alpha)(a^\dagger)^k|0\rangle \\ &= k!\langle k|D^\dagger(\alpha)\rho D(\alpha)|k\rangle = k!\langle\alpha,k|\rho|\alpha,k\rangle \ ,\end{aligned}$$

where $|\alpha,k\rangle$ are the displaced number states [6]. Substituting the last expression in Eq. (12) we obtain the Wigner function as an infinite series in terms of the displaced number-state expectation values

$$W(\alpha) = \frac{1}{\pi}\sum_{n=0}^{\infty}2^{-n}\sum_{k=0}^{n}(-1)^k\begin{bmatrix} n \\ k\end{bmatrix}\langle\alpha,k|\rho|\alpha,k\rangle \ . \tag{13}$$

We can simplify Eq. (13) further by noting that the sum up to n may be extended to infinity (because $m! = -\infty$ for $m<0$). By doing so and by interchanging the order of the sums we obtain

$$\begin{aligned}W(\alpha) &= \frac{1}{\pi}\sum_{k=0}^{\infty}(-1)^k\langle\alpha,k|\rho|\alpha,k\rangle\sum_{n=0}^{\infty}2^{-n}\begin{bmatrix} n \\ k\end{bmatrix} \\ &= \frac{1}{\pi}\sum_{k=0}^{\infty}(-1)^k\langle\alpha,k|\rho|\alpha,k\rangle\sum_{n=k}^{\infty}2^{-n}\begin{bmatrix} n \\ k\end{bmatrix} \ . \end{aligned}\tag{14}$$

The second equality holds because of the same reason as above: a factorial of a negative integer diverges to minus infinity. So for $n<k$ all the components in the sum are zero. By using the relation [7]

$$\sum_{n=k}^{\infty}y^{n-k}\begin{bmatrix} n \\ k\end{bmatrix} = (1-y)^{-k-1} \ , \tag{15}$$

we can express Eq. (13) as

$$W(\alpha) = \frac{2}{\pi}\sum_{k=0}^{\infty}(-1)^k\langle\alpha,k|\rho|\alpha,k\rangle \ . \tag{16}$$

Provided the expectation values of the assumed known density matrix in a displaced number state basis can be calculated easily (and this is apparently usually so), the summation in (16) can be evaluated efficiently (especially numerically).

At this point we should note that, given the Q function is normally ordered, its natural states are the coherent states ($a|\alpha\rangle = \alpha|\alpha\rangle$). In this form, when in Eq. (7) we express the Wigner function in terms of the Q function, and, noting the Wigner function is symmetrically ordered, we obtain terms of the form

$$a^\dagger|\alpha\rangle = |\alpha,1\rangle + \alpha^*|\alpha\rangle \ , \tag{17}$$

or, in general, using

$$\begin{aligned}a^\dagger|n\rangle &= D^\dagger(\alpha)(a^\dagger-\alpha^*)D(\alpha)|n\rangle \\ &= (n+1)^{1/2}|n+1\rangle \\ &= (n+1)^{1/2}D^\dagger(\alpha)D(\alpha)|n+1\rangle\end{aligned}$$

it follows

$$a^\dagger|\alpha,n\rangle = (n+1)^{1/2}|\alpha,n+1\rangle + \alpha^*|\alpha,n\rangle \ , \tag{18}$$

thus making the displaced number states the natural states for the non-normally ordered quasiprobability distributions. We can generalize Eqs. (13) and (16) to obtain an expression for Eq. (1), i.e., for any s-parametrized quasiprobability distribution

$$F(\alpha,s) = \frac{1}{\pi}\sum_{n=0}^{\infty}[(s+1)/2]^n\sum_{k=0}^{n}(-1)^k\begin{bmatrix} n \\ k\end{bmatrix}\langle\alpha,k|\rho|\alpha,k\rangle \tag{19a}$$

or

$$F(\alpha,s) = \frac{2}{\pi}\sum_{k=0}^{\infty}(-1)^k\frac{(1+s)^k}{(1-s)^{k+1}}\langle\alpha,k|\rho|\alpha,k\rangle \ . \tag{19b}$$

Note that for $s=-1$, the only term that survives is $n=0$, and then we recover the definition for the Q function as an expectation value in terms of a coherent state.

We can illustrate our expression for the Wigner function by taking ρ to be that for a coherent state, $\rho = |\beta\rangle\langle\beta|$. For this choice of density matrix we obtain the displaced number-state expectation values

$$\langle \alpha,k|\rho|\alpha,k\rangle = |\langle k|\beta-\alpha\rangle|^2 = \exp(-|\beta-\alpha|^2)\frac{|\beta-\alpha|^{2k}}{k!} , \quad (20)$$

and by substituting Eq. (20) into Eq. (16), we obtain the Wigner function as

$$W(\alpha)=\frac{2}{\pi}\exp(-2|\beta-\alpha|^2) . \quad (21)$$

Analogously, for the P function, i.e., with $s=1$ in Eq. (19b), we obtain

$$P(\alpha)=\frac{1}{\pi}\exp(-|\beta-\alpha|^2) \times \lim_{r\to 1-}\exp[-|\beta-\alpha|^2 r/(1-r)]/(1-r) , \quad (22)$$

where $r\to 1-$ means that r tends to one from the left in Eq. (21), for the P function, is a positive singular function of the form obtained by Cahill and Glauber [2], who related this to the two-dimensional δ function expected for a coherent state.

The series forms in Eqs. (16) and (19) will allow a straightforward computation of the field quasiprobabilities in cases where we know the field density matrix but wish to avoid phase-space integration, for example in the Schrödinger-cat states of the field in the Jaynes-Cummings model [8]. Often of course we do *not* know the field density matrix, but instead use Fokker-Planck or other techniques to obtain it from an appropriate quasiprobability evolution. In this case our results are not of value. However, there are many cases as we have pointed out where the converse is true and our series solution may be helpful.

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[1] M. Hillery, R. F. O'Connell, M. O. Scully, and E. P. Wigner, Phys. Rep. **106**, 121 (1984); C. W. Gardiner, *Quantum Noise* (Springer-Verlag, Berlin, 1991).
[2] K. E. Cahill and R. J. Glauber, Phys. Rev. **177**, 1857 (1969); **177**, 1882 (1969).
[3] A. Wünsche, Quantum Opt. **3**, 359 (1991).
[4] R. J. Glauber, Phys. Rev. **130**, 2529 (1963).
[5] W. H. Louisell, *Quantum Statistical Properties of Radiation* (Wiley, New York, 1973).
[6] F. A. M. de Oliveira, M. S. Kim, P. L. Knight, and V. Buzek, Phys. Rev. A **41**, 2645 (1990); P. Král, J. Mod. Opt. **37**, 889 (1990); S. M. Roy and Virendra Singh, Phys. Rev. D **25**, 3413 (1982); M. V. Satyanarayana, *ibid.* **32**, 400 (1985); M. Boiteux and A. Levelut, J. Phys. A **6**, 596 (1973).
[7] *Handbook of Mathematical Functions,* edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1972).
[8] J. Gea-Banacloche, Phys. Rev. Lett. **65**, 3385 (1990); Phys. Rev. A **44**, 5913 (1991); S. J. D. Phoenix and P. L. Knight, Phys. Rev. Lett. **66**, 2833 (1991); Phys. Rev. A **44**, 6203 (1991); V. Buzek, H. Moya-Cessa, P. L. Knight, and S. J. D. Phoenix, *ibid.* **45**, 8190 (1992).