Coherent states on a circle and quantum interference

J. Janszky, P. Domokos, and P. Adam
Crystal Physics Laboratory, Hungarian Academy of Sciences, P.O. Box 132, H-1504 Budapest, Hungary
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As a generalization of the optical Schrödinger cats, discrete sets of coherent states are considered on a circle in the α plane. It is shown that simple superpositions of Schrödinger cats exhibit amplitude squeezing, similarly to the case of a superposition of several coherent states along a straight line that shows quadrature squeezing. The interference fringes between the coherent states form the annuli of the Fock states in the Wigner-function picture. It is also shown that a continuous superposition of coherent states on a circle can serve as a basis for the representation of any state.

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I. INTRODUCTION

There has been considerable interest in the properties of superpositions of macroscopically distinguishable quantum states of light known as "Schrödinger-cat states." Various nonclassical effects emerge due to the quantum interference between the components of such superposition states, e.g., quadrature squeezing, sub-Poissonian photon statistics, oscillation in photon-number distribution, and amplitude squeezing [1–8]. The most elementary cases, i.e., the Yurke-Stoler state [1] and the even and odd coherent state [2], were widely discussed in the literature. The superposition of two coherent states, symmetric to the real axis with identical mean numbers of photons, was investigated by Schleich, Pernigo, and Kien [3].

On the other hand, superposition of coherent states on a one-dimensional manifold in phase space is an alternative representation of the quantum states of light. The number states can be represented on circles with arbitrary radius by continuous superposition. The modulation determines the number state in the following way,

$$|n\rangle=\frac{1}{2\pi}e^{R^2/2}\sqrt{n!}R^{-n}\int e^{-in\varphi}|\mathrm{R}e^{i\varphi}\rangle d\varphi\ . \qquad (1)$$

It was shown that a continuous Gaussian superposition of coherent states on a circle describes an amplitude-squeezed state [4],

$$|\alpha_0,u,\delta\rangle=c\int\exp(-\tfrac{1}{2}u^2\varphi^2-i\delta\varphi)|\alpha_0e^{i\varphi}\rangle d\varphi\ , \qquad (2)$$

where c is an insignificant normalization coefficient, u determines the width of the distribution, α_0 its origin, and δ is a free modulation constant. For $u\to 0$ and $\delta=n$, we get back the expression of number state n in Eq. (1). For $u\to\infty$, the distribution contracts into the coherent state.

We note that a similar distribution on a straight line leads to a quadrature-squeezed state [5]. The k orthonormalized eigenstates of the higher-order powers a^k $(k\geq 3)$ of the annihilation operator a can be represented by a superposition of k coherent states on a circle [8].

Several nonlinear optical processes seem to be suitable for generating "Schrödinger-cat states" [9]. Quantum-nondemolition and back-action-evading measurements [10,11] can also produce nonclassical superposition states. For example, in the nondemolition measurement proposed by Brune and co-workers [11], the detection of atoms crossing a cavity converts the initial coherent state of the field into a series of coherent states on a circle.

In this paper, we will examine more general cases involving the previously mentioned states. We shall deal with quantum interference of coherent states $|\alpha\rangle$ with the same amplitude $|\alpha|=R$ in different arrangements. In Sec. II, we will investigate the interference between coherent states symmetrically separated from each other on the circle, having modulation factors correspondingly to the continuous case [see Eq. (1)]. Situating more and more coherent states on the circle this way, it becomes clear how the number state is built up. Expanding this superpose state to the sum of Fock states, we will describe how this state converges, by increasing the number of constituent coherent states, to a given Fock state depending on the applied modulation. Then in Sec. III, we will construct a superposition state comprised of a given finite number of coherent states on the circle by choosing the phases and weight factors so that the state has the minimal uncertainty in the amplitude. We will compare the symmetric and asymmetric states (defined in Sec. II and Sec. III) with respect to the amplitude variances. Proceeding with this generalization in Sec. IV, the coherent states on a circle will be proved to serve as a potential basis to represent any state of the light. We will furnish the connection of this circle representation with the well-known Glauber representation.

In our calculations, we shall use the Wigner-function approach. To find the Wigner function, it is convenient to obtain the normally ordered characteristic function first,

$$\chi(\eta)=\mathrm{Tr}[\rho\exp(\eta a^\dagger)\exp(-\eta^*a)]\ , \qquad (3)$$

where ρ is the density operator of the system, and a and $a^\dagger$ are the annihilation and creation operators. Performing the integration in Eq. (3) over the complex plane, we get the Wigner function of the system

$$W(\alpha)=\frac{1}{\pi^2}\int \exp(\eta^*\alpha-\eta\alpha^*-\tfrac{1}{2}|\eta|^2)\chi(\eta)d\eta^2 \ . \quad (4)$$

The Wigner function offers a convenient description of the system, e.g., the expectation values of the coordinate and momentum are proportional to integrals of the Wigner function over the imaginary and real axes, respectively ($h/m\omega=1$),

$$|\psi_q|^2=W(q)\equiv\int W(\alpha)d(\mathrm{Im}\alpha), \quad q=\sqrt{2}\,\mathrm{Re}(\alpha) \ , \quad (5)$$

$$|\psi_p|^2=W(p)\equiv\int W(\alpha)d(\mathrm{Re}\alpha), \quad p=\sqrt{2}\,\mathrm{Im}(\alpha) \ . \quad (6)$$

For coherent states, both $W(q)$ and $W(p)$ are Gaussian functions with equal variances. A state is squeezed if one of its $W(q)$ or $W(p)$ has a variance less than that of the coherent state.

II. UNWEIGHTED SUPERPOSITIONS

Let us first consider a state consisting of $N+1$ coherent states that are equidistantly separated from each other along a circle with modulation factor $(\epsilon^*)^n$ of unit absolute value,

$$|n,N\rangle=c_{n,N}\sum_{k=0}^{N}(\epsilon^*)^{nk}|R\epsilon^k\rangle \ , \quad (7)$$

where

$$\epsilon=e^{2\pi i/(N+1)} \ ,$$
$$c_{n,N}^{-2}=\sum_{l,k=0}^{N}(\epsilon^*)^{nk}(\epsilon)^{nl}\exp(R^2\epsilon^{*l}\epsilon^k-R^2) \ . \quad (8)$$

In the photon-number representation, this state has the coefficients

$$\langle m|n,N\rangle=c_{n,N}e^{-(1/2)R^2}\frac{R^m}{\sqrt{m!}}\sum_{k=0}^{N}(\epsilon^*)^{nk}(\epsilon)^{mk} \ . \quad (9)$$

The characteristic function has the form

$$\chi(\eta)=c_{n,N}^2\sum_{l,k=0}^{N}(\epsilon^*)^{nk}(\epsilon)^{nl}\exp(R^2\epsilon^{*l}\epsilon^k-R^2-\eta^*R\epsilon^k+\eta R\epsilon^{*l}) \ . \quad (10)$$

The corresponding Wigner function is

$$W(\alpha)=c_{n,N}^2\frac{2}{\pi}\sum_{l,k=0}^{N}(\epsilon^*)^{nk}(\epsilon)^{nl}\times\exp[R^2\epsilon^{*l}\epsilon^k-R^2-2(\alpha-R\epsilon^k)(\alpha^*-R\epsilon^{*l})] \ . \quad (11)$$

The simplest choices are $N=1$, $n=0$ and 1. The first describes the even, the second one the odd coherent state.

The Wigner function of the simplest superposition state, the even coherent state, shows a strong quantum-mechanical interference. The two Gaussian bells in Fig. 1 represent the coherent states and the fringe between them occurs due to the interference. The wavelength of the fringe is reciprocally proportional to the distance between the coherent states. The closer the coherent states to each other, the less peaks are significant in the fringe. In the case when the fringe partly merges with the two bells, the spread of the Wigner function in the direction of the imaginary axis becomes less than that of a coherent state, i.e., the state is squeezed in momentum [4] (Fig. 2).

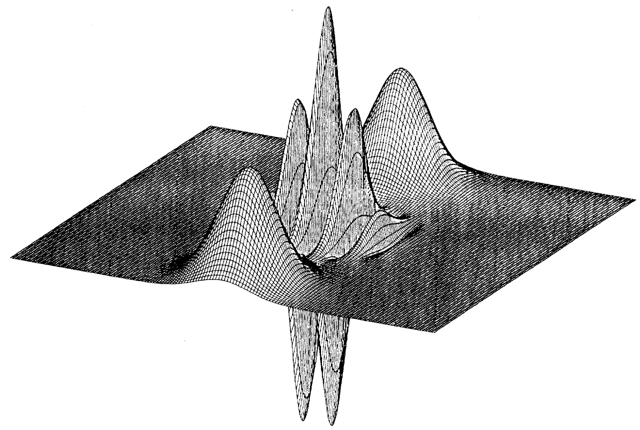

FIG. 1. Wigner function of the even coherent states situated along the real axis with a distance 6 between them. The two bells correspond to the coherent states themselves, while the fringe between them emerges from quantum interference of the two states. If the coherent states are far away, the fringe has many well-pronounced peaks situated near to each other. On the contrary, if the coherent states are near enough, the fringe has only several peaks, which, partly merging with the bells of the coherent states, decrease the uncertainty of one of the quadratures below the vacuum level.

The fringe structure gets more complicated as the

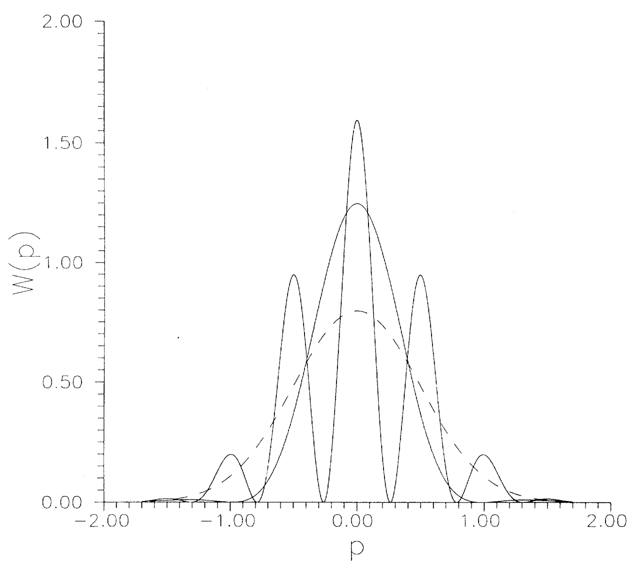

FIG. 2. Shape of the truncated-pulse Wigner function $W(p)$ for the vacuum (dashed line), for real-axis even coherent states. The curve with many peaks corresponds to coherent states with distance 6 (note that its width is practically the same as that of the vacuum), while the other curve with narrow width describes an even coherent state with distance 1.6.

number of the coherent states taking part in the superposition increases. In the case of a few coherent states, they preserve their individuality and the even coherent-state feature is also maintained in pairs. Nevertheless, the effect of the collective interference can be seen as the inner peaks curve to form rings around the center of the phase space (Fig. 3). This fact will be exploited in Sec. III to generate amplitude-squeezed states by superposing only a few coherent states. For a large number of superposed coherent states [m in Eq. (3)], the resulting state will approximate a number state with photon number equal to the modulation number (Fig. 4). It is interesting to note that by adding just one coherent state to a state with a very large photon-number uncertainty, one may convert it into an almost number state (Fig. 5).

The photon statistics of the state consisting of $N+1$ coherent states and defined in Eq. (7) originates from the usual Poisson distribution decimated by a factor strongly depending on n and $N+1$ [see Eq. (9)]. At a given photon number m, this factor is $\sum_{k=0}^{N} e^{i(m-n)k}$. It does not diminish only in the cases if: (a) $n=m$ independently on $N+1$, (b) $m-n=p(N+1)$, where p is an integer. This shows that if we expand the state in Eq. (7) into the series of Fock states, due to the modulation, the photon number n always contributes to the expansion with a finite coefficient [case (a)]. The other remaining Fock states, assigned to the integers $m=n+p(N+1)$ [case (b)] get further and further from the n by increasing $N+1$, hence the sharply decreasing Poisson envelope reduces their contribution in the expansion. This consideration makes it clear how the photon-number state n is built up with an increasing number of coherent states on the circle.

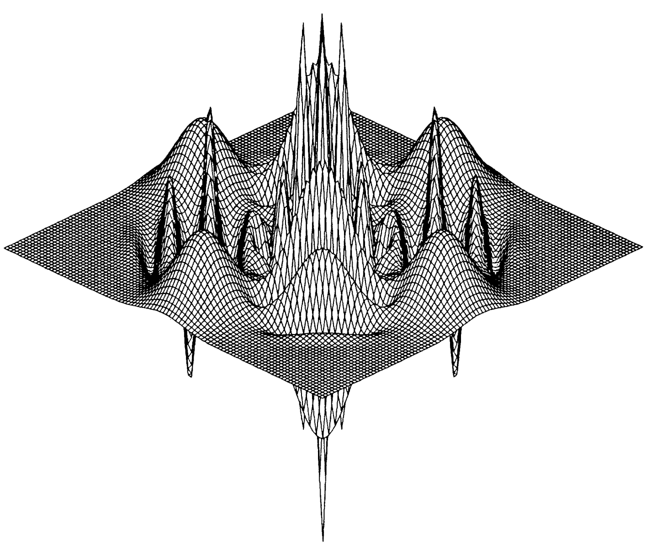

FIG. 3. Wigner function of a superposition of four coherent states evenly distributed along a circle with radius 3. One can still distinguish the bells of the coherent states and the fringes between the nearest neighbors, but around the center of the picture the effect of the collective interference can be noticed to begin forming rings.

III. WEIGHTED SUPERPOSITIONS

As we could see in Sec. II, it is possible to generate practically pure number states by superposing a finite number of coherent states (see also [11]). In this section, we shall find the optimal arrangement of a given number of coherent states to achieve the minimum uncertainty of photon numbers.

Let us consider a superposition of m coherent states on a circle

$$|n,m,\{p,\varphi\}\rangle = c_{n,m,p,\varphi} \sum_{k=-j}^{j} p_k(\delta_k^*)^n |R\delta_k\rangle \ , \tag{12}$$

where

$$\delta_k = e^{i\varphi_k}, \quad \delta_{-k}=\delta_k^*, \quad p_{-k}=p_k, \quad k\geq 0, \quad j=\left[\frac{m}{2}\right] .$$

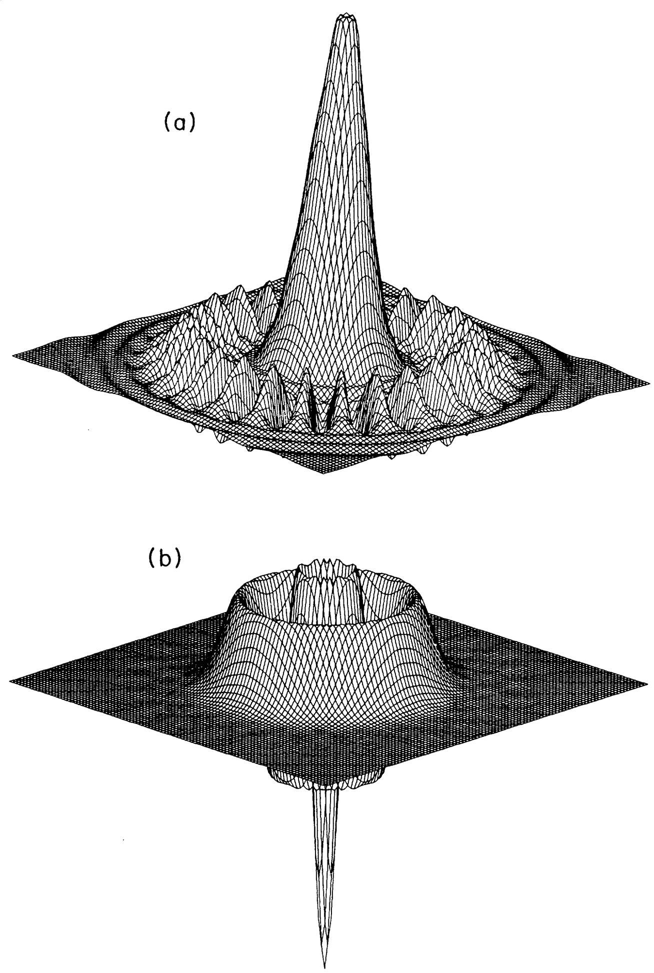

FIG. 4. Wigner function of states formed by 25 coherent states along a circle with radius 3. The individual coherent states are indistinguishable and the emerging state is near to a Fock state. The photon number is determined by the modulation factor n of Eq. (7), which is 0 and 3 for Fig. 4(a) and 4(b), correspondingly. With a higher number of coherent states taking part in forming the state, it will coincide with a number state in every practical purpose.

Here, if m is even, then $p_0=0$, and

$$c_{n,m,p,\varphi}^{-2}=\sum_{l,k=-j}^{j} p_l p_k (\delta_k^*)^n (\delta_l)^n \exp(R^2\delta_l^*\delta_k - R^2) , \quad (13)$$

where p_k's are real coefficients, the angles φ_k define the place of the coherent states on the circle, and the modulation factors δ_k^*'s were chosen so that they correspond to the continuous superposition leading to the number state n [see Eq. (1) and Sec. IV]. Since the distribution function is symmetric in the continuous case in Eq. (1), we supposed that $\varphi_k=-\varphi_{-k}$ and $p_k=p_{-k}$.

This state can be decomposed into the sum of $|k\rangle$ number states,

$$\langle k|n,m,\{p,\varphi\}\rangle = c_{n,m,p,\varphi} e^{-(1/2)R^2} \frac{R^k}{\sqrt{k!}} \left[p_0 + 2\sum_{l=1}^{j} p_l \cos[(k-n)\varphi_l] \right] , \quad (14)$$

where oscillation in the photon-number distribution occurs in a striking way due to the quantum interference. Analogously as in Sec. II, we get the normally ordered characteristic function and the Wigner function of the state in Eq. (12):

$$\chi(\eta)=c_{n,m,p,\varphi}^2 \sum_{l,k=-j}^{j} p_k p_l (\delta_k^*)^n (\delta_l)^n \times \exp(R^2\delta_l^*\delta_k - R^2 - \eta^* R\delta_k + \eta R\delta_l^*) , \quad (15)$$

$$W(\alpha)=c_{n,m,p,\varphi}^2 \frac{2}{\pi} \sum_{l,k=-j}^{j} p_k p_l (\delta_k^*)^n (\delta_l)^n \times \exp[R^2\delta_l^*\delta_k - R^2 - 2(\alpha - R\delta_k) \times (\alpha^* - R\delta_l^*)] . \quad (16)$$

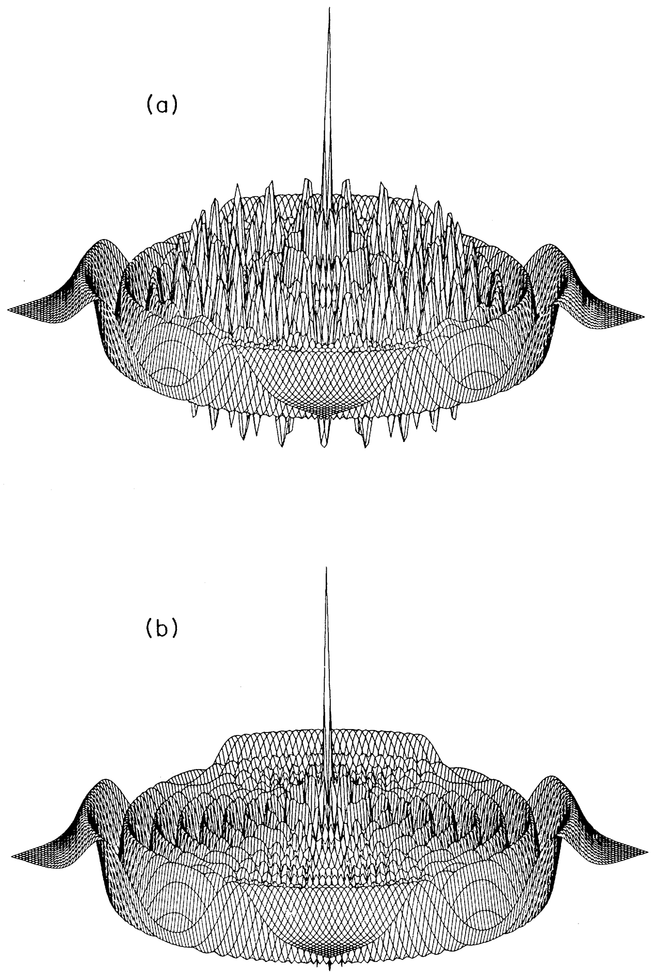

FIG. 5. Transition between a rather complicated state with high photon-number uncertainty and a practically number state sometimes is quite dramatic. Figure 5(a) shows a state consisting of 20 coherent states along a circle with modulation factor $n=20$. Its mean photon number is 16.67 and its square photon-number uncertainty is 55.58. Just adding one coherent state and keeping the same modulation, its mean photon number becomes 20 and the square uncertainty less than 10^{-7} [Fig. 5(b)].

Tables I and II contain the optimized values of the weights and positions of m coherent states and the corresponding minimized variances of the photon numbers for a circle with radius $R=3$. We can see that the phase difference between two adjacent coherent states is approximately 30°. With increasing m, the phase differences get smaller, as we can see in a φ_k column. The further from the center the coherent states are, the larger is the phase difference that can be experienced at a given m number.

For $m=2$ to $m=11$, the values of the variances Δn^2 are absolute minima. In Fig. 6, it can be seen how the fringes of the Schrödinger-cat state evolve into the system of annulus segments of the amplitude-squeezed state. In the case of $m \geq 12$, the equidistantly separated coherent states have lower photon-number variance (Fig. 7) than the asymmetrically situated ones, though for $m=12$ we still found a local minimum ($\Delta n^2=0.33$ while the value of the absolute minimum was 0.29). This transition between the asymmetric and symmetric distribution of the coherent states can be understood easily as a consequence of the interference of the outside states, because with an increasing number of coherent states, the crescent begins to close (for $m=12$ the outside states are at the angles $\pm 137.5°$ at the local minimum).

It is worth noting that the variances rapidly diminished with increasing m. The best fit was a power function with an exponent -1.347.

TABLE I. For odd m, the optimized values of the weights and the positions of m coherent states and the corresponding minimized variances of the photon numbers for a circle with radius $R=3$ are listed.

m	p_0	p_1	p_2	p_3	p_4	p_5	φ_1	φ_2	φ_3	φ_4	φ_5	Δn^2
1	1											9
3	0.57	0.43					29.2					2.497
5	0.43	0.39	0.25				26.9	55.6				1.235
7	0.36	0.34	0.28	0.17			25.2	51.6	80.3			0.752
9	0.32	0.30	0.27	0.21	0.13		24.7	49.3	75.1	103.2		0.512
11	0.28	0.27	0.25	0.22	0.17	0.11	23.5	47.6	72.2	98.0	126.1	0.373

IV. CONTINUOUS SUPERPOSITIONS

Let us now consider a state emerging from a continuous superposition of coherent states lying on the same circle

$$|F,R\rangle=\frac{e^{R^2/2}}{2\pi}\int F(\varphi)|Re^{i\varphi}\rangle d\varphi \ ; \tag{17}$$

in order to simplify the following expressions, we defined the factor before the integral here. For the Fock state $|n\rangle$, we have a simple weight function

$$F(n,\varphi)=\sqrt{n!}R^{-n}e^{-in\varphi} \ . \tag{18}$$

It means that for a state with a Fock-state representation

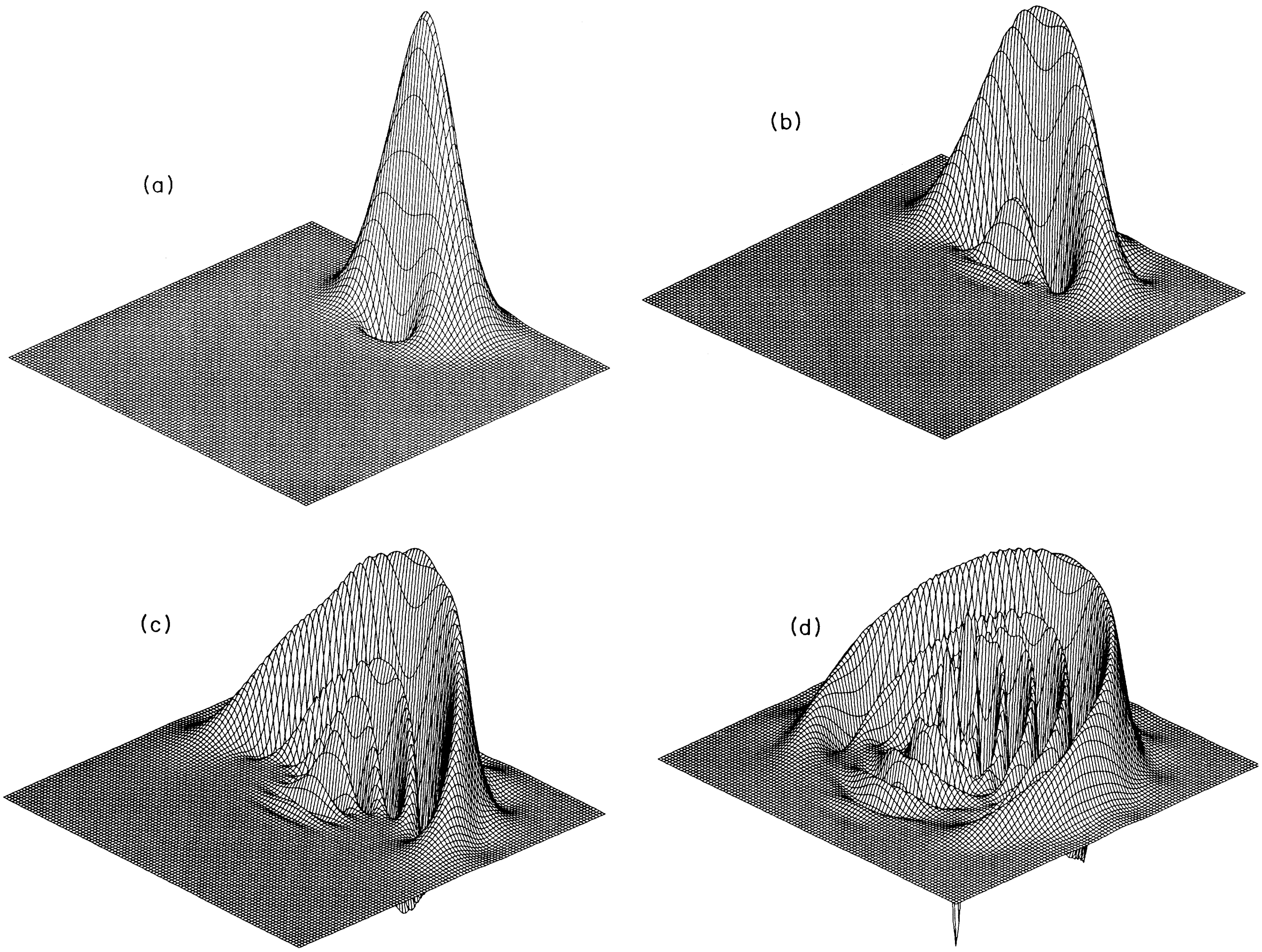

FIG. 6. Wigner function of photon-number-optimized coherent-state superpositions along a circle with radius 3. The number of coherent states is two [Fig. 6(a)], four [Fig. 6(b)], seven [Fig. 6(c)], and 12 [Fig. 6(d)]. With an increasing number of coherent states, the interference fringes between the coherent states form the segments of the annuli of the amplitude-squeezed state.

TABLE II. For even m, the optimized values of the weights and the positions of m coherent states and the corresponding minimized variances of the photon numbers for a circle with radius $R=3$ are listed.

m	p_1	p_2	p_3	p_4	p_5	p_6	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	Δn^2
2	0.63						15.5						4.154
4	0.47	0.32					13.8	42.4					1.693
6	0.38	0.33	0.21				13.2	39.5	68.2				0.947
8	0.33	0.30	0.24	0.15			12.6	37.8	63.6	91.6			0.613
10	0.29	0.27	0.24	0.19	0.11		12.0	36.1	60.7	86.5	114.6		0.434
12	0.26	0.26	0.23	0.20	0.15	0.09	11.5	35.0	59.0	83.7	109.4	137.5	0.326

$$|\,\rangle=\sum_{n=0}^{\infty} c_n|n\rangle\ , \tag{19}$$

the $F(\varphi)$ weight function of the circle representation has Fourier coefficients

$$F_n=\sqrt{n!}R^{-n}c_n\ , \tag{20}$$

where

$$F(\varphi)=\sum_{n=0}^{\infty} e^{-in\varphi}F_n\ . \tag{21}$$

For a coherent state, we find the circle weight function from Eq. (17)

$$F(\alpha,\varphi)=\frac{Re^{i\varphi}e^{-|\alpha^2|/2}}{Re^{i\varphi}-\alpha},\quad |\alpha|<R\ . \tag{22}$$

For a state with Glauber's analytic function $f(\alpha^*)$,

$$|f\rangle=\frac{1}{\pi}\int f(\alpha^*)e^{-|\alpha|^2/2}|\alpha\rangle d^2\alpha\ . \tag{23}$$

If we chose the radius of the circle big enough so that the integration over the α plane in Eq. (23) can be replaced by an integration inside the circle,

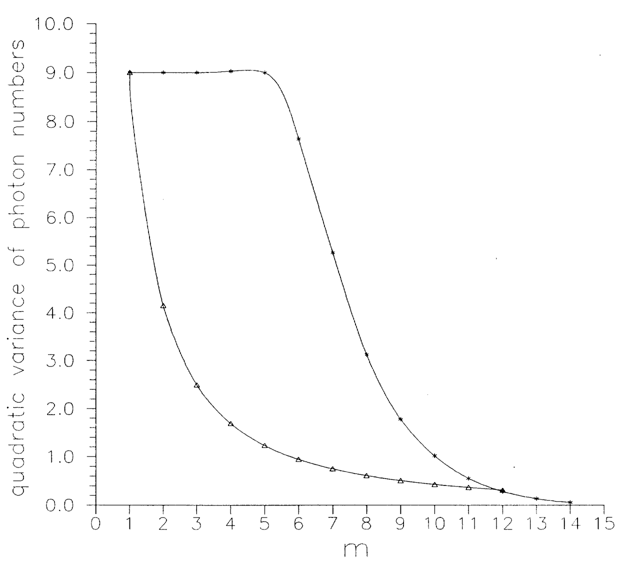

FIG. 7. Square of the uncertainty of photon number as a function of the coherent-state number for unweighted (marked by quadrangles) and optimized weighted (marked by asterisks) distribution of coherent states. For small numbers, the weighted distribution has much less uncertainty than the evenly distributed one.

$$|f\rangle=\frac{1}{\pi}\int_{|\alpha|<R} f(\alpha^*)e^{-|\alpha|^2/2}|\alpha\rangle d^2\alpha\ , \tag{24}$$

then from Eq. (22) we can find the connections between the circle weight function $F(\varphi)$ and the analytic function $f(\alpha^*)$,

$$F(\varphi)=\frac{Re^{i\varphi}}{\pi}\int_{|\alpha|<R} f(\alpha^*)\frac{e^{-|\alpha|^2/2}}{Re^{i\varphi}-\alpha}d^2\alpha\ , \tag{25}$$

and

$$f(\alpha^*)=\int F(\varphi)e^{\alpha^* Re^{i\varphi}}d\varphi\ . \tag{26}$$

This way we have shown that, using only those coherent states that lie on a circle with a given radius, one can construct any physical state. The equal distribution of coherent states leads to a Fock state; here the photon number is determined by the modulation factor. Another simple case is the Gaussian distribution resulting in amplitude squeezing.

V. CONCLUSIONS

We examined the problem of quantum-mechanical interference of multiple Schrödinger-cat states consisting of coherent states on the same circle. It was shown that, with the increase of the number of the coherent states on the circle, the fringes between two coherent states in a Schrödinger cat can interfere constructively, forming concentric annuli that lead to a reduction in the variance of the photon number. We found the optimal distribution of a given number of coherent states on a circle having the maximal amplitude squeezing. in the case of a few coherent states, the optimal distribution was asymmetric, while above a certain threshold in the number of coherent states, depending on the radius of the circle, it turned out to be symmetric.

We considered the continuous superpositions and found that coherent states with the same amplitude form a complete set to describe any state of the quantum oscillator. We determined the connection between this circle representation on one hand and the Glauber-Klauder-Sudarshan representation and the photon-number representation on the other hand.

ACKNOWLEDGMENT

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