Phase-equivalent potentials for arbitrary modifications of the bound spectrum

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An analytic expression for a potential with the same phase shifts as a given potential but an arbitrarily different bound spectrum is derived with supersymmetric factorizations. The wave functions of the phase-equivalent potential can be calculated from the wave functions of the original potential. From their expressions, a general form of the integral kernel appearing in the Abraham-Moses method for deriving phase-equivalent potentials is established, with which the equivalence between the two approaches is proved. Different particular cases are considered: removing (or adding) one or several bound states at arbitrary negative energies and modifying the energy of a bound state. The similarity between the expressions obtained for removing or adding an excited state with those obtained in earlier works for the ground state leads to an extension of supersymmetric factorizations where potentials which are singular at finite distance are employed in intermediate steps of the calculation.

I. INTRODUCTION

Supersymmetric [1–4] or Darboux [5] factorizations allow one to construct phase-equivalent potentials [6] whose spectra differ by a number of bound states [7–9]. However, until now, the modifications only concern the lowest part of the bound spectrum. A number of *lowest* bound states can be suppressed or new bound states located *below* the ground state can be added. This restriction of the supersymmetric approach is not necessarily shared by other techniques for modifying the bound spectrum (see, e.g., Ref. [10]). Recently, iterations of the supersymmetric technique [9] have opened a way towards more general modifications of the spectrum. This paper is devoted to their derivation and an analysis of their consequences.

Sukumar [3,4] has classified different types of supersymmetric factorizations. When these factorizations are employed for modifying the bound spectrum of a given potential, the phase shifts are modified. This modification is unavoidable because of the Levinson theorem. Starting from Sukumar's results, I have shown that a phase-equivalent potential can be obtained after removal of the ground state of a given potential V_0 by performing *two* successive supersymmetric transformations [7]. The potential resulting from the second transformation reads [8]

$$V_2 = V_0 - 2\frac{d^2}{dr^2}\ln \int_0^r \psi_0^{(0)2}dt, \qquad (1)$$

where $\psi_0^{(0)}$ is the ground-state wave function of V_0. This potential is singular, in agreement with generalized versions of the Levinson theorem [11]. As this procedure can easily be iterated, it allowed me to solve the ambiguity problem of different $\alpha + \alpha$ potentials. The wave functions of the equivalent potentials are different and this differ-

ence should be observable in a bremsstrahlung reaction [12,13]. In a similar way, two successive supersymmetric transformations allow one to add a bound state below the ground state of V_0 without modifying the phase shifts [8].

In Ref. [9] (to be referred to as AB in the following), Ancarani and I derive general analytic formulas for the suppression of the N lowest bound states of a spectrum. Phase-equivalent potentials are expressed with determinants of integrals similar to the integral appearing in (1), involving the N bound-state wave functions $\psi_0^{(i)}$ ($i = 0$ to $N-1$) of the initial Hamiltonian H_0. Although such a formula is probably not more efficient than the iterative procedure [7] in numerical calculations, it is helpful for clarifying the comparison with other techniques. The addition of N bound states below the ground state of V_0 and the general form of phase-equivalent potentials sharing the same bound spectrum as V_0 are also considered in AB. In fact, as we shall see below, these formulas provide an ideal starting point for extensions to fully arbitrary modifications of the bound spectrum. By combining them, one can indeed remove a number of bound states, add new bound states, and reintroduce some of the removed states to reach a completely different spectrum.

Before generalizing the results of the supersymmetric approach, let me come back to the Gel'fand-Levitan technique described by Abraham and Moses in Ref. [10] (to be referred to as AM). These authors establish a general algorithm for deriving phase-equivalent potentials but do not provide analytic expressions for these potentials. Different authors have compared these two methods whose spirit is rather different [14,4,15,16]. The equivalence between the AM procedure and two successive supersymmetric transformations, when the ground state is removed, was soon realized [14,4,16]. In that case, the AM equivalent potential (50) offers a striking similarity with (1), although it is not presented in the context of the ra-

dial equation. However, contrary to the supersymmetric approach, the AM technique is not restricted to removing the ground state and the formula (50) in AM is also valid for removing any excited state. This puzzling fact indicates that the problem is not fully understood yet in the supersymmetric framework. The present generalization and extension of the supersymmetric approach provides a better basis for a comparison with AM.

In Sec. II a formula valid for the most general modification of the energy bound spectrum is established. This formula is applied in Sec. III to a comparison with the AM method. A general solution of the AM integral equation is derived. Section IV describes the removal or addition of any number of arbitrary bound states. The important particular case of modifying the energy of a bound state is also considered. In Sec. V, I come back to the principle of supersymmetric factorization in order to understand some results of Sec. IV. Concluding remarks are presented in Sec. VI.

II. GENERAL SPECTRUM MODIFICATION

For a fixed bound spectrum, finding the most general phase-equivalent potential is a well-understood problem [17] (see also AB for the supersymmetric approach). Therefore, I focus here on finding *one* potential whose energy spectrum has been modified in a given way. Other potentials with the same spectrum can then be derived with the standard technique in a further step. This will allow me to choose a definite value for some arbitrary constants which appear in the procedure in order to simplify as much as possible the final result.

First, let me define the notations and conventions. The states belonging to the bound spectrum of the initial Hamiltonian H_0 are numbered with integers in the order of increasing energies: $E^{(0)} < E^{(1)} < E^{(2)} < \cdots < 0$. Among these bound states, M states with labels $s_1 < s_2 < \cdots < s_M$ will be suppressed. Let S_0 be the set of these labels. Then auxiliary sets S_k, depending on some positive integer k, are defined as

$$S_k = \{i \in S_0 | \ i \geq k\}. \tag{2}$$

The number of elements in S_k is denoted as σ_k. Important particular values are $\sigma_0 = M$ and $\sigma_n = 0$ for $n > s_M$.

Simultaneously, N states numbered from -1 to $-N$ with increasing energies $E^{(-1)} < E^{(-2)} < \cdots < E^{(-N)} < 0$ are added to the bound spectrum. The set of their indices is called A_0. The added energies $E^{(i)}$ $(i < 0)$ and the energies $E^{(i)}$ $(i \geq 0)$ of the initial spectrum must be different but they are not related by any imposed order relationship. Finally, let $n > s_M$ be some integer satisfying (if a bound state with index n exists) the condition $E^{(n)} > E^{(-N)}$.

The principle of the calculation is very simple. First, the n states with labels 0 to $n-1$ are suppressed with the iterative procedure described in Ref. [7]. This provides a potential V_{2n}. Second, $n + N - M$ states are introduced with the technique explained in AB, i.e., the N states

to be added and the $n - M$ states which should not be removed (whose labels do not belong to S_0). Auxiliary sets A_k containing indices of added states are now defined for $k \geq 0$ as

$$A_k = \{i \notin S_0 | \ i < k\}. \tag{3}$$

The A_k and S_k will be useful in recurrence proofs. The number of elements in A_k is $k + \sigma_k + N - M$.

Let me denote as V_{2k} the potential obtained after removing the k lowest bound states of V_0. This potential does not possess bound states at energies $E^{(i)}$ with $i < k$ and more generally at energies $E^{(i)}$ with $i \in A_k$. Hence, at those energies, square integrable solutions of the Schrödinger equation do not exist. Let the functions $f_{2k}^{(i)}$ be solutions, bounded at infinity, of the differential equation

$$\left[-\frac{d^2}{dr^2} + V_{2k}(r) \right] f_{2k}^{(i)}(r) = E^{(i)} f_{2k}^{(i)}(r) \quad (i \in A_k). \tag{4}$$

Following AB, a matrix $\boldsymbol{F}_{2k}$ is defined whose elements read

$$F_{2k}^{(i,j)}(r) = \alpha_i \delta_{ij} + \int_r^\infty f_{2k}^{(i)} f_{2k}^{(j)} dt \quad (i,j \in A_k), \tag{5}$$

where the α_i are positive parameters. As mentioned above, I choose $\alpha_i = 1$ when $i \geq 0$ in order to simplify the calculation. The parameters α_i of the added states $(i \in A_0)$ do not introduce any complication and are left free. Notice that indices i and j in the definition (5) do not take all possible positive values smaller than k and can take negative values. The indices in $\boldsymbol{F}_{2k}$ are therefore defined in a generalized sense.

Equation (41) of AB leads to the potential

$$V = V_{2n} - 2 \frac{d^2}{dr^2} \ln \det \boldsymbol{F}_{2n}, \tag{6}$$

where $\boldsymbol{F}_{2n}$ is a matrix whose dimension is $n + N - M$. The potential defined in (6) is already the final equivalent potential I am looking for. The N states -1 to $-N$ belong to its spectrum and the M states s_1 to s_M are suppressed. However, V is expressed as a function of solutions corresponding to the potential V_{2n} and not to V_0. Therefore, V must now be expressed in a more convenient way. To this end, let me introduce the auxiliary symmetric matrices $\boldsymbol{X}_{2k}$ with elements

$$X_{2k}^{(i,j)} = \begin{cases} F_{2k}^{(i,j)} & (i,j \in A_k) \\ \int_r^\infty f_{2k}^{(i)} \psi_{2k}^{(j)} dt & (i \in A_k, j \in S_k) \\ -\Psi_{2k}^{(i,j)} & (i,j \in S_k), \end{cases} \tag{7}$$

where the $\psi_{2k}^{(j)}$ are physical eigenfunctions of potential V_{2k} at energy $E^{(j)}$. Indeed, the states belonging to S_k are not suppressed for V_{2k}. In the last line of (7), a notation already employed in AB represents

$$\Psi_{2k}^{(i,j)}(r) = \int_0^r \psi_{2k}^{(i)} \psi_{2k}^{(j)} dt \quad (i,j \in S_k). \tag{8}$$

The dimension of matrix $\boldsymbol{X}_{2k}$ is given by

$$d_k = k + 2\sigma_k + N - M \tag{9}$$

and depends therefore on k. The matrices $\boldsymbol{X}_{2k}$ present some resemblance with the matrices $\boldsymbol{M}_{2k}$ defined in Eq. (53) of AB. This is not surprising as the same type of algorithm is employed in that work to construct a general form of phase-equivalent potential with the same bound spectrum as a given potential. However, important differences between them are that the dimension of $\boldsymbol{X}_{2k}$ is variable and that not all values of the indices are allowed in the definition of $\boldsymbol{X}_{2k}$. For example, index k appears in $\boldsymbol{X}_{2k}$ if and only if $k \in S_0$. These differences make the properties of $\boldsymbol{X}_{2k}$ more complicated than those of $\boldsymbol{M}_{2k}$.

For $k = n$, the definition (7) provides

$$\boldsymbol{X}_{2n} = \boldsymbol{F}_{2n}. \tag{10}$$

Moreover, the matrices $\boldsymbol{X}_{2k}$ satisfy the recurrence relations

$$\det \boldsymbol{X}_{2k+2} = \begin{cases} -[\Psi_{2k}^{(k,k)}]^{-1} \det \boldsymbol{X}_{2k} & (k \in S_0) \\ +[\Psi_{2k}^{(k,k)}]^{-1} \det \boldsymbol{X}_{2k} & (k \notin S_0) \end{cases} \tag{11}$$

which are proved in the Appendix. In spite of their similarity, the two equations (11) describe rather different behaviors. When $k \in S_0$, the dimension of $\boldsymbol{X}_{2k}$ exceeds the dimension of $\boldsymbol{X}_{2k+2}$ by one unit because k belongs to S_k but not to S_{k+1}. The opposite occurs when $k \notin S_0$ because k belongs to A_{k+1} but not to A_k. According to (1), the potentials V_{2k+2} and V_{2k} are related by

$$V_{2k+2} = V_{2k} - 2\frac{d^2}{dr^2} \ln \Psi_{2k}^{(k,k)}. \tag{12}$$

Equations (6), (10), (11), and (12) show that the relation

$$V = V_{2k} - 2\frac{d^2}{dr^2} \ln \det \boldsymbol{X}_{2k} \tag{13}$$

holds for any k. The sign in (11) does not play any role in a logarithmic derivative. Hence, for $k = 0$, the phase-equivalent potential reads

$$V = V_0 - 2\frac{d^2}{dr^2} \ln \det \boldsymbol{X}_0. \tag{14}$$

This equation is the central result of this paper and its consequences are discussed in Secs. III and V while particular cases are analyzed in Sec. IV. According to (7), the matrix $\boldsymbol{X}_0$ contains integrals involving the wave functions $\psi_0^{(i)}$ of the suppressed states ($i \in S_0$) and the functions $f_0^{(i)}$, bounded at infinity, corresponding to the added bound states ($i \in A_0$). The striking property of (14) is that it does not depend at all on the states which are removed and reintroduced in order to comply with the requirements of the supersymmetric approach. This simplicity is related to the choice $\alpha_i = 1$ for $i \notin S_0$ and $i \geq 0$.

The wave functions corresponding to the final potential V can now be derived as a function of the eigenfunctions of V_0 by using a similar algorithm based on Eqs. (42) and

(43) of AB. To this end, let me introduce the matrices $\boldsymbol{Y}_{2k}(\varphi_{2k})$ which are obtained by adding one row and one column (denoted with index d_k+1) to $\boldsymbol{X}_{2k}$. They depend on a function φ_{2k}, which may represent an unspecified ψ_{2k} or f_{2k}. When $\varphi_{2k} = \psi_{2k}$, it may represent either a bound state or a scattering state. Let me also define the functions

$$\varphi_{2k}^{(i)} = \begin{cases} f_{2k}^{(i)} & (i \in A_k) \\ \psi_{2k}^{(i)} & (i \in S_k) \end{cases} \tag{15}$$

which correspond to energy $E^{(i)}$. The elements of $\boldsymbol{Y}_{2k}(\varphi_{2k})$ are then defined as

$$Y_{2k}^{(i,j)}(\varphi_{2k}) = \begin{cases} X_{2k}^{(i,j)} & (i, j \leq d_k) \\ \varphi_{2k}^{(i)} & (i \leq d_k, j = d_k + 1) \\ \int_r^\infty \varphi_{2k}^{(j)} \varphi_{2k} dt & (i = d_k + 1, j \leq d_k) \\ \varphi_{2k} & (i = j = d_k + 1). \end{cases} \tag{16}$$

This nonsymmetric matrix resembles a matrix whose determinant is displayed in Eq. (63) of AB. In addition, let me define the matrices $\boldsymbol{X}_{2k(m)}$ with elements

$$X_{2k(m)}^{(i,j)} = \begin{cases} X_{2k}^{(i,j)} & (j \neq m) \\ \varphi_{2k}^{(i)} & (j = m). \end{cases} \tag{17}$$

Up to a permutation, these matrices are obtained by suppressing the last row and the mth column in $\boldsymbol{Y}_{2k}$. Notice that these matrices only exist if m appears as an allowed index in $\boldsymbol{X}_{2k}$. Their existence is therefore subject to the conditions $k \leq m$ if $m \in S_0$ and $k > m$ if $m \notin S_0$.

As shown in the Appendix, Eqs. (42) and (43) of AB lead to

$$\psi(r) = [\det \boldsymbol{X}_0]^{-1} \det \boldsymbol{Y}_0(\psi_0) \tag{18}$$

for arbitrary physical wave functions ψ_0 of V_0, i.e., for scattering states or for nonsuppressed bound states, including those which are suppressed and reintroduced. For suppressed states $\psi_0^{(m)}$ ($m \in S_0$), ψ vanishes because identical rows appear in $\det \boldsymbol{Y}_0(\psi_0^{(m)})$ if $\int_r^\infty \psi_0^{(j)} \psi_0 dt$ is understood as $-\int_0^r \psi_0^{(j)} \psi_0 dt$. The wave functions of added states read

$$\psi^{(m)}(r) = \alpha_m^{1/2} [\det \boldsymbol{X}_0]^{-1} \det \boldsymbol{X}_{0(m)} \quad (m \in A_0) \tag{19}$$

as also proved in the Appendix.

III. COMPARISON WITH THE METHOD OF ABRAHAM AND MOSES

Starting from (18), one can make a detailed comparison with the AM method. To this end, I expand $\det \boldsymbol{Y}_0$ with respect to its last row and rewrite (18) as Eq. (12) of AM under the form

$$\psi(r) = \psi_0(r) - \int_r^\infty K(r,t)\psi_0(t)dt, \tag{20}$$

where

$$K(r,t) = \sum_{i \in A_0} \alpha_i^{-1/2} \psi^{(i)}(r) f_0^{(i)}(t) + \sum_{i \in S_0} f^{(i)}(r) \psi_0^{(i)}(t).$$

$$(21)$$

In (21), the physical wave functions $\psi^{(i)}(r)$ defined in (19) are employed. These functions are known expressions of the $f_0^{(i)} (i \in A_0)$ and $\psi_0^{(i)} (i \in S_0)$. For symmetry reasons, I introduce the additional functions

$$f^{(m)}(r) = [\det \boldsymbol{X}_0]^{-1} \det \boldsymbol{X}_{0(m)} \quad (m \in S_0). \qquad (22)$$

Their expression is completely similar to (19) but cannot describe a bound state of the final potential $V(r)$ as the states $m \in S_0$ have been suppressed. A function $f^{(m)}(r)$ is a nonphysical solution of the Schrödinger differential equation with the potential $V(r)$ given in (14), at energy $E^{(m)}$. Proving this statement requires general expressions for nonphysical functions which are not available in AB. However, one can show in a completely parallel way that Eq. (42) of AB remains valid for a nonphysical function f_0. Starting from this modified equation, it is proved in the Appendix that $f^{(m)}(r)$ in (22) is really what its notation represents, i.e., a solution, bounded at infinity, of the Schrödinger differential equation.

The expression (21) displays an elegant symmetry between the physical and nonphysical solutions of the initial and final potentials. I shall now prove that it corresponds to the kernel required in the AM method. First, let me calculate $K(r,r)$. With (19), (22), and the definition of the derivative of a determinant, one immediately obtains

$$K(r,r) = -\frac{d}{dr} \ln \det \boldsymbol{X}_0(r). \qquad (23)$$

Introducing this result in (14) provides the AM formula for a phase-equivalent potential.

To complete the proof, one must show that $K(r,t)$ satisfies an equation of the Gel'fand-Levitan type with the kernel

$$\Omega(r,t) = \sum_{j \in A_0} \alpha_j^{-1} f_0^{(j)}(r) f_0^{(j)}(t)$$

$$- \sum_{j \in S_0} \psi_0^{(j)}(r) \psi_0^{(j)}(t) \qquad (24)$$

defined in Eq. (9) of AM. This is most easily done by introducing (21) and (24) into $\int_r^\infty K(r,t)\Omega(t,u)dt$. The integrals over t can be expressed with (5) and (8) as a function of $F_0^{(i,j)}$ and $\Psi_0^{(i,j)}$. The δ_{ij} symbols which then appear provide $-K(r,u)$. The remaining expression contains the sum

$$(\det \boldsymbol{X}_0)^{-1} \sum_i X_0^{(i,j)} \det \boldsymbol{X}_{0(i)} = \varphi_0^{(j)} \qquad (25)$$

which is proved by elementary determinant properties. Hence the kernel $K(r,u)$ verifies the equation

$$K(r,u) = \Omega(r,u) - \int_r^\infty K(r,t)\Omega(t,u)dt \qquad (26)$$

as in Eq. (10) of AM.

In the present section, I not only prove the equivalence between the supersymmetric and AM approaches in a general case but I also provide the analytic solution (21) for the AM kernel. This solution can be written either in mixed notation as in (21), or only as a function of the initial solutions with the help of (19) and (22). Although both methods are equivalent, phase-equivalent potentials are not represented by the same formula. The supersymmetric expression (14) is more convenient than the equivalent expression (11) in AM.

IV. PARTICULAR CASES AND DISCUSSION

A. Suppression of M arbitrary bound states

The result (30) in AB provides the potential obtained after removing the lowest bound states of a spectrum. From (14) it can be generalized by the amazingly simple expression

$$V = V_0 - 2\frac{d^2}{dr^2} \ln \det \boldsymbol{\Psi}_0, \qquad (27)$$

where $\boldsymbol{\Psi}_0$ is an $M \times M$ matrix whose matrix elements [Eq. (8)] depend on indices belonging to S_0. The difference with Eq. (30) of AB is only that the suppressed states which appear in $\boldsymbol{\Psi}_0$ can be arbitrary bound states. The apparent obligation of sticking to the lowest bound states in the supersymmetric approach disappears.

Equation (27) can be particularized to the suppression of an arbitrary bound state $\psi_0^{(i)}$ as

$$V_2 = V_0 - 2\frac{d^2}{dr^2} \ln \int_0^r \psi_0^{(i)2} dt. \qquad (28)$$

This is nothing but Eq. (1), valid for any bound state. I shall come back on this extended validity in Sec. V. If one has to remove M bound states, one can either apply (27) or apply (28) iteratively to a succession of potentials $V_2, V_4, \ldots$. In numerical calculations, the iterative scheme is very easy to program. It does not involve a determinant calculation as in (27). Therefore, it should be recommended.

Finally, Eq. (27) or the iterative scheme provide a single potential with M bound states removed. Other potentials can be obtained by applying to this potential the transformation (58) of AB.

B. Addition of N arbitrary bound states

The result (41) in AB is immediately generalized from (14) to

$$V = V_0 - 2\frac{d^2}{dr^2} \ln \det \boldsymbol{F}_0, \qquad (29)$$

where $F_0^{(i,j)}$ is defined by (5). There is no apparent difference between (29) and the AB result. The generalization arises in the fact that the added negative energies $E^{(i)} (i < 0)$ are not restricted any more to values smaller than $E^{(0)}$. Again the generalization is extremely sim-

ple and consists in forgetting the usual supersymmetry restriction of adding states only below the ground state.

Equation (29) can be particularized to the addition of a single bound state at an arbitrary negative energy $E^{(-1)}$ as

$$V_2 = V_0 - 2\frac{d^2}{dr^2}\ln\left(\alpha_{-1} + \int_r^\infty f_0^{(-1)2}dt\right). \tag{30}$$

This generalizes Eq. (3.8) of Ref. [8]. This equation can be the starting point of an iterative process equivalent to Eq. (29). However, phase equivalence is in that case restricted by condition (50) of AB, which assumes that the potential is singular enough to allow additional bound states.

$$V = V_0 - 2\frac{d^2}{dr^2}\ln\left|\begin{array}{cc}\alpha_{-1} + \int_r^\infty f_0^{(-1)2}dt & \int_r^\infty f_0^{(-1)}\psi_0^{(i)}dt \\ \int_r^\infty f_0^{(-1)}\psi_0^{(i)}dt & -\int_0^r \psi_0^{(i)2}dt\end{array}\right|. \tag{31}$$

This expression involves the bound-state wave function $\psi_0^{(i)}$ at the initial energy $E^{(i)}$ and the nonphysical solution $f_0^{(-1)}$ bounded at infinity, at the final energy $E^{(-1)}$. The parameter α_{-1} provides some freedom in the potential.

The potential V in (31) can also be obtained in another way. First, (28) is employed to remove the state at energy $E^{(i)}$. Second, a state is introduced at energy $E^{(-1)}$ with (30). This requires calculating the function $f_2^{(-1)}$ for the intermediate potential V_2. In principle, the order of the operations could be reversed but then it would be subject to a validity condition in the first step (see Ref. [8] and Sec. IV B). The iterative algorithm is probably more accurate in practice than a direct use of (31).

V. EXTENDED SUPERSYMMETRIC FACTORIZATIONS

Usually, supersymmetric factorizations are not performed at energies larger than the ground-state energy $E^{(0)}$. In order to question this restriction, consider some bound state at energy $E^{(i)}$. A phase-equivalent potential is obtained after two supersymmetric factorizations denoted as $T_+^+(E^{(i)})$ and $T_-^0(E^{(i)})$ in AB. The initial Hamiltonian $H_0 = -d^2/dr^2 + V_0(r)$ is written as [3]

$$H_0 = A_0^+ A_0^- + E^{(i)}. \tag{32}$$

In this T_+^+ factorization, the linear differential operators A_0^+ and A_0^- read

$$A_0^+ = (A_0^-)^\dagger = \frac{d}{dr} + \frac{d}{dr}\ln\psi_0^{(i)}. \tag{33}$$

C. Moving bound states

While Secs. IV A and IV B provide natural extensions of cases which were considered in earlier works, the present section provides a new type of phase equivalence. Indeed, let me now consider how to move bound states from one energy to another one without affecting the phase shifts. This possibility might be useful in applications where a potential reproducing some phase shifts does not provide the correct bound spectrum of the system. This problem has been encountered, e.g., in a potential model for the study of radiative-capture reactions of astrophysical importance [18].

The general formula for moving N bound states is given by (14) with $M = N$. In order to have a better insight into its structure, let me now focus on the displacement of a single state. Then, Eq. (14) becomes

Then, the supersymmetric partner H_1 of H_0 is defined as

$$H_1 = A_0^- A_0^+ + E^{(i)} = -\frac{d^2}{dr^2} + V_1. \tag{34}$$

The expression (33) is usually restricted to the ground state $i = 0$ because the nodes of $\psi_0^{(i)}$ lead to finite-distance singularities in V_1 for an excited state.

However, phase equivalence requires two successive factorizations. In a second step T_-^0, one writes [7]

$$H_1 = A_1^+ A_1^- + E^{(i)}, \tag{35}$$

where the separation energy $E^{(i)}$ does not correspond any more to a physical state in the H_1 spectrum. The operators A_1^+ and A_1^- read [8]

$$A_1^+ = (A_1^-)^\dagger$$
$$= \frac{d}{dr} + \frac{d}{dr}\ln\left\{[\psi_0^{(i)}]^{-1}\int_0^r \psi_0^{(i)2}dt\right\}. \tag{36}$$

The supersymmetric partner H_2 of H_1 is defined as

$$H_2 = A_1^- A_1^+ + E^{(i)} \tag{37}$$

and provides the potential V_2 displayed in Eq. (28). Again the function appearing in the operators A_1^+ and A_1^- displays singularities at finite distance for excited states. However, these singularities do not arise in the positive function $\int_0^r \psi_0^{(i)2}dt$ which determines V_2.

The similarity between (28) and (1) indicates that an unnecessary restrictive assumption occurs in the derivation of (1). Indeed, Eq. (28) is derived after a long sequence of "standard" supersymmetric factorizations but

it is obtained in only two steps if V_1 is allowed to be singular. In fact, in the context of phase equivalence, only pairs of supersymmetric factorizations are considered. Intermediate steps (corresponding to odd values of the potential subscript) do not play any practical role. Hence, conditions on V_1 should be disregarded and the sequence of equations (32) to (37) provides the result (28) in a much faster way. Notice, however, that more general equations such as (27) still require the kind of recurrence proof performed in AB.

Equation (30) can also be obtained in a direct way generalizing the treatment presented in Ref. [8] for adding a bound state below the ground state. Starting with (30), Eq. (14) can be derived with a proof similar to the one employed in Sec. II.

In summary, one finds here an example of a case where the "no-node" restriction imposed to supersymmetric factorizations should be eliminated. Other cases, also involving combinations of several transformations, might exist where an intermediate singular potential would be acceptable.

VI. CONCLUSION

In this paper a potential providing the same phase shifts as a given potential but with arbitrary differences in the bound spectrum is derived with the supersymmetric factorization method. The result is simply expressed as a function of solutions of the initial differential equation. This compact analytic result may seem academic in the sense that iterative procedures provide more efficient algorithms for numerical applications. However, this is not the case at all since this analytic result gives rise to several interesting consequences.

First, the equivalence with the method of Abraham and Moses is now proved in a general case. The potentials derived in both methods, though presented in a very different way, are identical. The determinant expressions arising from the supersymmetric approach are simple and compact while the AM approach requires solving an integral equation. However, the main result of this comparison is that a general solution of the integral equation of the Gel'fand-Levitan type, appearing in the AM method, is now available. The expression of the integral kernel is simple in mixed notations (involving the initial and final problems simultaneously). In addition this kernel can be expressed as a function of the initial problem only, but in a more complicated way.

Notice that the general result (14) does not provide all the possible equivalent potentials compatible with the phase shifts and the bound spectrum, as stated at the beginning of Sec. II. A more general expression involving arbitrary parameters should exist combining the present result (14) and the expression (58) in Sec. VI of AB. In the AM approach, this would correspond to having common states in the two parts of the kernel Ω [Eq. (24)]. Establishing an equivalence between the two techniques in that case, i.e., calculating the kernel K, remains to be done.

Second, the simple problem of adding or removing a state anywhere in the bound spectrum receives an amazingly simple solution. The techniques for removing the ground state or adding a new one remain valid, with a simple modification of the relevant function. More generally, the formulas established in AB for removing the M lowest bound states or adding N new bound states below the ground state also remain valid for arbitrary sets of bound states.

Finally, the surprising result obtained for adding or removing a single bound state led me to reconsider the "no-node" restriction in the supersymmetric factorization process, i.e., the fact that only separation energies lower than or equal to the ground-state energy are considered in order to avoid finite-distance singularities in the potential of the supersymmetric partner. This condition does not necessarily apply if a succession of supersymmetric transformations is performed. In that case, a finite-distance singularity in an intermediate result may be acceptable provided that it leads to a nonsingular final result. This occurs in the construction of phase-equivalent potentials because only pairs of factorizations are needed. Hence, weakening this condition in other algorithms based on supersymmetry should be considered.

At the practical level, I think that the most interesting result, discussed in Sec. IV C, is the possibility of moving states in an almost arbitrary way in a bound spectrum. Of course, this modification of the bound-state energies is already available in Ref. [8] and in AB, but in a much more complicated way, by removing enough bound states and replacing them. Equations (28) and (30) [or Eq. (31)] now allow such modifications in a direct way, by correcting easily the energy location of a single state at each step, in an arbitrary order. This possibility might lead to the construction of potentials which not only reproduce given phase shifts but also a given bound spectrum.

ACKNOWLEDGMENTS

APPENDIX

In this appendix, I give proofs of some equations employed in the text. To this end, let me first recall a few recurrence relations satisfied by physical wave functions ψ_{2k} and nonphysical solutions f_{2k} associated with a potential V_{2k} of the iterative process. When constructing the phase-equivalent potential V_{2k+2}, the bound state of

V_{2k} at energy $E^{(k)}$ with a wave function $\psi_{2k}^{(k)}$ is eliminated. Equation (14) of AB provides

$$\psi_{2k+2} = \psi_{2k} - [\Psi_{2k}^{(k,k)}]^{-1}\psi_{2k}^{(k)}\int_0^r \psi_{2k}^{(k)}\psi_{2k}\,dt. \quad (A1)$$

Similarly, Eqs. (15) and (16) of AB read

$$f_{2k+2} = f_{2k} + [\Psi_{2k}^{(k,k)}]^{-1}\psi_{2k}^{(k)}\int_r^\infty \psi_{2k}^{(k)}f_{2k}\,dt \quad (A2)$$

and

$$f_{2k+2}^{(k)} = [\Psi_{2k}^{(k,k)}]^{-1}\psi_{2k}^{(k)}. \quad (A3)$$

Equation (A3) provides a nonphysical solution $f_{2k+2}^{(k)}$ at energy $E^{(k)}$ where the bound state has been suppressed. With the orthogonality between ψ_{2k} and $\psi_{2k}^{(k)}$, (A1) and (A2) can be combined into

$$\varphi_{2k+2} = \varphi_{2k} + [\Psi_{2k}^{(k,k)}]^{-1}\psi_{2k}^{(k)}\int_r^\infty \psi_{2k}^{(k)}\varphi_{2k}\,dt \quad (A4)$$

but (A4) is not valid for $\psi_{2k}^{(k)}$.

In order to prove the recurrence relation (11) for $\det \boldsymbol{X}_{2k}$ with these expressions, one has to discuss separately the cases $k \in S_0$ and $k \notin S_0$. When k belongs to S_0, all the functions $f_{2k+2}^{(i)}$ in $\boldsymbol{X}_{2k+2}$ correspond to functions $f_{2k}^{(i)}$ in $\boldsymbol{X}_{2k}$. This is not true if $k \notin S_0$, because $f_{2k+2}^{(k)}$ is related to $\psi_{2k}^{(k)}$ in $\boldsymbol{X}_{2k}$ as shown by (A3). For $k \in S_0$, a simple integration using (A4) immediately leads to

$$X_{2k+2}^{(i,j)} = X_{2k}^{(i,j)} - [X_{2k}^{(k,k)}]^{-1}X_{2k}^{(i,k)}X_{2k}^{(k,j)}. \quad (A5)$$

Equation (A4) applies for all $\varphi_{2k}^{(i)}$ because k does not appear as an index in $\boldsymbol{X}_{2k+2}$. Then the simple determinant property described in the appendix of AB provides the first part of Eq. (11) with $S_k = S_{k+1}$ and $A_k = \{k\} \cup A_{k+1}$.

For $k \notin S_0$, the calculation is slightly longer because $k \in A_{k+1}$. First, the proof leading to (A5) provides

$$X_{2k+2}^{(i,j)} = X_{2k}^{(i,j)} + [\Psi_{2k}^{(k,k)}]^{-1}\tilde{X}_{2k}^{(i,k)}\tilde{X}_{2k}^{(k,j)} \quad (i,j \neq k). \quad (A6)$$

Here a comment on the notation is necessary. Matrix $\boldsymbol{X}_{2k}$ does not contain elements with row or column index k. However, I employ the notation (7) anyway, but with a tilde. With (A3), even simpler integrations lead to

$$X_{2k+2}^{(i,k)} = [\Psi_{2k}^{(k,k)}]^{-1}\tilde{X}_{2k}^{(i,k)} \quad (i \neq k) \quad (A7)$$

and

$$X_{2k+2}^{(k,k)} = [\Psi_{2k}^{(k,k)}]^{-1}. \quad (A8)$$

By subtracting row k of $\det \boldsymbol{X}_{2k+2}$, multiplied by $\tilde{X}_{2k}^{(i,k)}$, from every row i, one obtains the second part of (11).

The proof of (18) follows a similar pattern. Equation (42) of AB can be rewritten as

$$\psi = [\det \boldsymbol{X}_{2n}]^{-1}\det \boldsymbol{Y}_{2n}(\psi_{2n}). \quad (A9)$$

When $k \in S_0$, the elements of $\boldsymbol{Y}_{2k}(\varphi_{2k})$ satisfy an extension of (A5)

$$Y_{2k+2}^{(i,j)}(\varphi_{2k+2}) = Y_{2k}^{(i,j)}(\varphi_{2k}) \\ -[X_{2k}^{(k,k)}]^{-1}Y_{2k}^{(i,k)}(\varphi_{2k})Y_{2k}^{(k,j)}(\varphi_{2k}). \quad (A10)$$

The proof involves a direct use of (A5) and (A4), as well as the calculation with a simple integration of $\int_r^\infty \varphi_{2k+2}^{(j)}\varphi_{2k+2}dt$. For $k \notin S_0$, (A6) to (A8) are extended to similar relations where X and $\tilde{X}$ are replaced, respectively, by Y and $\tilde{Y}$. Hence, one obtains

$$\det \boldsymbol{Y}_{2k+2}(\varphi_{2k+2})$$

$$= \begin{cases} -[\Psi_{2k}^{(k,k)}]^{-1}\det \boldsymbol{Y}_{2k}(\varphi_{2k}) & (k \in S_0) \\ +[\Psi_{2k}^{(k,k)}]^{-1}\det \boldsymbol{Y}_{2k}(\varphi_{2k}) & (k \notin S_0). \end{cases} \quad (A11)$$

By recurrence, (A9) and (A11) provide, for $0 \leq k \leq n$,

$$\psi = [\det \boldsymbol{X}_{2k}]^{-1}\det \boldsymbol{Y}_{2k}(\psi_{2k}) \quad (A12)$$

from which (18) follows.

The proof of Eq. (18) is not valid at energies where states have been added or have been suppressed and reintroduced. I now consider these cases by starting from Eq. (43) of AB which reads, after some permutation, as

$$\psi^{(m)} = \alpha_m^{1/2}[\det \boldsymbol{X}_{2n}]^{-1}\det \boldsymbol{X}_{2n(m)}, \quad (A13)$$

where $\boldsymbol{X}_{2n(m)}$ is defined in (17). From (A5) and (A4), the familiar-looking recurrence relation

$$X_{2k+2(m)}^{(i,j)} = X_{2k(m)}^{(i,j)} - [X_{2k(m)}^{(k,k)}]^{-1}X_{2k(m)}^{(i,k)}X_{2k(m)}^{(k,j)} \quad (A14)$$

is easily established for $k \in S_0$. Then, (A2) of AB provides the determinant relation

$$\det \boldsymbol{X}_{2k+2(m)} = -[\Psi_{2k}^{(k,k)}]^{-1}\det \boldsymbol{X}_{2k(m)} \quad (k \in S_0) \quad (A15)$$

which is an obvious extension of the first part of (A11). In a similar way, (A6) to (A8) can be extended for $k \notin S_0$ as

$$X_{2k+2(m)}^{(i,j)} = X_{2k(m)}^{(i,j)} + [\Psi_{2k}^{(k,k)}]^{-1}\tilde{X}_{2k}^{(i,k)}\tilde{X}_{2k(m)}^{(k,j)} \quad (i,j \neq k), \quad (A16)$$

$$X_{2k+2(m)}^{(i,k)} = [\Psi_{2k}^{(k,k)}]^{-1}\tilde{X}_{2k}^{(i,k)} \quad (i \neq k, k \neq m), \quad (A17)$$

$$X_{2k+2(m)}^{(k,j)} = [\Psi_{2k}^{(k,k)}]^{-1}\tilde{X}_{2k(m)}^{(k,j)} \quad (j \neq k), \quad (A18)$$

and

$$X_{2k+2(m)}^{(k,k)} = [\Psi_{2k}^{(k,k)}]^{-1} \quad (k \neq m), \quad (A19)$$

where $\tilde{X}_{2k(m)}^{(k,j)} = \tilde{X}_{2k}^{(k,j)}$ except for $j = m$ where $\tilde{X}_{2k(m)}^{(k,m)} = \psi_{2k}^{(k)}$. As earlier, one obtains from determinant manipulations the complement of (A15),

$$\det \boldsymbol{X}_{2k+2(m)} = [\Psi_{2k}^{(k,k)}]^{-1} \det \boldsymbol{X}_{2k(m)} \quad (k \notin S_0,\ k \neq m).$$

$$(\text{A}20)$$

However, for $k = m$, the elements of column m in $\boldsymbol{X}_{2m+2(m)}$ do not follow (A17) or (A19). Equation (A17) is replaced by (A16) with $j = m$. The matrix element $X_{2m+2(m)}^{(m,m)}$ is not equal to $X_{2m+2}^{(m,m)}$ but to $f_{2m+2}^{(m)}$ and verifies therefore a relation of the form (A18). By subtracting row m in $\det \boldsymbol{X}_{2m+2(m)}$, multiplied by $\tilde{X}_{2m}^{(i,m)}$, from the other rows, one obtains the relation

$$\det \boldsymbol{X}_{2m+2(m)} = [\Psi_{2m}^{(m,m)}]^{-1} \det \boldsymbol{Y}_{2m}(\psi_{2m}^{(m)}) \qquad (\text{A}21)$$

involving the matrix $\boldsymbol{Y}_{2m}$ defined in (16) but for the particular case $\varphi_{2m} = \psi_{2m}^{(m)}$. Now, Eqs. (A13), (A15), (A20), and (11) provide

$$\psi^{(m)} = \alpha_m^{1/2} [\det \boldsymbol{X}_{2k}]^{-1} \det \boldsymbol{X}_{2k(m)} \quad (k > m). \quad (\text{A}22)$$

When $m \in A_0$, $k = 0$ proves (19). For a reintroduced state, m does of course not belong to S_0 and to A_0. Then, in the recurrence process, k decreases to the value m where (A21) applies and gives

$$\psi^{(m)} = [\det \boldsymbol{X}_{2m}]^{-1} \det \boldsymbol{Y}_{2m}(\psi_{2m}^{(m)}) \qquad (\text{A}23)$$

with $\alpha_m = 1$. Hence, the general formula (A12) is recovered for the particular choice $\psi_{2m}^{(m)}$ of the wave function and (18) is valid. This was to be expected as the number n of removed states needs not be precisely fixed. The wave function of states whose removal and reintroduction is not essential should not depend on the fact that they have been considered or not.

Although all physical wave functions are now available, one still has to prove the expression (22) for nonphysical solutions corresponding to removed states, which is useful in establishing the connection with the AM method. This is a little more complicated because one has to start from a formula which is not derived in AB. However, since all proofs are almost identical to those of AB, or to earlier proofs in the present work, I only state a few important steps. For nonphysical solutions f, bounded at infinity, of the Schrödinger differential equation corresponding to the addition of $n + N - M$ bound states to potential V_{2n}, Eq. (42) of AB can be generalized into

$$f = [\det \boldsymbol{F}_{2n}]^{-1} \det \boldsymbol{Y}_{2n}(f_{2n}). \qquad (\text{A}24)$$

In (A24), the definition (16) for $\boldsymbol{Y}_{2n}$ is applied with φ_{2n} representing a nonphysical solution f_{2n}. We apply (A24) to calculate $f^{(m)}$ as a function of $f_{2n}^{(m)}$. Then, (A11) can be used provided that $k > m$. When $k = m$, (A3) leads to

$$\det \boldsymbol{Y}_{2m+2}(f_{2m+2}^{(m)}) = [\Psi_{2m}^{(m,m)}]^{-1} \det \boldsymbol{X}_{2m(m)} \qquad (\text{A}25)$$

which should be compared with (A21). The proof is rather simple because m does not appear as an index in $\boldsymbol{Y}_{2m+2}$ ($m \notin S_{m+1} \cup A_{m+1}$). When $k = m$, the recurrence provides

$$f^{(m)} = [\det \boldsymbol{X}_{2m}]^{-1} \det \boldsymbol{X}_{2m(m)} \qquad (\text{A}26)$$

with the help of (10) and (A25). Then, (11), (A15), and (A20) prove (22).

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